A 9 × 9 matrix representation of Temperley–Lieb algebra and corresponding entanglement*

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We present the matrix U. The 9×9 U matrix is a representation of specialized Temperley–Lieb algebra. Based on which, a unitary $\check{\boldsymbol{R}}$ matrix is generated via the Yang–Baxterization approach. The 9×9 $\check{\boldsymbol{R}}$ matrix, a solution of the Yang–Baxter equation, is universal for quantum entanglement.

Keywords: Temperley–Lieb algebra, Yang–Baxter equation, entanglement

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1. Introduction

Very recently, braiding operators and the Yang–Baxter equation (YBE) have been introduced to the field of quantum information and quantum computation.[1−11] The YBE originated from solving the one-dimensional δ-interacting models[12,13] and the statistical models.[14] A solution $\check{\boldsymbol{R}}$ matrix of the YBE can be obtained from a given braid group representation (BGR) by introducing a spectral parameter.[15,16] The authors in Ref. [2] has explored the role of unitary braiding operators in quantum computation. It has been shown that the unitary braid matrix can be identified as the universal quantum gate.[2−4] This motivates a novel way to study quantum entanglement and the Berry phase based on the theory of braiding operators, as well as the YBE. Along this direction, the first step is initiated by Zhang, Kauffman and Ge.[4] In Ref. [4], the Bell matrix, generating two-qubit entangled states, has been recognized to be a unitary braid transformation. Hereafter many researchers have devoted great effort to research of the topic.[5−11]

As is known, Temperley–Lieb algebra (TLA)[17] is intimately connected with braid group. In fact, TLA is the subalgebra of braid algebra, and any representation of TLA constructs automatically a representation of a corresponding braid group. A unitary solution of the YBE can also be constructed from a representation of TLA via Yang–Baxterization approach, so TLA has been widely used in the construction of YBE solutions.[18−22] Recently, the author in Ref. [23] showed that TLA is found to present a suitable mathematical framework for describing entangle swapping and universal quantum computation. In a very recent work, a reducible representation of TLA is constructed on the tensor product of n-dimensional space.[24] Due to the importance of TLA in quantum information processing, we shall seek for high dimensional matrix solutions of TLA. By means of the Yang–Baxterization approach, a matrix $\check{\boldsymbol{R}}$ associated with TLA can be constructed.

This paper is organized as follows. In Section 2, we first recall the method of constructing some $n^2 \times n^2$ matrix solutions of TLA, and then using this method, we obtain a 9×9 matrix representation of TLA. In Section 3, related to this U matrix, we derive a unitary matrix $\check{\boldsymbol{R}}$ and investigate the entanglement properties of it. We show that the arbitrary degree of entanglement for two qubits entangled states can be generated via the unitary $\check{\boldsymbol{R}}$ matrix acting on the standard basis. We end this paper with a brief conclusion.

2. A 9 × 9 matrix representation of TLA

We first briefly review the theory of TLA.[17] It is the algebra generated by U_i $(i = 1, 2, \ldots, n-1)$ which subjects to the following relations,

$$U_i U_{i\pm1} U_i = U_i, \quad 1 \leq i \leq n, \tag{1a}$$

$$U_i^2 = d\, U_i, \quad 1 \leq i \leq n-1, \tag{1b}$$

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$$U_iU_j = U_jU_i, \quad |i-j| \geq 2, \tag{1c}$$

where d is the single loop in the knot theory which does not depend on the sites of the lattices. The notation $U_i \equiv U_{i,i+1}$ is used, $U_{i,i+1}$ implies $I_1 \otimes \cdots \otimes I_{i-1} \otimes U \otimes I_{i+2} \otimes \cdots \otimes I_n$, and I_j represents the unit matrix of the j-th particle. We call U the representation of TLA.

In Ref. [24], a method of constructing some $n^2 \times n^2$ matrix solutions of TLA has been shown. Let us review it briefly. The representation is defined by two invertible $n \times n$ matrices $\boldsymbol{A} \in GL(n, \mathbf{C})$ and $\boldsymbol{B} \in GL(n, \mathbf{C})$, which can also be seen as an n^2-dimensional vector $A_{\mathrm{ab}} \in (\mathbf{C}^n \otimes \mathbf{C}^n)$ and an n^2-dimensional vector $B_{\mathrm{ab}} \in (\mathbf{C}^n \otimes \mathbf{C}^n)$, respectively. The generators U_i can be expressed as

$$(U_{\mathrm{i}})^{\mathrm{ab}}_{\mathrm{cd}} = A^{\mathrm{a}}_{\mathrm{b}} B^{\mathrm{c}}_{\mathrm{d}} \in \mathrm{Mat}\left(\mathbf{C}^n_i \otimes \mathbf{C}^n_{i+1}\right), \tag{2}$$

where one explicitly writes the indices corresponding to the factors in the tensor product space. The notations $U^{\mathrm{ab}}_{\mathrm{cd}} = U_{\mathrm{ab,cd}}$, $A^{\mathrm{a}}_{\mathrm{b}} \equiv A_{\mathrm{ab}}$, and $B^{\mathrm{c}}_{\mathrm{d}} \equiv B_{\mathrm{cd}}$ are used. In order to satisfy Eq. (1b), if and only if

$$(\boldsymbol{BA})^{\mathrm{T}} (\boldsymbol{AB}) = \boldsymbol{I}_{n\times n}, \tag{3}$$

equation (1a) determines the single loop d:

$$U_i^2 = U_i \mathrm{tr}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\right), \quad d = \mathrm{tr}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\right), \tag{4}$$

where $\mathrm{tr}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\right)$ denotes the trace of matrix $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}$, and $\boldsymbol{A}^{\mathrm{T}}$ denotes the transpose of matrix $\boldsymbol{A}$. By means of this method [Eq. (2)], one can construct lots of $n^2 \times n^2$ matrix $\boldsymbol{U}$ with $d = \mathrm{tr}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}\right)$ as long as $n \times n$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy Eq. (3). Especially, when $\boldsymbol{B} = \boldsymbol{A}^{-1}$, the generators U_i can be expressed as $(U_i)^{\mathrm{ab}}_{\mathrm{cd}} = A^{\mathrm{a}}_{\mathrm{b}} \left(A^{-1}\right)^{\mathrm{c}}_{\mathrm{d}}$.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ are 3×3 matrices. We set

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & r \\ 0 & s & 0 \\ t & 0 & 0 \end{pmatrix}. \tag{5}$$

According to the method of constructing some $n^2 \times n^2$ matrices solutions of TLA, we obtain a 9×9 matrix representation of TLA as follows:

$$\boldsymbol{U} = \begin{pmatrix} 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0\ 0\ 0 & q & -q^{1/2}\,\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)} & \mathrm{e}^{2\mathrm{i}\varphi_1} & 0\ 0\ 0 \\ 0\ 0\ 0 & -q^{1/2}\,\mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)} & 1 & -q^{-1/2}\,\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)} & 0\ 0\ 0 \\ 0\ 0\ 0 & \mathrm{e}^{-2\mathrm{i}\varphi_1} & -q^{-1/2}\,\mathrm{e}^{-\mathrm{i}(\varphi_1+\varphi_2)} & q^{-1} & 0\ 0\ 0 \\ 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \\ 0\ 0\ 0 & 0 & 0 & 0 & 0\ 0\ 0 \end{pmatrix}. \tag{6}$$

In the matrix $\boldsymbol{U}$, there are all three parameters q, φ_1, and φ_2. The parameter q originates from the q-deformation of the braiding operators, φ_1 and φ_2 are free parameters.

3. Yang–Baxterization and entanglement

We call the process of obtaining the matrix $\check{\boldsymbol{R}}$ from a given $\boldsymbol{U}$ matrix Yang–Baxterization.[16,25] Next we derive a unitary matrix $\check{\boldsymbol{R}}$ from the $\boldsymbol{U}$ matrix via the Yang–Baxterization approach. In this work, we are interested in the following relativistic YBE

$$\begin{aligned} &\check{R}_{12}(u)\,\check{R}_{23}\left(\frac{u+v}{1+\beta^2 uv}\right)\check{R}_{12}(v) \\ &= \check{R}_{23}(v)\,\check{R}_{12}\left(\frac{u+v}{1+\beta^2 uv}\right)\check{R}_{23}(u), \end{aligned} \tag{7}$$

where $\check{R}_{12} = \check{R} \otimes I$, $\check{R}_{23} = \boldsymbol{I} \otimes \check{\boldsymbol{R}}$. The spectral parameters u and v usually play an important role which are related to the one-dimensional momentum in some typical models.[12,13] Let the unitary Yang–Baxter matrix take the form

$$\check{\boldsymbol{R}}(u) = \rho(u)\left(\boldsymbol{I} + F(u)\,\boldsymbol{U}\right), \tag{8}$$

where $\rho(u)$ and $F(u)$ are two functions of u to be determined by Eq. (7). Let $\boldsymbol{A}^{\dagger}$ denote the transpose

conjugate of matrix $\boldsymbol{A}$. Through the unitary condition ($\check{\boldsymbol{R}}^{\dagger}(u)\check{\boldsymbol{R}}(u)=\check{\boldsymbol{R}}(u)\check{\boldsymbol{R}}^{\dagger}(u)=\boldsymbol{I}$) we can derive that q and $\varphi_i\,(i=1,2)$ are real.

Substituting Eq. (1) and Eq. (8) into Eq. (7), we easily obtain

$$F(u)+F(v)-F\left(\frac{u+v}{1+\beta^2uv}\right)+dF(u)F(v)+F(u)F(v)F\left(\frac{u+v}{1+\beta^2uv}\right)=0, \tag{9}$$

equation (9) has a solution[25]

$$F(u)=\frac{2\varepsilon\beta u}{\sqrt{d^2/4-1}\left(\beta^2u^2-2i\varepsilon\beta u\sqrt{d^2/(4-d^2)}+1\right)},\quad \varepsilon=\pm1. \tag{10}$$

Let $*$ denote conjugate. The unitary condition can be tenable only under the condition that $\rho^*(u)\rho(u)=1$ and $F(u)+F^*(u)+\mathrm{d}F(u)F^*(u)=0$. Taking account into these conditions, we obtain a solution

$$F(u)=\frac{\mathrm{e}^{-2\mathrm{i}\theta}-1}{d},$$

$$\mathrm{e}^{-2\mathrm{i}\theta}=\frac{\beta^2u^2+2\mathrm{i}\varepsilon\beta u\sqrt{d^2/(4-d^2)}+1}{\beta^2u^2-2\mathrm{i}\varepsilon\beta u\sqrt{d^2/(4-d^2)}+1}, \tag{11}$$

where the new parameter θ is real. Let $\rho(u)=\mathrm{e}^{\mathrm{i}\theta}$. The Yang–Baxter matrix can be rewritten in the following form

$$\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)=\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{I}-f(\theta)\boldsymbol{U}, \tag{12}$$

where $f(\theta)=2\mathrm{i}\sin\theta/d$. Generally a solution of $\check{\boldsymbol{R}}(u)$ depends on two parameters:[7] the first is θ (θ is time-independent), the second is φ_i contained in the braiding matrix. Alternatively, we can rewrite $\check{\boldsymbol{R}}(u)$ as $\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)$. In the matrix form, the $\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)$ can be recast as follows:

$$\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)=\mathrm{diag}\left(\check{\boldsymbol{R}}_1^{(1)},\check{\boldsymbol{R}}_2^{(2)},\check{\boldsymbol{R}}_3^{(3)},\check{\boldsymbol{R}}_2^{(4)},\check{\boldsymbol{R}}_1^{(5)}\right), \tag{13}$$

where

$$\check{\boldsymbol{R}}_1^{(1)}=\mathrm{e}^{\mathrm{i}\theta},$$

$$\check{\boldsymbol{R}}_2^{(2)}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta}&0\\0&\mathrm{e}^{\mathrm{i}\theta}\end{pmatrix},$$

$$\check{\boldsymbol{R}}_3^{(3)}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta}-f(\theta)q & f(\theta)q^{1/2}\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)} & -f(\theta)\mathrm{e}^{2\mathrm{i}\varphi_1}\\ f(\theta)q^{1/2}\mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)} & \mathrm{e}^{\mathrm{i}\theta}-f(\theta) & f(\theta)q^{-1/2}\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}\\ -f(\theta)\mathrm{e}^{-2\mathrm{i}\varphi_1} & f(\theta)q^{-1/2}\mathrm{e}^{-\mathrm{i}(\varphi_1+\varphi_2)} & \mathrm{e}^{\mathrm{i}\theta}-f(\theta)q^{-1}\end{pmatrix}.$$

Here, the block diagram represents the matrix in block form. That is $\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)=\oplus_{j=1}^{5}\check{\boldsymbol{R}}_k^{(j)}$ ($\check{\boldsymbol{R}}_k^{(j)}$ is a $k\times k$ matrix) and $\check{\boldsymbol{R}}_2^{(2)}=\check{\boldsymbol{R}}_2^{(4)}$, $\check{\boldsymbol{R}}_1^{(5)}=\check{\boldsymbol{R}}_1^{(1)}$.

In this paper we choose $\{|11\rangle,|10\rangle,|01\rangle,|1-1\rangle,|00\rangle,|-11\rangle,|0-1\rangle,|-10\rangle,|-1-1\rangle\}$ as the basis. When the unitary Yang–Baxter matrix $\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)$ acts on the basis, the matrix $\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)$ transfers the two-qubit product states to entangled states

$$\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)|1-1\rangle=[\mathrm{e}^{\mathrm{i}\theta}-f(\theta)q]|1-1\rangle+f(\theta)q^{1/2}\mathrm{e}^{-\mathrm{i}(\varphi_1-\varphi_2)}|00\rangle-f(\theta)\mathrm{e}^{-2\mathrm{i}\varphi_1}|-11\rangle, \tag{14a}$$

$$\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)|00\rangle=f(\theta)q^{1/2}\mathrm{e}^{\mathrm{i}(\varphi_1-\varphi_2)}|1-1\rangle+[\mathrm{e}^{\mathrm{i}\theta}-f(\theta)]|00\rangle+f(\theta)q^{-1/2}\mathrm{e}^{-\mathrm{i}(\varphi_1+\varphi_2)}|-11\rangle, \tag{14b}$$

$$\check{\boldsymbol{R}}(\theta,\varphi_1,\varphi_2)|-11\rangle=-f(\theta)\mathrm{e}^{2\mathrm{i}\varphi_1}|1-1\rangle+f(\theta)q^{-1/2}\mathrm{e}^{\mathrm{i}(\varphi_1+\varphi_2)}|00\rangle+[\mathrm{e}^{\mathrm{i}\theta}-f(\theta)q^{-1}]|-11\rangle. \tag{14c}$$

By means of concurrence, we study these entangled states. In Ref. [26], the generalized concurrence (or the degree of entanglement[27–32]) for two qubits is given by

$$\mathcal{C}=\sqrt{\frac{N}{N-1}(1-I_1)}, \tag{15}$$

where $I_1=\mathrm{Tr}\left[\rho_{\mathrm{A}}^2\right]=\mathrm{Tr}\left[\rho_{\mathrm{B}}^2\right]=|\kappa_0|^4+|\kappa_1|^4+\cdots+|\kappa_{\mathrm{d}-1}|^4$, with ρ_{A} and ρ_{B} are the reduced density matrices for the subsystems, and κ_{j}'s ($j=0,1,2,\ldots,N-1$) are the Schmidt coefficients (N is the N-dimensional space).

According to Eq. (15), we can obtain the entanglement degree of Eq. (14). The three entangled states have the same entanglement degree

$$\mathcal{C}(1)=\mathcal{C}(2)=\mathcal{C}(3)=\frac{2\sqrt{2}\,|\sin\theta|}{3}\sqrt{1+2\cos^2\theta}. \tag{16}$$

It is clear that the concurrence only depends on the parameter θ. In Fig. 1, we find that the concurrence ranges from 0 to 1 when the parameter θ runs

from 0 to π. However, for $\theta \in [0, \pi]$, the entanglement is not a monotonic function of θ. And when $\theta = \pi/3$, we will generate three complete and orthogonal maximally entangled states of two qubits. The entanglement does not depend on the parameters φ_1 and φ_2.

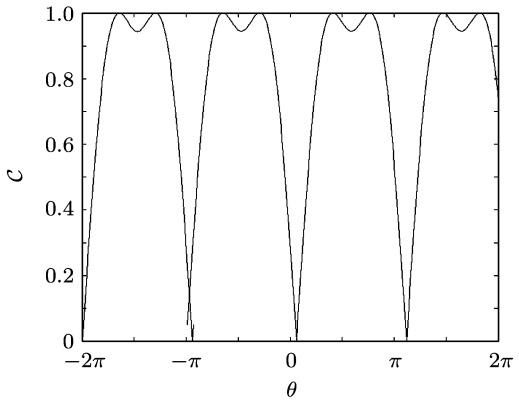


Fig. 1. The concurrence in the two-qubit model is plotted versus θ.

4. Conclusion

In this paper, using the method shown in Ref. [24], we obtained the matrix $\boldsymbol{U}$ which satisfies TLA relations. Then we obtained a 9×9 matrix representation of TLA.

We showed a unitary $\check{\boldsymbol{R}}$ matrix via Yang–Baxterization of the $\boldsymbol{U}$ matrix. Finally, we investigated the entanglement properties of $\check{\boldsymbol{R}}$ matrix, and it is shown that the arbitrary degree of entanglement for two-qubit entangled states can be generated via the unitary matrix $\check{\boldsymbol{R}}$ matrix acting on the standard basis.

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