Symmetry groups and Gauss kernels of Schrödinger equations*

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The relationship between symmetries and Gauss kernels for the Schrödinger equation $\mathrm{i}\,u_t = u_{xx} + f(x)u$ is established. It is shown that if the Lie point symmetries of the equation are nontrivial, a classical integral transformations of the Gauss kernels can be obtained. Then the Gauss kernels of Schrödinger equations are derived by inverting the integral transformations. Furthermore, the relationship between Gauss kernels for two equations related by an equivalence transformation is identified.

Keywords: Schrödinger equation, symmetry group, Gauss kernel, equivalence transformation

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1. Introduction

Schrödinger equations appear in various domains in the field of mathematical physics, for example they are the basic equations in quantum mechanics.[1] The linear time-dependent equation

$$\mathrm{i}\,u_t + \Delta u = 0,$$

is the Schrödinger equation for "free particles", when there is no force acting on the particle. If the particle moves in a force field given by a potential V, such as the electric potential, as proposed by Schrödinger, the equation describing the evolution of a quantum particle in space in the presence of a potential V is

$$\mathrm{i}\,\hbar u_t + \frac{\hbar^2}{2m}\Delta u = Vu, \tag{1}$$

where $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$ is the wave function of a quantum particle, $\hbar$ is Planck's constant, m is the mass of the particle, and the potential function V is assumed to depend only on the spatial variable x.

For Eq. (1), if we consider the case of (1+1) dimensions, its solution for the Cauchy problem with the initial condition

$$u(x,0) = \phi(x), \tag{2}$$

is given by

$$u(x,t) = \int_{-\infty}^{\infty} p(t,x,z)\phi(z)\mathrm{d}z, \tag{3}$$

where $p(t,x,z)$ is the Gauss kernel for Eq. (1). It follows from Eq. (3) that solutions to the Cauchy problem of the linear evolutionary equations can be directly obtained by convolution of initial data and Gauss kernels of the equations under study. Based on the Gauss kernels of the linear equations, one can represent the solutions of semi-linear equations by integral equations with Gauss kernels as the kernel. Such a representation is helpful for the analysis or estimate to solutions of semi-linear evolutionary equations.

This study is devoted to the construction of Gauss kernels for the (1+1)-dimensional linear Schrödinger equations

$$\mathrm{i}\,u_t = u_{xx} + f(x)u, \quad x \in \mathbb{R} \tag{4}$$

by using Lie symmetry groups. Lie's point symmetries of partial differential equations (PDEs) describe their geometric or basic physical properties. The method of determining Lie's point symmetries was introduced first by Lie[2] who performed classification of Lie's point symmetries of the one-dimensional heat equation, which is also called the classical method. Today,

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Lie's symmetry method is one of the most powerful tools for solving linear and nonlinear PDEs.[3−9] It provides us with a powerful approach for obtaining exact analytic solutions of PDEs. It is worthwhile noting that many of these special solutions illustrate some important physical phenomena. More importantly, exact solutions stemming from symmetry methods have been used frequently to study various properties of the equations.[10−16]

It was shown by many studies[17−25] that Gauss kernels to linear PDEs are closely related to their Lie symmetries. Lie's symmetry method provides us with an efficient method to obtain Gauss kernels of linear PDEs. Bluman and Cole[19] constructed the Gauss kernel of the Fokker–Planck equation by using the symmetry of the equation. Goard[20] used the so-called method of reduction to canonical form to obtain Gauss kernels for a class of parabolic PDEs. Recently, Craddock *et al.* suggested an efficient method to obtain Gauss kernels for a wide class of parabolic equations by using Lie symmetries admitted by the equations under study, which was first introduced in Ref. [21] and further developed in Refs. [22]–[24]. Their approach relies on the fact that if a parabolic PDE admits at least four-dimensional nontrivial Lie symmetry groups, then it is always possible to construct, or up to a slight modification of infinitesimal parameter and variables, classical integral transformations of the Gauss kernels of the PDE by utilizing a time-independent solution. More recently, Craddock and Lennox[25] extended their method to obtain fundamental solutions of multi-dimensional parabolic PDEs.

The aim of this paper is to develop a method due to Craddock to obtain the Gauss kernels of the linear Schrödinger equation (4). We will identify the potential function f so that the corresponding equation admits nontrivial Lie point symmetries which naturally yield some integral transformations of Gauss kernels. In this paper, both Laplace transformation and Fourier transformation are explored to obtain the Gauss kernels. Furthermore, the connection between the Gauss kernels of two equations related by an equivalence transformations is established.

2. Symmetry groups and the Gauss kernel

Let us now consider the n-th order (1+1) dimensional linear PDEs

$$\Delta(x,t,u,u^{(1)}(x,t),\ldots,u^{(n)}(x,t))=0, \tag{5}$$

where $(x,t)\in\mathbb{R}\times\mathbb{R}^{+}$ are independent variables, u is the dependent variable, and $u^{(j)}$ denotes the set of all the partial derivatives of order j of u, $j=1,\ldots,n$.

Let $\boldsymbol{G}$ be a Lie symmetry group generated by the vector field of the form

$$V=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u, \tag{6}$$

where $\partial/\partial_x=\partial_x$, ξ, τ, and ϕ are infinitesimals corresponding to x, t, and u, respectively. According to Lie's infinitesimal criterion for symmetry,[3−6] V generates a symmetry of equation (5) if and only if

$$V^{(n)}(\Delta(x,t,u,u^{(1)}(x,t),\ldots,u^{(n)}(x,t)))|_M=0, \tag{7}$$

where $V^{(n)}$ is the n-th prolongation of V on the n-th jet space of (x,t,u), M denotes the solution manifold of equation (5). Equation (7) yields the system of over-determined equations for infinitesimals ξ, τ, and ϕ. Thus the problem of constructing a symmetry group admitted by Eq. (5) is equivalent to finding its symmetry algebra through solving an over-determined system of equations. Then the transformation group admitted by Eq. (5) is obtained by exponentiating the infinitesimal generators which is carried out by solving the system

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\epsilon}=\xi(\tilde{x},\tilde{t},\tilde{u}),\ \frac{\mathrm{d}\tilde{t}}{\mathrm{d}\epsilon}=\tau(\tilde{x},\tilde{t},\tilde{u}),\ \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\epsilon}=\phi(\tilde{x},\tilde{t},\tilde{u}), \tag{8}$$

with the initial condition

$$\tilde{x}(0)=x,\quad \tilde{t}(0)=t,\quad \tilde{u}(0)=u. \tag{9}$$

Here we denote the action of the symmetry generated by V as

$$\tilde{u}_\epsilon(x,t)=\sigma(x,t;\epsilon)u(a_1(x,t;\epsilon),a_2(x,t;\epsilon)), \tag{10}$$

where the real number ϵ is the group parameter. We call σ the multiplier, a_1 and a_2 the change of variables (see Olver's book[4] for more details on the theory of Lie symmetries). It turns out that if $u(x,t)$ is a solution of Eq. (5), then equation (10) is also a solution, at least for sufficiently small ϵ. This allows us to connect

the solutions (3) and (10). By choosing a known analytic solution $u(x,t)$ and plugging the resulting $\tilde{u}_\epsilon(x,t)$ in Eq. (3), we obtain the relation

$$\int_{-\infty}^{+\infty} p(t,x,z)\sigma(z,0;\epsilon)u(a_1(z,0;\epsilon),a_2(z,0;\epsilon))\mathrm{d}z = \tilde{u}_\epsilon(x,t). \tag{11}$$

Since σ, a_1, and a_2 are known, equation (11) is a family of integral equations for $p(t,x,z)$. For example, if $\sigma(z,0,\epsilon)u(a_1(z,0,\epsilon),a_2(z,0,\epsilon)) = \mathrm{e}^{-\epsilon z}\phi(z)$, where ϕ is some function of z, equation (11) implies $\tilde{u}_\epsilon(x,t)$ is a Laplace transformation of $p(t,x,z)$. It was proved in Refs. [21], [22], and [24] that if equation (11) holds for some distribution p, then p is the Gauss kernel of Eq. (5). It follows that the Gauss kernel can be recovered by inverting the integral transformation of $\tilde{u}_\epsilon(x,t)$, which is said to be the characteristic solution for the equation. In Refs. [22] and [24], instead of the Laplace transformation, other classical transformations including the generalized Laplace, Whittaker, Hankel, and Fourier transformations are also used to derive Gauss kernels of linear PDEs. They showed that if the symmetry group of the PDE is nontrival, the solution $\tilde{u}_\epsilon(x,t)$ corresponding to some symmetry, or a slight modification of the infinitesimal parameter is always a classical integral transformation of the Gauss kernel of the PDE. Therefore, to apply Craddock's approach for the construction of Gauss kernels of the linear Schrödinger equations of the form (4), we need to look for the proper integral transformation of the Gauss kernel by symmetry transformation.

3. Gauss kernels of Eq. (4)

3.1. Lie point symmetries

First, we classify Lie point symmetries of Eq. (4). Following the general Lie's algorithm, we look for the vector field (6) admitted by Eq. (4). The criterion for Eq. (4) to be invariant with respect to operator V reads as

$$V^{(2)}(\mathrm{i}\,u_t - u_{xx} - f(x)u)|_{\Delta} = 0, \qquad \Delta = \mathrm{i}\,u_t - u_{xx} - f(x)u, \tag{12}$$

where $V^{(2)}$ is the second prolongation of the operator V, and it has the following form

$$V^{(2)} = V + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}},$$

where ϕ^t and ϕ^{xx} depend on ξ, τ, and ϕ, and can be determined by Lie's recursive prolongation formulae. It is easy to show that τ only depends on t, and then the standard symmetry group calculation leads to the equation

$$\begin{aligned}\mathrm{i}\,\phi_t &- \phi_{xx} - f\phi + [(\phi_u - \tau'(t))f - \xi f'(x)]u \\ &+ (\xi_{xx} - \mathrm{i}\,\xi_t - 2\phi_{xu})u_x + (2\xi_{xu} - \phi_{uu})u_x^2 \\ &+ (2\xi_x - \tau'(t))u_{xx} + 2\xi_u u_x u_{xx} \\ &- f\xi_u u u_x = 0.\end{aligned} \tag{13}$$

From Eq. (13) we can read off individual equations for ξ, τ, and ϕ by equating the coefficients of the derivatives of u. First, in view of the coefficients of u_xu_{xx}, we find ξ does not depend on u. And then the coefficient of u_{xx} gives

$$\xi = \frac{1}{2}\tau'(t)x + \rho(t),$$

where ρ is some function of t. Examining the term involving u_x^2, we see that

$$\phi_{uu} = 0,$$

which implies that ϕ must be linear in u, namely

$$\phi(x,t,u) = \alpha(x,t)u + \beta(x,t),$$

for some functions α and β. The coefficient of u_x leads to

$$\alpha(x,t) = -\frac{\mathrm{i}}{8}\tau''(t)x^2 - \frac{\mathrm{i}}{2}\rho(t)x + \eta(t),$$

where η depends on t only. Finally, plugging these results into Eq. (13), we find that β is any solution of Eq. (4) and the functions τ, ρ, and η satisfy the following equation:

$$\begin{aligned}&\frac{1}{8}\tau'''(t)x^2 + \frac{1}{2}\rho''(t)x + \mathrm{i}\left[\eta'(t) + \frac{1}{4}\tau''(t)\right] \\ &- \rho(t)f'(x) - \tau'(t)\left[\frac{1}{2}f'(x)x + f\right] = 0.\end{aligned} \tag{14}$$

It follows that functions τ, ρ, and η and then the symmetries of the equation are determined by the potential function f. Differentiating it with respect to x third yields

$$f^{(4)}(x)\rho(t) + \left(\frac{5}{2}f'''(x) + \frac{1}{2}f^{(4)}(x)x\right)\tau'(t) = 0. \tag{15}$$

Therefore, we need to distinguish two cases for further discussion: $\tau'(t) \neq 0$ and $\tau'(t) = 0$.

Case 1 $\tau'(t) \neq 0$

In this case, equation (15) implies there are two possible forms for f, which are

1a) $f = A(x+a)^2 + B(x+a) + \frac{C}{(x+a)^2} + D$,

1b) $f = Ax^2 + Bx + C$,

where A, B, C, D, and a are arbitrary constants. We now consider each case separately.

Case 1a $f = A(x+a)^2 + B(x+a) + \frac{C}{(x+a)^2} + D$.

In this subcase, we arrive at $B = 0$, otherwise it leads to $\tau'(t) = 0$ which contradicts the assumption in this case. To solve equation (14), there are three cases to be considered.

Case 1a-1 $A > 0$. The corresponding vector fields of the symmetries to the corresponding equation are

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \exp(4\sqrt{A}t)\{2\sqrt{A}(x+a)\partial_x + \partial_t\\
&\quad - u[\mathrm{i}(2Ax^2 + 4Aax + 2Aa^2 + D) + \sqrt{A}]\partial_u\},\\
V_4 &= -\exp(-4\sqrt{A}t)\{2\sqrt{A}(x+a)\partial_x - \partial_t\\
&\quad + u[\mathrm{i}(2Ax^2 + 4Aax + 2Aa^2 + D) - \sqrt{A}]\partial_u\}.
\end{aligned}$$

Case 1a-2 $A < 0$. The Lie algebra is generated by the vector fields

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= 2\sqrt{-A}\cos(4\sqrt{-A}t)(x+a)\partial_x + \sin(4\sqrt{-A}t)\partial_t\\
&\quad - u[\mathrm{i}\sin(4\sqrt{-A}t)(2Ax^2 + 4Aax + 2Aa^2 + D)\\
&\quad + \sqrt{-A}\cos(4\sqrt{-A}t)]\partial_u\},\\
V_4 &= -\{2\sqrt{-A}\sin(4\sqrt{-A}t)(x+a)\partial_x - \cos(4\sqrt{-A}t)\partial_t\\
&\quad + u[\mathrm{i}\cos(4\sqrt{-A}t)(2Ax^2 + 4Aax + 2Aa^2 + D)\\
&\quad - \sqrt{-A}\sin(4\sqrt{-A}t)]\partial_u\}.
\end{aligned}$$

Case 1a-3 $A = 0$. A basis for the Lie algebra consists of

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= (x+a)\partial_x + 2t\partial_t - 2D\,\mathrm{i}\,ut\partial_u,\\
V_4 &= (x+a)t\partial_x + t^2\partial_t\\
&\quad - \left[\mathrm{i}\left(\frac{1}{4}x^2 + \frac{a}{2}x + Dt^2\right) + \frac{t}{2}\right]u\partial_u.
\end{aligned}$$

Case 1b $f = Ax^2 + Bx + C$.

In this subcase, analogous to the earlier analysis, there are three cases to be considered.

Case 1b-1 $A > 0$. In this case, the corresponding vector fields of the symmetries to the corresponding equation are

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \exp(4\sqrt{A}t)\left\{\frac{2Ax+B}{\sqrt{A}}\partial_x + \partial_t - \frac{u}{4A}[\mathrm{i}(8A^2x^2\right.\\
&\quad \left. + 8ABx + B^2 + 4AC) + 4A^{3/2}]\partial_u\right\},\\
V_4 &= -\exp(-4\sqrt{A}t)\left\{\frac{2Ax+B}{\sqrt{A}}\partial_x - \partial_t + \frac{u}{4A}[\mathrm{i}(8A^2x^2\right.\\
&\quad \left. + 8ABx + B^2 + 4AC) - 4A^{3/2}]\partial_u\right\},\\
V_5 &= \exp(2\sqrt{A}t)\left[\partial_x - \frac{\mathrm{i}\,u}{2}\left(2Ax + \frac{B}{\sqrt{A}}\right)\partial_u\right],\\
V_6 &= \exp(-2\sqrt{A}t)\left[\partial_x + \frac{\mathrm{i}\,u}{2}\left(2Ax + \frac{B}{\sqrt{A}}\right)\partial_u\right].
\end{aligned}$$

Case 1b-2 $A < 0$. The Lie algebra is generated by the vector fields

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \frac{\sin(4\sqrt{-A}t)}{\sqrt{-A}}(2Ax+B)\partial_x + \cos(4\sqrt{-A}t)\partial_t\\
&\quad - \frac{u}{4A}[\mathrm{i}\cos(4\sqrt{-A}t)(8A^2x^2 + 8ABx + B^2 + 4AC)\\
&\quad + 4(-A)^{3/2}\sin(4\sqrt{-A}t)]\partial_u,\\
V_4 &= -\left\{\frac{\cos(4\sqrt{-A}t)}{\sqrt{-A}}(2Ax+B)\partial_x - \sin(4\sqrt{-A}t)\partial_t\right.\\
&\quad + \frac{u}{4A}[\mathrm{i}\sin(4\sqrt{-A}t)(8A^2x^2 + 8ABx + B^2 + 4AC)\\
&\quad \left. - 4(-A)^{3/2}\cos(4\sqrt{-A}t)]\partial_u\right\},\\
V_5 &= \sin(2\sqrt{-A}t)\partial_x + \frac{(2Ax+B)\,\mathrm{i}\,u\cos(2\sqrt{-A}t)}{2\sqrt{-A}}\partial_u,\\
V_6 &= \cos(2\sqrt{-A}t)\partial_x - \frac{(2Ax+B)\,\mathrm{i}\,u\sin(2\sqrt{-A}t)}{2\sqrt{-A}}\partial_u.
\end{aligned}$$

Case 1b-3 $A = 0$. A basis for the Lie algebra consists of

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= (x + 3Bt^2)\partial_x + 2t\partial_t\\
&\quad - \mathrm{i}\,u(B^2t^3 + 3Bxt + 2Ct)\partial_u,\\
V_4 &= (x + Bt^2)t\partial_x + t^2\partial_t\\
&\quad - \frac{u}{4}[\mathrm{i}(B^2t^4 + 4Ct^2 + 6Bxt^2 + x^2) + 2t]\partial_u,\\
V_5 &= t\partial_x - \frac{\mathrm{i}\,u}{2}(x + Bt^2)\partial_u, \quad V_6 = \partial_x - \mathrm{i}\,But\partial_u.
\end{aligned}$$

Case 2 $\tau'(t) = 0$.

If $\tau'(t) = 0$, equation (14) now reads as

$$f'(x)\rho(t) - \eta'(t)\,\mathrm{i} - \frac{1}{2}\rho''(t)x = 0. \tag{16}$$

Two cases arise:

Case 2a $\rho \neq 0$. In this case, it is easy to show that f must be of the form $f = Ax^2 + Bx + C$. Actually, this subcase can be reduced to the subcase 1b.

Case 2b $\rho = 0$. If $\rho = 0$, from Eq. (16) we see that η is an arbitrary constant and then equation (14) holds identically. Therefore, f is an arbitrary function of x. The symmetry algebra has the basis: $V_1 = \partial_t$, $V_2 = u\partial_u$, and $V_\beta = \beta(x,t)\partial_u$.

3.2. Gauss kernels on Case 1b

In Ref. [21], Laplace transformation was used to derive Gauss kernels of the parabolic equations over the positive half-line. For equations (4) defined over the whole of $\mathbb{R}$, we can obtain their Gauss kernels by inverting the Fourier transformation. In the following, we will illustrate our method for Case 1b. In this case, the corresponding equations admit six-dimensional symmetry groups, the symmetries can be used to produce Fourier transformations of Gauss kernels. To compute the Gauss kernels, we need the following fact.

Proposition 1 For linear Schrödinger equations (4), let the potential function f be of the form $f(x) = Ax^2 + Bx + C$, where A, B, and C are arbitrary constants and $A \geq 0$. Then, if $A \neq 0$, the corresponding equation has the symmetry solution

$$\begin{aligned}\tilde{u}_\epsilon(x,t) = \exp\Big[& -2\,\mathrm{i}\,\epsilon\sqrt{A}\cosh(2\sqrt{A}t)x \\ & + \mathrm{i}\,\epsilon^2\sqrt{A}\sinh(4\sqrt{A}t) \\ & + \frac{\mathrm{i}\,B\epsilon}{\sqrt{A}}[1-\cosh(4\sqrt{A}t)]\Big] \\ & \times u(x - 2\epsilon\sinh(2\sqrt{A}t), t); \end{aligned} \tag{17}$$

if $A = 0$, equation (4) has the symmetry solution

$$\tilde{u}_\epsilon(x,t) = \exp\left[\frac{\mathrm{i}}{4}\epsilon(\epsilon t - 2x - 2Bt^2)\right]u(x-\epsilon t, t), \tag{18}$$

where ϵ is a sufficiently small parameter and u is an arbitrary solution of the corresponding equation.

Proof If $A > 0$, it follows from the vector fields V_5 and V_6 in subcase 1b-1 that equation (4) admits the symmetry

$$V = 2\sinh(2\sqrt{A}t)\partial_x - \mathrm{i}\,u\left(2\sqrt{A}x + \frac{B}{\sqrt{A}}\right)\cosh(2\sqrt{A}t)\partial_u.$$

Note that the group action is generated by the vector field (6), which is obtained by solving the first-order system of ODEs (8) subject to the initial condition (9). Solving the system (8) for V and multiplying the result by the constant $\exp(\mathrm{i}\,B\epsilon/\sqrt{A})$ produces Eq. (17).

If $A = 0$, the corresponding equation in Case 1a-2 admits the vector field V_5. Exponentiating V_5 leads to Eq. (18). □

Notice that in both cases we have $\tilde{u}_\epsilon(x,0) = \mathrm{e}^{-\mathrm{i}\,\epsilon\gamma x}u(x,0)$ for some constant γ. Thus by equation (11), we can seek a Gauss kernel with the property that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\lambda z}u(z,0)p(t,x,z)\mathrm{d}z = \tilde{u}_{\lambda/\gamma}(x,t), \tag{19}$$

which implies that $\tilde{u}_{\lambda/\gamma}(x,t)$ is the Fourier transformation of $u(z,0)p(t,x,z)$. In other words, the Gauss kernel can be obtained by inverting the Fourier transformation of $\tilde{u}_{\lambda/\gamma}(x,t)$. In the following, we present two examples to illustrate our approach.

Example 1 Consider

$$\mathrm{i}\,u_t = u_{xx} + ax^2u, \quad x \in \mathbb{R},\ a > 0. \tag{20}$$

By Proposition 1, it has the symmetry

$$\begin{aligned}\tilde{u}_\epsilon(x,t) = \exp\Big[& -2\,\mathrm{i}\,\epsilon\sqrt{a}\cosh(2\sqrt{a}t)x \\ & + \mathrm{i}\,\epsilon^2\sqrt{a}\sinh(4\sqrt{a}t)\Big] \\ & \times u(x - 2\epsilon\sinh(2\sqrt{a}t), t), \end{aligned}$$

which implies that if $u(x,t)$ is a solution of Eq. (20), then so is $\tilde{u}_\epsilon(x,t)$, for sufficiently small ϵ. Here we take

$$u(x,t) = \exp\left(-\frac{\sqrt{a}\,\mathrm{i}}{2}x^2 + \sqrt{a}t\right)$$

as the initial solution, and then equation (19) becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\lambda z}\exp\left(-\frac{\sqrt{a}i}{2}z^2\right)p(t,x,z)\mathrm{d}z \\ &= \exp\Big\{-\frac{1}{4\sqrt{a}}[\mathrm{i}\,\lambda^2(1-\mathrm{e}^{-4\sqrt{a}t}) \\ &\quad + 4\,\mathrm{i}\,\lambda\sqrt{a}x\,\mathrm{e}^{-2\sqrt{a}t} + 2a(\mathrm{i}\,x^2 + 2t)]\Big\}. \end{aligned}$$

Using the inverse Fourier transformation, we obtain the Gauss kernel given by

$$p(t,x,z) = \frac{a^{1/4}\exp\left[-\dfrac{\sqrt{a}(x^2+z^2)\,\mathrm{i}}{2\tan(2\sqrt{a}t)} + \dfrac{\sqrt{a}xz\,\mathrm{i}}{\sin(2\sqrt{a}t)}\right]}{\sqrt{-2\pi\,\mathrm{i}\sin(2\sqrt{a}t)}}.$$

Example 2 Consider

$$\mathrm{i}\,u_t = u_{xx} - axu, \quad x \in \mathbb{R}. \tag{21}$$

It follows that if u solves the equation, then so does

$$\tilde{u}_\epsilon(x,t) = \exp\left(\frac{\mathrm{i}}{4}\epsilon^2 t - \frac{\mathrm{i}}{2}\epsilon x + \frac{a\,\mathrm{i}\,\epsilon}{2}t^2\right)u(x-\epsilon t, t).$$

We choose

$$u(x,t) = \exp\left[(axt + \frac{1}{3}a^2t^3)\,\mathrm{i}\right]$$

and take $\epsilon = 2\lambda$, so the Gauss kernel $p(t,x,z)$ satisfies

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\,\lambda z}p(t,x,z)\mathrm{d}z \\ &= \exp\left[\frac{1}{3}a^2t^3 + \mathrm{i}\,\lambda^2 t - \lambda(x+at^2) + axt\right]. \end{aligned}$$

We can thus apply the inverse Fourier transformation to recover the Gauss kernel, namely

$$p(t,x,z) = \frac{1}{2\sqrt{-\pi t i}} \exp\left\{\left[\frac{a^2t^3}{12} + \frac{a(x+z)}{2}t - \frac{(x-z)^2}{4t}\right] \mathrm{i}\right\}.$$

For equations belonging to Case 1a, they admit four-dimensional non-trivial Lie point symmetries. However, we cannot find a proper symmetry that can generate the Fourier transformation of the Gauss kernel. Furthermore, since these equations are defined over the whole of $\mathbb{R}$, the method given in Ref. [21] cannot be directly used to compute the Gauss kernels. Note that equation

$$\mathrm{i}\, u_t = x u_{xx} + \frac{1}{2} u_x + f(x)u, \quad x \in \mathbb{R}^+ \tag{22}$$

is related with the equation

$$\mathrm{i}\, u_\tau = u_{yy} + 4f(y^2)u, \quad y \in \mathbb{R} \tag{23}$$

by the equivalence transformation

$$x = y^2, \quad t = 4\tau. \tag{24}$$

A natural question now arises: Is there any relationship between the Gauss kernels for the two classes of equations? In the following, we will establish the connections between Gauss kernels for the equations related by the equivalence transformation, by which the fundamental solutions of Eq. (4) belong to Case 1a can be obtained by the Gauss kernels of the corresponding equations (22) related to Eq. (4) through the equivalence transformations (24).

4. Gauss kernels on Case 1a

4.1. Equivalence transformations and fundamental solutions

Now we study the relationship of Gauss kernels of two equations related by equivalence transformations, thus we can obtain the Gauss kernel from a given Gauss kernel of another equation which is related to the equation under study by an equivalence transformation. We first restrict our consideration to an invertible point transformation of the following form

$$\tau = T(t), \quad y = X(x,t), \quad \tilde{u} = r(x,t)u. \tag{25}$$

Requiring invertibility of Eqs. (25) yields $X_x \neq 0$, $\dot{T}(t) \neq 0$, and $r \neq 0$. Assume that transformation (25) is an equivalence transformation of a class of linear equations

$$E(x,t,u) = 0, \ \ x \in \Omega, \ \ t \in \mathbb{R}^+, \tag{26}$$

so that the transformed equations

$$E(y,\tau,\tilde{u}) = 0, \ \ y \in \tilde{\Omega}, \ \ \tau \in \mathbb{R}^+ \tag{27}$$

belong to the same class of equations as the initial one. Denote the inverse transformations of X and T by

$$x = Y(y,\tau), \quad t = T^{-1}(\tau).$$

It follows from the above assumption that if $u(x,t)$ solves the initial equations (26), then

$$\tilde{u}(y,\tau) = r(Y(y,\tau), T^{-1}(\tau))u(Y(y,\tau), T^{-1}(\tau)) \tag{28}$$

is a solution of the transformed system (27). The relationship between Gauss kernels of equations (26) and (27) is governed by the following theorem.

Theorem 1 Assume that linear equation (26) can be transformed into equation (27) by equivalence transformation (25), and the compatibility condition $\tau(0) = 0$ is satisfied, if $p(t,x,z)$ is the Gauss kernel of the initial equation (26), then the Gauss kernel of the transformed equation (27) is

$$\tilde{p}(\tau,y,\tilde{z}) = r(Y(y,\tau), T^{-1}(\tau))p(T^{-1}(\tau), Y(y,\tau), Y(\tilde{z},\tau))r^{-1}(Y(\tilde{z},0),\tau(0))Y_{\tilde{z}}(\tilde{z},\tau). \tag{29}$$

Proof The fact that $p(t,x,z)$ solves Eq. (26) for each fixed z implies that $\tilde{p}(\tau,y,\tilde{z})$ of the form (29) solves equation (27) for each fixed $\tilde{z}$. Furthermore,

$$\begin{aligned}
&\int_{\tilde{\Omega}} \tilde{p}(\tau,y,\tilde{z})\tilde{u}(\tilde{z},0)\mathrm{d}\tilde{z} \\
&= \int_{\tilde{\Omega}} r(Y(y,\tau), T^{-1}(\tau))p(T^{-1}(\tau), Y(y,\tau), Y(\tilde{z},\tau))r^{-1}(Y(\tilde{z},0),\tau(0))Y_{\tilde{z}}(\tilde{z},\tau)\tilde{u}(\tilde{z},0)\mathrm{d}\tilde{z} \\
&= \int_{\Omega} r(x,t)p(t,x,z)u(z,0)Y_{\tilde{z}}(X(z,t),T(t))X_z(z,t)\mathrm{d}z
\end{aligned}$$

$$= r(x,t)\int_{\Omega} p(t,x,z)u(z,0)\mathrm{d}z = r(x,t)u(x,t) = \tilde{u}(y,\tau).$$

Therefore, if $u(x,t)$ is the solution of the Cauchy problem of Eq. (26) with the initial data $u(x,0)$, then the integral $\int_{\tilde{\Omega}} \tilde{p}(\tau,y,\tilde{z})\tilde{u}(\tilde{z},0)\mathrm{d}\tilde{z}$ is a solution of the Cauchy problem for Eq. (27) with the initial data $\tilde{u}(y,0)$. Consequently, $\tilde{p}(\tau,y,\tilde{z})$ is a Gauss kernel of the transformed system. □

4.2. Classification of Lie symmetries for Eq. (22)

Since the classification of Lie point symmetries for equations of the form (23) has been given in Section 2, we can directly use these classification results. Omitting the detailed calculation, we just present a complete list of Lie point symmetries admitted by Eq. (22).

Case I $\tau'(t) \neq 0$.

Case Ia $f = \frac{A}{4}(\sqrt{x}+a)^2 + \frac{C}{4(\sqrt{x}+a)^2} + \frac{D}{4}$.

Case Ia-1 $A > 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \mathrm{e}^{\sqrt{A}t}\{4\sqrt{A}(\sqrt{x}+a)\sqrt{x}\partial_x + 4\partial_t\\
&\quad - u[\mathrm{i}(2Ax + 4Aa\sqrt{x} + 2Aa^2 + D) + \sqrt{A}]\partial_u\},\\
V_4 &= -\mathrm{e}^{-\sqrt{A}t}\{4\sqrt{A}(\sqrt{x}+a)\sqrt{x}\partial_x - 4\partial_t\\
&\quad + u[\mathrm{i}(2Ax + 4Aa\sqrt{x} + 2Aa^2 + D) - \sqrt{A}]\partial_u\}.
\end{aligned}$$

Case Ia-2 $A < 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= 4\sqrt{-A}\cos(\sqrt{-A}t)(\sqrt{x}+a)\sqrt{x}\partial_x\\
&\quad + 4\sin(\sqrt{-A}t)\partial_t\\
&\quad - u[\mathrm{i}\sin(\sqrt{-A}t)(2Ax + 4Aa\sqrt{x}\\
&\quad + 2Aa^2 + D) + \sqrt{-A}\cos(\sqrt{-A}t)]\partial_u,\\
V_4 &= -\{4\sqrt{-A}\sin(\sqrt{-A}t)(\sqrt{x}+a)\sqrt{x}\partial_x\\
&\quad - 4\cos(\sqrt{-A}t)\partial_t\\
&\quad + u[\mathrm{i}\cos(\sqrt{-A}t)(2Ax + 4Aa\sqrt{x}\\
&\quad + 2Aa^2 + D) - \sqrt{-A}\sin(\sqrt{-A}t)]\partial_u\}.
\end{aligned}$$

Case Ia-3 $A = 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= 2(\sqrt{x}+a)\sqrt{x}\partial_x + 2t\partial_t - \frac{D\,\mathrm{i}\,ut}{2}\partial_u,\\
V_4 &= 2(\sqrt{x}+a)\sqrt{x}t\partial_x + t^2\partial_t\\
&\quad - \left[\mathrm{i}\left(x + 2a\sqrt{x} + \frac{D}{4}t^2\right) + \frac{t}{2}\right]u\partial_u.
\end{aligned}$$

Case Ib $f = \frac{A}{4}x + \frac{B}{4}\sqrt{x} + \frac{C}{4}$.

Case Ib-1 $A > 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \mathrm{e}^{\sqrt{A}t}\left\{\frac{2(2A\sqrt{x}+B)}{\sqrt{A}}\sqrt{x}\partial_x + 4\partial_t - \frac{u}{4A}[\mathrm{i}(8A^2x\right.\\
&\quad \left. + 8AB\sqrt{x} + B^2 + 4AC) + 4A^{3/2}]\partial_u\right\},\\
V_4 &= -\mathrm{e}^{-\sqrt{A}t}\left\{\frac{2(2Ax+B)}{\sqrt{A}}\sqrt{x}\partial_x - 4\partial_t + \frac{u}{4A}[\mathrm{i}(8A^2x^2\right.\\
&\quad \left. + 8ABx + B^2 + 4AC) - 4A^{3/2}]\partial_u\right\},\\
V_5 &= \exp\left(\frac{\sqrt{A}}{2}t\right)\left[2\sqrt{x}\partial_x - \frac{\mathrm{i}\,u}{2\sqrt{A}}(2A\sqrt{x}+B))\partial_u\right],\\
V_6 &= \exp\left(-\frac{\sqrt{A}}{2}t\right)\left[2\sqrt{x}\partial_x + \frac{\mathrm{i}\,u}{2\sqrt{A}}(2A\sqrt{x}+B)\partial_u\right].
\end{aligned}$$

Case Ib-2 $A < 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= \frac{\sin(\sqrt{-A}t)}{\sqrt{-A}}2(2A\sqrt{x}+B)\sqrt{x}\partial_x + 4\cos(\sqrt{-A}t)\partial_t\\
&\quad - \frac{u}{4A}[\mathrm{i}\cos(\sqrt{-A}t)(8A^2x + 8AB\sqrt{x}\\
&\quad + B^2 + 4AC) + 4(-A)^{3/2}\sin(\sqrt{-A}t)]\partial_u,\\
V_4 &= -\left\{\frac{\cos(\sqrt{-A}t)}{\sqrt{-A}}2(2A\sqrt{x}+B)\partial_x - 4\sin(\sqrt{-A}t)\partial_t\right.\\
&\quad + \frac{u}{4A}[\mathrm{i}\sin(\sqrt{-A}t)(8A^2x + 8AB\sqrt{x}\\
&\quad \left. + B^2 + 4AC) - 4(-A)^{3/2}\cos(\sqrt{-A}t)]\partial_u\right\},\\
V_5 &= 2\sin\left(\frac{\sqrt{-A}}{2}t\right)\sqrt{x}\partial_x\\
&\quad + \frac{(2A\sqrt{x}+B)}{2\sqrt{-A}}\,\mathrm{i}\,u\cos\left(\frac{\sqrt{-A}}{2}t\right)\partial_u,\\
V_6 &= 2\cos\left(\frac{\sqrt{-A}}{2}t\right)\sqrt{x}\partial_x\\
&\quad - \frac{(2A\sqrt{x}+B)}{2\sqrt{-A}}\,\mathrm{i}\,u\sin\left(\frac{\sqrt{-A}}{2}t\right)\partial_u.
\end{aligned}$$

Case Ib-3 $A = 0$.

$$\begin{aligned}
V_1 &= \partial_t, \quad V_2 = u\partial_u, \quad V_\beta = \beta(x,t)\partial_u,\\
V_3 &= 2\left(\sqrt{x} + \frac{3B}{16}t^2\right)\sqrt{x}\partial_x + 2t\partial_t\\
&\quad - \frac{\mathrm{i}\,u}{4}\left(\frac{B^2}{16}t^3 + 3B\sqrt{x}t + 2Ct\right)\partial_u,\\
V_4 &= 2\left(\sqrt{x} + \frac{B}{16}t^2\right)\sqrt{x}t\partial_x + t^2\partial_t\\
&\quad - u\left[\mathrm{i}\left(\frac{B^2}{256}t^4 + \frac{C}{4}t^2 + \frac{3B}{8}\sqrt{x}t^2 + x\right) + \frac{t}{2}\right]\partial_u,
\end{aligned}$$

$$V_5 = \sqrt{x}t\partial_x - \mathrm{i}\,u\left(\sqrt{x} + \frac{B}{16}t^2\right)\partial_u,$$
$$V_6 = 2\sqrt{x}\partial_x - \frac{\mathrm{i}\,B}{4}ut\partial_u.$$

Case II $\tau'(t) = 0$. In this case, f is an arbitrary function of x. The symmetry algebra is generated by $V_1 = \partial_t$, $V_2 = u\partial_u$, and $V_\beta = \beta(x,t)\partial_u$.

4.3. Gauss kernels on Case Ia

We now compute Gauss kernels for equations

$$\mathrm{i}\,u_t = xu_{xx} + \frac{1}{2}u_x + \frac{C}{4x}u, \quad x \in \mathbb{R}^+, \tag{30}$$

and

$$\mathrm{i}\,u_t = xu_{xx} + \frac{1}{2}u_x + \left(\frac{A}{4}x + \frac{C}{4x} + \frac{D}{4}\right)u, \quad x \in \mathbb{R}^+,\ A > 0, \tag{31}$$

which belong to Cases Ia-3 and Ia-1, respectively. Then applying Theorem 1, the Gauss kernels of equations

$$\mathrm{i}\,u_\tau = u_{yy} + \frac{C}{y^2}u, \quad y \in \mathbb{R}, \tag{32}$$

and

$$\mathrm{i}\,u_\tau = u_{yy} + \left(Ay^2 + \frac{C}{y^2} + D\right)u, \quad y \in \mathbb{R} \tag{33}$$

can be obtained through the transformation (24).

We now employ the method given in Ref. [21] to compute the Gauss kernel for Eq. (30). To obtain its fundamental solution, the following proposition is needed.

Proposition 2 The linear Schrödinger equation

$$\mathrm{i}\,u_t = xu_{xx} + \frac{1}{2}u_x + \left(\frac{C}{4x} + \frac{D}{4}\right)u, \quad x \in \mathbb{R}^+ \tag{34}$$

has the symmetry

$$\tilde{u}_\epsilon(x,t) = \frac{1}{\sqrt{1+\epsilon t}}\exp\left[-\frac{\mathrm{i}\,\epsilon(Dt^2+4x)}{4(1+\epsilon t)}\right] \times u\left(\frac{x}{(1+\epsilon t)^2}, \frac{t}{1+\epsilon t}\right), \tag{35}$$

where ϵ is a sufficiently small parameter and u is an arbitrary solution of Eq. (34). If $u(x,t) = u_0(x)$ with u_0 an analytic, stationary solution then there is a Gauss kernel $p(t,x,y)$ of Eq. (34) satisfying

$$\int_0^{+\infty} \mathrm{e}^{-\mathrm{i}\,\epsilon z}p(t,x,z)u_0(z)\mathrm{d}z = \tilde{u}_\epsilon(x,t). \tag{36}$$

Proof The above classification results show that equation (34) admits the infinitesimal symmetry of the form

$$V = 2xt\partial_x + t^2\partial_t - u\left[\mathrm{i}(x + \frac{D}{4}t^2) + \frac{t}{2}\right]\partial_u.$$

Exponentiating this vector field and applying it to a solution $u(x,t)$ yields $\tilde{u}_\epsilon(x,t)$ which has the property that $\tilde{u}_\epsilon(x,0) = \mathrm{e}^{-\mathrm{i}\,\epsilon x}u_0(x)$. Then by Theorem 1 in Ref. [22], Proposition 2 is then proved. □

For the Gauss kernel of Eqs. (30) and (32), we have the following result.

Theorem 2

The Gauss kernels for Eqs. (30) and (32) are given respectively by

$$p(t,x,z) = \frac{\mathrm{i}}{t}\left(\frac{x}{z}\right)^{1/4}\exp\left[-\frac{\mathrm{i}(x+z)}{t}\right] \times \mathrm{I}_{2d-1/2}\left(\frac{2\,\mathrm{i}\sqrt{xz}}{t}\right), \tag{37}$$

and

$$\tilde{p}(\tau,y,\tilde{z}) = \frac{\mathrm{i}}{2\tau}\sqrt{y\tilde{z}}\exp\left[-\frac{\mathrm{i}(y^2+\tilde{z}^2)}{4\tau}\right] \times \mathrm{I}_{2d-1/2}\left(\frac{\mathrm{i}\,y\tilde{z}}{2\tau}\right), \tag{38}$$

where I_γ is the modified Bessel function of the first kind with order γ.

Proof It follows from Proposition 2 that equation (30) has the symmetry solution

$$\tilde{u}_\epsilon(x,t) = \frac{1}{\sqrt{1+\epsilon t}}\exp\left(-\frac{\mathrm{i}\,\epsilon x}{1+\epsilon t}\right) \times u\left(\frac{x}{(1+\epsilon t)^2}, \frac{t}{1+\epsilon t}\right).$$

Clearly, equation (30) has the solution

$$u_0(x) = x^d, \quad d = \frac{1}{4}(1 + \sqrt{1-16C}).$$

Substituting it into Eq. (36) yields

$$\int_0^{+\infty} \mathrm{e}^{-\mathrm{i}\,\epsilon z}p(t,x,z)z^d\mathrm{d}z = \frac{x^d}{(1+\epsilon t)^{2d+1/2}}\exp\left(-\frac{\mathrm{i}\,\epsilon x}{1+\epsilon t}\right). \tag{39}$$

Inverting equation (39) gives immediately Gauss kernel (37).

Similarly, one can show that the Gauss kernel of Eq. (32) is given by Eq. (38). □

For Eq. (31), we can use a group-invariant solution to construct its Gauss kernel. A group-invariant solution is a solution which is left invariant under the action of a group transformation. A detailed and rigorous discussion on group-invariant solutions and their

applications can be found in Olver's book[4] (see also in Refs. [3] and [23]). The following theorem given in Ref. [23] is needed for our approach.

Theorem 3 Consider a boundary value problem for a k-th order scalar PDE $F(\boldsymbol{x}, D^\alpha u) = 0$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ defined on a domain $\Omega \subset \mathbb{R}^n$ with boundary conditions $B_\alpha(\boldsymbol{x}, u, u^{k-1)}) = 0$ prescribed on boundary surfaces $\omega_\alpha(\boldsymbol{x}) = 0$, $\alpha = 1, 2, \ldots, s$. A vector field V is admitted by the BVP if:

(i) $V^k[F(\boldsymbol{x}, D^\alpha u)] = 0$ when $F(\boldsymbol{x}, D^\alpha u) = 0$;

(ii) $V(\omega_\alpha(\boldsymbol{x})) = 0$ when $\omega_\alpha(\boldsymbol{x}) = 0$;

(iii) $V^{n-1}[B_\alpha(\boldsymbol{x}, u, u^{k-1)})] = 0$ when $B_\alpha(\boldsymbol{x}, u, u^{k-1)}) = 0$ on $\omega_\alpha(\boldsymbol{x}) = 0$, $\alpha = 1, 2, \ldots, s$.

If the BVP admits a vector field V, then the solution of the BVP is a group-invariant solution with respect to the symmetries generated by V.

Applying Theorem 2 and Theorem 3 to the construction of Gauss kernels of Eqs. (31) and (33), one obtains the following result.

Theorem 4 The Gauss kernel for Eqs. (31) and (33) are given respectively by

$$\begin{aligned} &p(t,x,z)\\ &= \frac{\sqrt{A}}{\sinh(\frac{\sqrt{A}}{2}t)} \exp\left[-\left(\frac{\sqrt{A}\,\mathrm{i}(x+z)}{2\tanh(\frac{\sqrt{A}}{2}t)} + \frac{D}{4}t\,\mathrm{i}\right)\right]\\ &\quad\times\left(\frac{x}{z}\right)^{1/4} z\,\mathrm{I}_{\sqrt{1-4C}/2}\left(\frac{\mathrm{i}\sqrt{Axz}}{\sinh\frac{\sqrt{A}}{2}t}\right), \end{aligned} \tag{40}$$

and

$$\begin{aligned} &\tilde{p}(\tau,y,\tilde{z})\\ &= \frac{2\sqrt{A}}{\sinh(2\sqrt{A}\tau)} \exp\left[-\left(\frac{\sqrt{A}\,\mathrm{i}(y^2+\tilde{z}^2)}{2\tanh(2\sqrt{A}\tau)} + D\tau\,\mathrm{i}\right)\right]\\ &\quad\times\left(\frac{y}{\tilde{z}}\right)^{1/2} \tilde{z}^3\mathrm{I}_{\sqrt{1-4C}/2}\left(\frac{\mathrm{i}\sqrt{A}y\tilde{z}}{\sinh(2\sqrt{A}\tau)}\right). \end{aligned} \tag{41}$$

Proof Equation (30) admits a Lie algebra of symmetries spanned by $\{V_1 - V_4,\ V_\beta\}$ in Case Ia-1 with $a = 0$. To solve

$$\begin{aligned} &\mathrm{i}\,u_t = xu_{xx} + \frac{1}{2}u_x + \left(\frac{A}{4}x + \frac{C}{4x} + \frac{D}{4}\right)u,\\ &x \in \mathbb{R}^+,\ A > 0,\\ &u(x,0) = \delta(x-z), \end{aligned}$$

we need to look for symmetries of the form $V = \sum_{k=1}^{4} c_k V_k$ which preserve the boundary conditions. By Theorem 3, we have:

(I) $V(x) = 0$, when $x = 0$;

(II) $V(t) = 0$, when $t = 0$;

(III) $V(u - \delta) = 0$, when $u(x,0) = \delta(x - z)$.

It is easy to show that these conditions hold if and only if $c_3 = c_4$, $c_1 = -8c_3$, and $c_2 = 2(2Ay + D)\,\mathrm{i}\,c_3$. We then compute the invariants for the action of

$$V = c_3[-8V_1 + V_3 + V_4 + 2(2Ay + D)\,\mathrm{i}\,V_2].$$

The corresponding invariant solution has the invariant form

$$\begin{aligned} u = &\exp\left\{-\frac{\sqrt{A}(x+z)\,\mathrm{i}}{2\tanh(\frac{\sqrt{A}}{2}t)} - \frac{D}{4}t\,\mathrm{i}\right\}\left(\frac{z}{x}\right)^{1/4}\\ &\times F(\zeta), \end{aligned} \tag{42}$$

where $F(\zeta)$ is an arbitrary function of the similarity variable

$$\zeta = \frac{x}{4\sinh^2(\frac{\sqrt{A}}{2}t)}.$$

Substituting Eq. (42) into Eq. (31), we find that $F(\zeta)$ satisfies the ODE $18\zeta^2F''(\zeta) + (3 + 16Az\zeta + 4c)F = 0$. And hence

$$\begin{aligned} F(\zeta) = &\,2\sqrt{Az\zeta}[c_1(z)\,\mathrm{I}_{\sqrt{1-4C}/2}(2\,\mathrm{i}\sqrt{Az\zeta})\\ &+ c_2(z)\,\mathrm{I}_{-\sqrt{1-4C}/2}(2\,\mathrm{i}\sqrt{Az\zeta})], \end{aligned}$$

where c_1 and c_2 are arbitrary functions of z. Here we take $c_1(z) = 1$ and $c_2(z) = 0$, and then the Gauss kernel (40) for Eq. (31) follows.

Subsequently, an application of Theorem 2 leads to the Gauss kernel for Eq. (33). □

5. Concluding remarks

In this paper, we developed Craddock's method to construct the Gauss kernels for Schrödinger equations with some potential functions. The relationship between the Lie point symmetries and Gauss kernels to Schrödinger equations was identified. Explicit Gauss kernels to several specified forms of equations were obtained by inverting Laplace or Fourier transformations of certain group-invariant solutions. We also established the relationship between Gauss kernels of two systems which are related by equivalence transformations. Several examples are used to illustrate such a relationship.

Integrable PDEs are divided into two classes: S-integrable and C-integrable.[26] If a nonlinear PDE can be linearized through transformations of independent and dependent variables, then it is said to be C-integrable. It has been shown in Refs. [27] and [28] (see, also, the relevant references therein) that

a number of scalar or systems of nonlinear evolutionary equations can be linearized by using hodograph transformations. Motivated by the above results, the Cauchy problem of nonlinear Schrödinger and parabolic PDEs can be solved explicitly provided that the equations can be linearized through hodograph transformations.

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