Static spherically symmetric star in Gauss–Bonnet gravity*

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We explore static spherically symmetric stars in Gauss–Bonnet gravity without a cosmological constant, and present an exact internal solution which attaches to the exterior vacuum solution outside stars. It turns out that the presence of the Gauss–Bonnet term with a positive coupling constant completely changes thermal and gravitational energies, and the upper bound of the red shift of spectral lines from the surface of stars. Unlike in general relativity, the upper bound of the red shift is dependent on the density of stars in our case. Moreover, we have proven that two theorems for judging the stability of equilibrium of stars in general relativity can hold in Gauss–Bonnet gravity.

Keywords: Gauss–Bonnet gravity, static spherically symmetric star, red shift

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1. Introduction

Up to now, many quantum theories of gravity have been proposed. The most promising candidate among them is the superstring/M-theory. Following the advent of string theory, extra dimensions were promoted from an interesting curiosity to a theoretical necessity since superstring theory requires an eleven-dimensional space-time to be consistent from a quantum point of view.[1−4] It caused a renewed interest about general relativity in more than four dimensions. In recent years, renewed interest has also grown in higher order gravity, which involves higher derivative curvature terms. Among the higher curvature gravities, the most extensively studied theory is the so-called Gauss–Bonnet gravity,[5,6] which naturally emerges when we want to generalize Einstein's theory in higher dimensions by keeping all characteristics of usual general relativity excepting the linear dependence of the Riemann tensor. In addition, the Gauss–Bonnet term occurs in the effective low-energy action of superstring theory.[7−10] Hence, Gauss–Bonnet gravity provides a promising framework to study curvature corrections to the Einstein–Hilbert action. In Gauss–Bonnet gravity, the exact static and spherically symmetric black hole solutions were found in Refs. [11], [12], and [13], their thermodynamics have been investigated[14−17] and slowly rotating black hole solutions have been obtained in Refs. [18]–[21].

As is well known, in general relativity, there is an exact solution of gravitational fields with a spherically symmetric static star, which takes the form of the exterior Schwarzschild solution of the vacuum outside the star.[22] It implies that the exterior Schwarzschild solution can naturally emerge through a suitable distribution of matter. Thus, we are convinced of the existence of the Schwarzschild black hole, since it is formed from the gravitational collapse of a heavy star. An interesting question will be: what is the solution of gravitational fields with a spherically symmetric static star in higher dimensional Gauss–Bonnet gravity, which relates to the solution of black holes, if matter can disperse to the extra dimension?

The objective of this paper is to find a solution and to discuss the physical properties of static spherically symmetric stars in Gauss–Bonnet gravity without a cosmological constant. The dimension of space-time is assumed to be $D \geq 5$ since the Gauss–Bonnet term yields nontrivial dynamics. We try to investigate the spherically symmetric system consisting of static perfect fluid, and give an exact solution. In order

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to find the effect of Gauss–Bonnet term, we discuss thermal and gravitational energies, the distribution of pressure of stars, and the upper bound of the red shift of spectral lines from their surface. Meanwhile, we also attempt ot find whether two theorems for judging the stability of equilibrium in general relativity hold in $D \geq 5$ Gauss–Bonnet gravity.

This paper is organized as follows. In Section 2, we give the equations for the structure of stars in $D \geq 5$ Gauss–Bonnet gravity without the cosmological constant, and find a solution. In Section 3, we investigate the effects of Gauss–Bonnet term. The equilibrium stability of stars is demonstrated in Section 4. Some remarks are given in Section 5.

2. Differential equations and solutions for star structure

The action of Gauss–Bonnet gravity in D-dimensional space-time can be written in the form of

$$S = -\frac{1}{2\kappa}\int \mathrm{d}^D x\sqrt{-g}(R + \alpha\mathcal{L}_2) + S_{\mathrm{matter}}, \quad (1)$$

where the coupling constant α can be regarded as the inverse string tension and we assume $\alpha > 0$ in this paper. Here the second-order (Gauss–Bonnet) term of the Lagrangian is given by

$$\mathcal{L}_2 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2.$$

Varying the action Eq. (1), we obtain the field equations as

$$G_{ab} = G^{(1)}_{ab} + \alpha G^{(2)}_{ab} = \kappa T_{ab}, \quad (2)$$

where

$$\begin{aligned} G^{(1)}_{ab} &= R_{ab} - \frac{1}{2}Rg_{ab}, \\ G^{(2)}_{ab} &= 2(-R_{acde}R^{dec}{}_{b} - 2R_{acbd}R^{cd} \\ &\quad - 2R_{ac}R^{c}{}_{b} + RR_{ab}) - \frac{1}{2}\mathcal{L}_2 g_{ab}. \end{aligned}$$

For future simplicity, we take the coefficient α as $\alpha = \tilde{\alpha}/(D-3)(D-4)$.

The metric for a static spherically symmetric system will be taken in the form of

$$\mathrm{d}s^2 = -B(r)\mathrm{d}t^2 + A(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{D-2}, \quad (3)$$

where A and B are functions of r, and $\mathrm{d}\Omega^2_{D-2}$ represents the line element of a $(D-2)$-dimensional unit sphere with $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$. The energy–momentum tensor is assumed to be that for a perfect fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \quad (4)$$

with p the proper pressure, ρ the proper total energy density, and U_μ the velocity four-vector, defined so that $U_\mu U^\mu = -1$. Since the fluid is at rest, we take

$$U^\mu = \frac{1}{\sqrt{B(r)}}\delta^\mu_t.$$

We assume p and ρ are positive throughout the star. Our assumptions of time independence and spherical symmetry imply that p and ρ are functions only of the radial coordinate r.

Inserting Eq. (3) and Eq. (4) into Eq. (2), we find that the equations read

$$-G^t_t = \frac{D-2}{2}\frac{\tilde{\alpha}[(D-5)(A-1)^2A + 2(A-1)A'r] + (D-3)(A-1)A^2r^2 + AA'r^3}{A^3r^4} = \kappa\rho, \quad (5)$$

$$G^r_r = \frac{D-2}{2}\frac{\tilde{\alpha}[-(D-5)(A-1)^2B + 2(A-1)B'r] - (D-3)(A-1)ABr^2 + AB'r^3}{A^2Br^4} = \kappa p, \quad (6)$$

where a prime denotes the derivative with respect to r. Here we omit the third equation. Instead, we use the equation for hydrostatic equilibrium which is equivalent to the equation of energy–momentum conservation

$$\frac{B'}{B} = -\frac{2p'}{p+\rho}. \quad (7)$$

From Eq. (5) we find that the solution with finite $A(0)$ is

$$A(r) = \frac{1}{1 + \dfrac{r^2}{2\tilde{\alpha}} - \sqrt{\dfrac{r^4}{4\tilde{\alpha}^2} + \dfrac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}}, \quad (8)$$

where the mass function $M(r)$ is defined as

$$M(r) = \Omega_{D-2}\int_0^r \rho(r')r'^{(D-2)}\mathrm{d}r'.$$

Here we have abandoned another solution since it does not recover the solution in general relativity in the

limit $\alpha \to 0$. The function $M(r)$ is not the total energy of the matter since it is defined as the integral of the energy density $\rho(r)$ of the matter alone thus does not include the energy of the gravitational field. The thermal and gravitational energies of the star will be discussed in the next section. Using Eqs. (6), (7), and (8), we obtain

$$p' = -r(p+\rho)\frac{\frac{2}{D-2}p + 2\sqrt{\frac{1}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-1}}} + \frac{2(D-5)\kappa M}{(D-2)\Omega_{D-2}r^{D-1}} - \frac{1}{\tilde{\alpha}}}{4\tilde{\alpha}\sqrt{\frac{1}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-1}}}\left(1 + \frac{r^2}{2\tilde{\alpha}} - \sqrt{\frac{r^4}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}\right)}, \tag{9}$$

and the solution with $B(\infty) = 1$ is

$$B(r') = \exp\left[-\int_{r'}^{+\infty} r\frac{\frac{2}{D-2}p + 2\sqrt{\frac{1}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-1}}} + \frac{2(D-5)\kappa M}{(D-2)\Omega_{D-2}r^{D-1}} - \frac{1}{\tilde{\alpha}}}{2\tilde{\alpha}\sqrt{\frac{1}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-1}}}(1 + \frac{r^2}{2\tilde{\alpha}} - \sqrt{\frac{r^4}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}})}\mathrm{d}r\right]. \tag{10}$$

Outside the star, $p(r)$ and $\rho(r)$ vanish, and $M(r)$ is the constant $M(R)$, therefore Eqs. (8) and (10) give

$$\begin{aligned} B(r) &= A^{-1}(r) \\ &= 1 + \frac{r^2}{2\tilde{\alpha}} - \sqrt{\frac{r^4}{4\tilde{\alpha}^2} + \frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}} \end{aligned} \tag{11}$$

for arbitrarily $r \geq R$. Equation (11) is the exterior solution of the static spherically symmetric black hole in a flat space. Hence, if the matter can disperse to the extra dimension, in the $D \geq 5$ spherically symmetric space-time, there is an interior solution which describes a perfect fluid relating to the exterior solution of the static spherically symmetric black hole. Thus, Eq. (11) makes the solution of black holes more believable since such exterior solution can be naturally emerged by a suitable distribution of matter.

Equation (9) is the generalization of Tolman–Oppenheimer–Volkoff (TOV) equation in the $D \geq 5$ Einstein–Gauss–Bonnet (EGB) theory. One can check that in the general relativity limit $\alpha \to 0$, such an equation in a 5D space-time will be written in the form of

$$p' = -r(p+\rho)\frac{p + \frac{(D-3)\kappa M}{\Omega_{D-2}r^{D-1}}}{D-2-\frac{2\kappa M}{\Omega_{D-2}r^{D-3}}}, \tag{12}$$

which is similar to the TOV equation in a 4D case.[23–25] As in general relativity, we can use this equation to obtain $\rho(r)$, $M(r)$, $p(r)$ throughout a static spherically symmetric star in the 5D space-time. The pressure p may in general be expressed as a function of the density ρ, the entropy per nucleon s, and the chemical composition. We can assume that the entropy per nucleon s does not vary throughout the star, and the star we considered has a chemical composition that is constant throughout for simplicity, that is, $p(r)$ may be regarded as a function of $\rho(r)$ alone. The definition of $M(r)$ provides an initial condition $M(0) = 0$. Equation (9) together with an equation of state giving $p(\rho)$, serves to determine $\rho(r)$, $M(r)$, $p(r)$ of the star for arbitrary r, once we specify the other initial condition for the value of $\rho(0)$. Equation (9) should be integrated out from the center of the star, until $p(\rho(r))$ drops to zero at some point $r = R$, which we then interpret as the radius of the star.

3. Effect of Gauss–Bonnet term

This section is devoted to the influence on a spherically symmetric star caused by Gauss–Bonnet term. First we investigate the thermal and gravitational energies of the star. As in general relativity, we can compare $M(R)$ with the energy M_0 that the matter of the star would have dispersed to infinity in $D \geq 5$ space-time. This is simply $M_0 = m_N N$, where m_N is the rest mass of a nucleon, N is the number of nucleons in the star which is given by

$$\begin{aligned} N &= \int_\Sigma \sqrt{-g} J_N^0 \,\mathrm{d}r\,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{D-2} \\ &= \Omega_{D-2}\int_0^R \sqrt{A(r)B(r)} J_N^0 r^{D-2}\,\mathrm{d}r, \end{aligned}$$

where J_N^μ is the conserved nucleon current, $\theta_1, \ldots, \theta_{D-2}$ are spherical coordinates on S_{D-2}, Σ

is a Cauchy surface of the space-time. J^0_N can be expressed in terms of the proper nucleon number density n, which is $n = -U_\mu J^\mu_N = \sqrt{B}J^0_N$. The expression of N then becomes

$$N = \Omega_{D-2}\int_0^R \sqrt{A(r)}n(r)r^{D-2}\mathrm{d}r. \tag{13}$$

The proper number density $n(r)$ and N are fixed for a star with a given constant s and chemical composition, once we choose $\rho(0)$. The internal energy of the star is now defined by $E = M - m_N N$. We can define the proper internal material energy density as $e(r) = \rho(r) - m_N n(r)$, and decompose the internal energy E as the sum of the thermal and gravitational energies T and V, respectively

$$\begin{aligned} T &= \Omega_{D-2}\int_0^R \sqrt{A}e(r)r^{D-2}\mathrm{d}r, \\ V &= \Omega_{D-2}\int_0^R (1-\sqrt{A})\rho(r)r^{D-2}\mathrm{d}r. \end{aligned} \tag{14}$$

Using Eq. (8), we obtain that

$$\begin{aligned} T &= \Omega_{D-2}\int_0^R \frac{1}{\sqrt{1+\dfrac{r^2}{2\tilde{\alpha}}-\sqrt{\dfrac{r^4}{4\tilde{\alpha}^2}+\dfrac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}}}e(r)r^{D-2}\mathrm{d}r, \\ V &= \Omega_{D-2}\int_0^R \left(1-\frac{1}{\sqrt{1+\dfrac{r^2}{2\tilde{\alpha}}-\sqrt{\dfrac{r^4}{4\tilde{\alpha}^2}+\dfrac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}}}\right)\rho(r)r^{D-2}\mathrm{d}r. \end{aligned} \tag{15}$$

It is straightforward to check that

$$\frac{\mathrm{d}A}{\mathrm{d}\tilde{\alpha}} = -\frac{1}{2\tilde{\alpha}\sqrt{\dfrac{r^4}{4\tilde{\alpha}^2}+\dfrac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}}\left(1-\frac{1}{1+\dfrac{r^2}{2\tilde{\alpha}}-\sqrt{\dfrac{r^4}{4\tilde{\alpha}^2}+\dfrac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}}\right)^2 < 0, \tag{16}$$

which leads to $\partial T/\partial\tilde{\alpha} < 0$ and $\partial V/\partial\tilde{\alpha} > 0$. Hence, the Gauss–Bonnet term with positive α will decrease the thermal energy and increase the gravitational energy.

Next, we consider a star with uniform density, consisting of incompressible fluids with $\rho = \text{const}$. In this case Eq. (9) may be written as

$$p'(r) = -r(p+\rho)\frac{\dfrac{2}{D-2}p + 2\sqrt{\dfrac{1}{4\tilde{\alpha}^2}+\dfrac{2\kappa\rho}{(D-1)(D-2)\tilde{\alpha}}} + \dfrac{2(D-5)\kappa\rho}{(D-1)(D-2)} - \dfrac{1}{\tilde{\alpha}}}{4\tilde{\alpha}\sqrt{\dfrac{1}{4\tilde{\alpha}^2}+\dfrac{2\kappa\rho}{(D-1)(D-2)\tilde{\alpha}}}\left[1+r^2\left(\dfrac{1}{2\tilde{\alpha}}-\sqrt{\dfrac{1}{4\tilde{\alpha}^2}+\dfrac{2\kappa\rho}{(D-1)(D-2)\tilde{\alpha}}}\right)\right]}. \tag{17}$$

The solution with $p(R) = 0$ is

$$p(r) = \frac{\rho\Phi - \rho\Phi\Xi(r)}{\kappa\rho\Xi(r) - \Phi}, \tag{18}$$

here we define

$$\begin{aligned} \Phi &= (D-2)\sqrt{\frac{1}{4\tilde{\alpha}^2}+\frac{2\kappa\rho}{(D-1)(D-2)\tilde{\alpha}}} + \frac{(D-5)\kappa\rho}{D-1} - \frac{D-2}{2\tilde{\alpha}}, \\ \Theta &= \frac{1}{2\tilde{\alpha}} - \sqrt{\frac{1}{4\tilde{\alpha}^2}+\frac{2\kappa\rho}{(D-1)(D-2)\tilde{\alpha}}}, \end{aligned}$$

$$\Xi(r) = \sqrt{\frac{1+\Theta R^2}{1+\Theta r^2}}.$$

Such a solution does not make sense for all values of ρ and R. The pressure will become infinite at a point r_∞ where

$$r_\infty^2 = \frac{(1+\Theta R^2)\kappa^2\rho^2 - \Phi^2}{\Theta\Phi^2}.$$

An infinity in $p(r)$ cannot be blamed on an injudicious choice of coordinate system since the pressure is a scalar. In order to ensure that $p(r)$ is not singular

throughout the star, we find that

$$1+\Theta R^2 > \frac{\Phi^2}{\kappa^2\rho^2}, \tag{19}$$

which leads to $r_\infty^2 < 0$.

Hence, the function $B(R) = 1+\Theta R^2$ has a lower limit. This result has a profound influence on the red shift of spectral lines from the surface of stars. The red shift z will have an upper bound $z_{\max}$ since $z = \triangle\lambda/\lambda = B^{-1/2}(R) - 1$, which gives

$$z_{\max} = \frac{\kappa\rho}{\Phi} - 1. \tag{20}$$

In the general relativistic limit, we have $\lim\limits_{\alpha\to 0} z_{\max} = 2/(D-3)$. Obviously, a non-negative coefficient α will change the upper bound of the red shift, which is given in Fig. 1. The most interesting consequence of the second-order curvature corrections is that $z_{\max}$ is dependent on the value of ρ while it is a constant in general relativity case.[26]

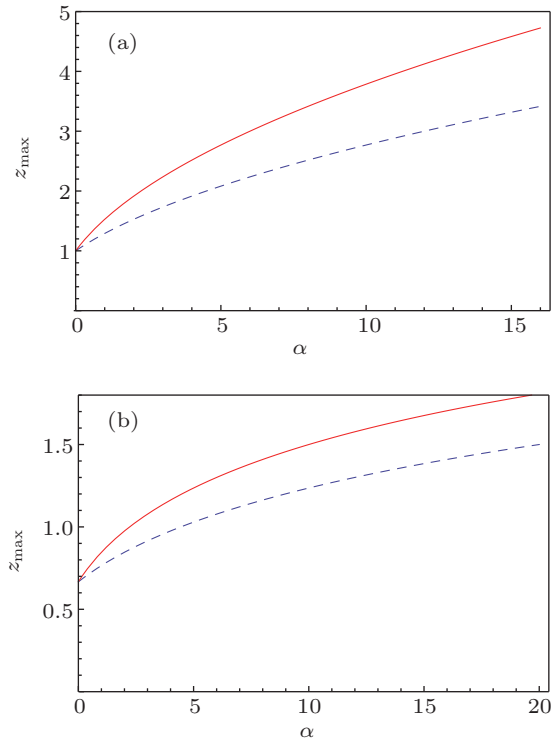


Fig. 1. (colour online) Relation between α and the upper bound of $z_{\max}$ in 5D (a) and 6D (b) Gauss–Bonnet gravity: the solid and dotted lines represent $\rho = 2$ and $\rho = 1$, respectively. In general, relativity limit $\alpha \to 0$ two lines have the same limit.

4. Stability of equilibrium

Our solution represents an equilibrium state of the star, but it may be stable or unstable. For most purposes, we are only concerned with the stable solution. In order to investigate the stability of a particular configuration, it would be necessary to compute the frequencies ω_n of all normal modes of the configuration. If the frequency ω_n has a positive imaginary part, the factor $\exp(-\mathrm{i}\,\omega_n t)$ would grow exponentially, and the system would be unstable. However, it is often possible to obtain from the equilibrium solution alone whether the corresponding configuration is stable, since we have following two theorems:

Theorem 1 A star consisting of a perfect fluid with a constant chemical composition and entropy per nucleon can only pass from stability to instability with respect to some particular radial normal mode, at a value of the central density $\rho(0)$ for which the equilibrium energy E and nucleon number N are stationary under the conditions that

$$\begin{aligned}\frac{\partial E(\rho(0), s\ldots)}{\partial\rho(0)} &= 0,\\ \frac{\partial N(\rho(0), s, \ldots)}{\partial\rho(0)} &= 0,\end{aligned} \tag{21}$$

by a radial normal mode, that is, a mode of oscillation in which the density perturbation $\delta\rho$ is a function of t and r alone, and in which nuclear reactions, viscosity, heat conduction, and radiative energy transfer play no role.

Theorem 2 A particular star configuration with uniform entropy per nucleon and chemical composition will satisfy Eq. (9) for equilibrium, if and only if the quantity M defined by $M = \Omega_{D-2}\int\rho(r)r^3\mathrm{d}r$ is stationary with respect to all variations of $\rho(r)$ that leave the quantity $N = \Omega_{D-2}\int\sqrt{A}n(r)r^3\mathrm{d}r$ unchanged and that leave the entropy per nucleon and the chemical composition uniform and unchanged. The equilibrium is stable with respect to radial oscillations if and only if M or equivalently E is a minimum with respect to all such variations.

These theorems have been proven in general relativity in $4D$ space-time.[27,28] The proof of Theorem 1 is unconcerned with dimensions and the metric of space-time, therefore it is obviously appropriate for our $D \geq 5$ star in Gauss–Bonnet gravity. Now we are going to prove Theorems 2 in the present case by using the Lagrange multiplier method. M will be stationary with respect to all variations that leave N fixed if and only if there exists a constant λ for which $M - \lambda N$ is stationary with respect to all variations. In general, the change in $M - \lambda N$ for a given variation $\delta\rho(r)$ is

$$\delta M-\lambda\delta N=\Omega_{D-2}\int_0^\infty\delta\rho(r)r^{D-2}\mathrm{d}r-\lambda\Omega_{D-2}\int_0^\infty\frac{\delta n(r)r^{D-2}\mathrm{d}r}{\sqrt{1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M)}}$$
$$-\lambda\Omega_{D-2}\int_0^\infty\frac{n(r)\delta M(r)r^3\mathrm{d}r}{2(D-2)\tilde{\alpha}\Omega_{D-2}\Sigma(r,M)(1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M))^{3/2}},\tag{22}$$

where

$$\Sigma(r,M)=\sqrt{\frac{r^4}{4\tilde{\alpha}^2}+\frac{2\kappa M}{(D-2)\tilde{\alpha}\Omega_{D-2}r^{D-5}}}.$$

Here the integrals are carried out to infinity for notational convenience since they vanish outside a radius $R+\delta R$. These variations are supposed not to change the entropy per nucleon, that is $0=\delta(\rho/n)+p\delta(1/n)$, therefore

$$\delta n(r)=\frac{n(r)}{p(r)+\rho(r)}\delta\rho(r).$$

Obviously,

$$\delta M(r)=\Omega_{D-2}\int_0^r\delta\rho(r')r'^{D-2}\mathrm{d}r'.$$

Thus, $\delta M-\lambda\delta N$ will vanish for all $\delta\rho(r)$ if and only if

$$\frac{1}{\lambda}=\frac{n}{(p+\rho)\sqrt{1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M)}}+\Omega_{D-2}\int_r^\infty\frac{n(r')r'^3\mathrm{d}r'}{2(D-2)\tilde{\alpha}\Omega_{D-2}\Sigma(r',M)\left(1+\dfrac{r'^2}{2\tilde{\alpha}}-\Sigma(r',M)\right)^{3/2}}.\tag{23}$$

This will be the case for some Lagrange multiplier λ if and only if the right-hand side is independent of r that

$$0=\left[\frac{n'}{p+\rho}-\frac{n(p'+\rho')}{(p+\rho)^2}\right]\frac{1}{\sqrt{1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M)}}+\frac{n}{p+\rho}\frac{\dfrac{r^3}{2\tilde{\alpha}}+\dfrac{\rho r^3}{D-2}-\dfrac{(D-5)\kappa M}{(D-2)\Omega_{D-2}r^{D-4}}-r\Sigma(r,M)}{2\tilde{\alpha}\Sigma(r,M)\left(1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M)\right)^{3/2}}$$
$$-\frac{nr^3}{2(D-2)\tilde{\alpha}\Sigma(r,M)\left(1+\dfrac{r^2}{2\tilde{\alpha}}-\Sigma(r,M)\right)^{3/2}}.\tag{24}$$

The condition of uniform entropy per nucleon gives $0=(\rho/n)'+p(1/n)'$ and therefore $n'(r)=n(r)\rho'(r)/[p(r)+\rho(r)]$. Substituting the expression of $n'(r)$ into Eq. (24), we obtain Eq. (9). Consequently δM vanishes for all $\delta\rho(r)$ under a given N and Eq. (9). Thus, the stability is determined by the sign of the second order of δM. Namely, the star is stable if $\delta^2M(r)/\delta\rho(r)^2\geq 0$ and unstable if $\delta^2M(r)/\delta\rho(r)^2\leq 0$.

5. Conclusion

In this paper, we have investigated static spherically symmetric stars in $D\geq 5$ Gauss–Bonnet gravity without the cosmological constant. It is shown that there is an exact solution of such a system relating to the exterior solution of the static spherically symmetric black hole, and the Gauss–Bonnet interaction decreases or enhances the thermal energy and increases the gravitational energy of stars respectively. The Gauss–Bonnet terms will modify the upper bound of the red shift of spectral lines from the surface of stars of uniform density, and the upper bound is dependent on the value of density rather than a constant in general relativity case. In addition, two theorems for judging the stability of equilibrium of stars in $D\geq 5$ Gauss–Bonnet gravity are proved, which are the natural generalization of the ones in general relativity.

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