Modified variational iteration method for an El Niño Southern Oscillation delayed oscillator*

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This paper studies a delayed air–sea coupled oscillator describing the physical mechanism of El Niño Southern Oscillation. The approximate expansions of the delayed differential equation's solution are obtained successfully by the modified variational iteration method. The numerical results illustrate the effectiveness and correctness of the method by comparing with the exact solution of the reduced model.

Keywords: air–sea coupling, nonlinear oscillator, modified variational iteration method, delayed differential equation

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1. Introduction

The El Niño Southern Oscillation (ENSO) is an abnormal and interannual physical phenomenon seen on earth, which brings disaster to global economic development and human life. Because the ENSO is a complex and nonlinear system, it is difficult to predict when it will happen and its strength accurately.[1] Since the work of Bjerknes,[2] a lot of studies have investigated and proposed the dynamics of the ENSO.[3−7] It is now widely accepted that the ENSO is an oscillation phenomenon repeated by adjustment processes of the coupled ocean–atmosphere system in the tropical Pacific. The adjustment processes examined extensively thus far are the equatorial wave propagations with a time delay[1] and the basin-wide heat and mass exchange between the equator and off-equator.[6,7] The behaviour of the ENSO is a very interesting subject to study in the international scientific community. A great deal of progress has been made through these studies.[1−27] Lin *et al.* investigated the related issues in atmospheric physics, ocean climate, dynamical systems, among others.[8−11] Mo *et al.*[15−26] have studied a number of nonlinear problems in atmospheric physics using a variational iteration method (VIM), generalized VIM, and homotopic analysis method[27,28] (HAM). The VIM was first proposed by He and has been proved by many authors to be a powerful mathematical tool for all kinds of nonlinear problem.[29−36] The solution procedure of this method is very simple and occurs by means of variational theory, and only a few iterations lead to highly accurate solutions that are valid for the whole solution domain.

Insight into the solution procedure of the VIM shows some disadvantages, namely, repeated computations and calculation of unneeded terms.[30] In Refs. [34] and [35], the modified VIM (MVIM) was proposed to overcome these disadvantages. In this paper, the MVIM is applied to solve a delayed differential equation for the typical ENSO air–sea oscillator. The numerical results illustrate the reliability and performance of the MVIM.

2. The ENSO delayed oscillator

Let us consider the following delayed air–sea oscillator model describing the physical mechanism of the ENSO. The model is given as a first-order, nonlinear differential equation[37−40]

$$\begin{aligned} &\frac{\mathrm{d}T}{\mathrm{d}t} = AT - \varepsilon T^3 - BT(t-\sigma), \quad t \in (0, +\infty), \\ &T(t)|_{t=0} = T(0), \end{aligned} \tag{1}$$

where T denotes the sea surface temperature (SST) anomaly, that is, the deviation from a suitably defined long term average temperature. Equation (1) is the conceptual oscillator model which emphasizes the

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ocean and atmosphere interactions in the equatorial eastern Pacific and considers anomaly variations only in this coupling region. The first term on the right-hand side of Eq. (1) represents a strong positive feedback within a coupled ocean–atmosphere system. The second term is a nonlinear damping term which is presented to limit the growth of unstable perturbations. This term represents various ocean advective processes and moist processes which stop temperatures from increasing or decreasing without bound. The third term considers equatorially trapped ocean waves propagating across the Pacific, before interacting back with the central Pacific region after a time delay σ. The $T(0)$ is the initial value of SST. The model parameters A, B, σ, and ε are positive real variables, their definition and physical meaning can be seen from Refs. [37]–[40].

In Eq. (1), on the assumption that σ is small enough, the first order approximation $T(t)-\sigma T'(t)$ can be used instead of $T(t-\sigma)$.[39] Then, equation (1) has the form

$$(1-B\sigma)\frac{\mathrm{d}T}{\mathrm{d}t} = (A-B)T - \varepsilon T^3, \quad t \in (0,+\infty). \tag{2}$$

3. Approximate solution by the MVIM

The correction functional of nonlinear dynamical system (2) by the variational iteration method can be written as follows:

$$\begin{aligned} &T_{n+1}(t) \\ &= T_n(t) + \int_0^t \lambda\Big\{(1-B\sigma)\frac{\mathrm{d}T_n(s)}{\mathrm{d}s} \\ &\quad + (B-A)T_n(s) + \varepsilon[\bar{T}_n(s)]^3\Big\}\mathrm{d}s, \end{aligned} \tag{3}$$

where λ is a generalized Lagrange multiplier. The Lagrange multiplier can be identified optimally via variational theory.[29–36] Here T_n denotes the n-th order approximate solution and $\bar{T}_n(t)$ is considered as restricted variations, i.e., $\delta\bar{T}_n(t)=0$. It is easy to observe that we use restricted variation only in the nonlinear term. The $T_0(t)$ is the initial approximate solution or trial function through careful selection.

Making the correction functional, Eq. (3), stationary with respect to T_n, and noting that $\delta\bar{T}_n(t)=0$, we construct

$$\begin{aligned} \delta T_{n+1}(t) = \delta T_n(t) + \delta\int_0^t \Big\{\lambda\Big[(1-B\sigma)\frac{\mathrm{d}T_n(s)}{\mathrm{d}s} \\ + (B-A)T_n(s)\Big]\Big\}\mathrm{d}s. \end{aligned} \tag{4}$$

After integration by parts, we can rewrite Eq. (4) as follows:

$$\begin{aligned} &\delta T_{n+1}(t) \\ &= \delta T_n(t) + [(1-B\sigma)\lambda\delta T_n(s)]\,|_{s=0}^{s=t} \\ &\quad - \int_0^t \Big[(1-B\sigma)\frac{\mathrm{d}\lambda}{\mathrm{d}s} + (A-B)\lambda\Big]\delta T_n(s)\mathrm{d}s \\ &= 0, \end{aligned} \tag{5}$$

which yields the following stationary conditions:

$$(1-B\sigma)\frac{\mathrm{d}\lambda}{\mathrm{d}s} + (A-B)\lambda = 0,$$

$$\lambda(t) = -1/(1-B\sigma).$$

Therefore, if we assume that $(1-B\sigma)\neq 0$, the general Lagrange multiplier λ can be readily identified as

$$\lambda(s) = -\frac{1}{(1-B\sigma)}\exp\left[\frac{A-B}{B\sigma-1}(s-t)\right].$$

Substituting the identified Lagrangian multiplier into functional (3) results in the following variational iteration formula:

$$\begin{aligned} &T_{n+1} \\ &= T_n - \int_0^t \frac{1}{(1-B\sigma)}\exp\left[\frac{A-B}{B\sigma-1}(s-t)\right] \\ &\quad \times \left[(1-B\sigma)\frac{\mathrm{d}T_n}{\mathrm{d}s} + (B-A)T_n + \varepsilon T_n^3\right]\mathrm{d}s. \end{aligned} \tag{6}$$

The second term on the right is called the correction term. The successive approximation T_n, $n\geq 0$ of the solution T will be readily obtained by using the determined Lagrange multiplier and any selective function T_0. Consequently, the solution is given by $T=\lim\limits_{n\to\infty} T_n$. For estimated error and convergence of the variational iteration method, see Ref. [36].

In the iteration formula (6), it can be observed that there are repeated and unneeded calculations in each step. To overcome this disadvantage of the VIM, a function $u(t)$ can be introduced and is defined as follows:

$$u(t) = -\int_0^t \frac{\varepsilon T^3}{(1-B\sigma)}\exp\left[\frac{A-B}{B\sigma-1}(s-t)\right]\mathrm{d}s. \tag{7}$$

Differentiating both sides of Eq. (7) with respect to t results in

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} \\ &= \frac{\varepsilon T^3(s)}{(B\sigma-1)}\exp\left[\frac{A-B}{B\sigma-1}(s-t)\right]\Big|_{s=t} \\ &\quad - \frac{A-B}{B\sigma-1}\int_0^t \frac{\varepsilon T_n^3}{(B\sigma-1)}\exp\left[\frac{A-B}{B\sigma-1}(s-t)\right]\mathrm{d}s. \end{aligned}$$

Substituting Eq. (7) into the above equation, and after a simple manipulation, we have

$$(1-B\sigma)\frac{\mathrm{d}u}{\mathrm{d}t} = (A-B)u - \varepsilon T^3, \tag{8}$$

this is equivalent to Eq. (2). That means $u(t)$ is a special solution of the ENSO delayed air–sea oscillator model (2). Accordingly we can construct a very simple iteration formulation in the form

$$T_{n+1}(t) = T_0(t) + u_n(t), \tag{9}$$

where T_0 is the initial solution with or without unknown parameters. In the case of no unknown parameters, T_0 should satisfy the initial conditions. When some unknown parameters are involved in T_0, the unknown parameters can be identified by the initial conditions after a few iterations.

In view of Eq. (7), the iteration is reduced to a simple formulation, which reads

$$\begin{aligned} &T_{n+1}(t) \\ &= T_0(t) - \int_0^t \frac{\varepsilon T_n^3(s)}{(1-B\sigma)} \exp\left[\frac{A-B}{B\sigma-1}(s-t)\right] \mathrm{d}s. \end{aligned} \tag{10}$$

Equation (10) is the recursive formula of the ENSO delayed air–sea oscillator model (2) by the MVIM. It can be observed that the repeated calculations are canceled in formula (10). And this modification may lead to faster convergence to the exact solution in general.

In order to give the initial approximation solution, at first we consider the linear system for the ENSO delayed air–sea oscillator model (2):

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = \frac{A-B}{1-B\sigma}T_0. \tag{11}$$

Then the following analytical solution can be readily obtained as

$$T_0(t) = C_1 \exp(rt), \tag{12}$$

where $r = (A-B)/(1-B\sigma)$ and C_1 is an arbitrary constant. Substituting initial approximate solution (12) into Eq. (10), we can obtain the first order approximate solution of the model (2), it is

$$\begin{aligned} T_1(t) = {} & C_1\left[1 - \frac{\varepsilon C_1^2}{2r(B\sigma-1)}\right]\exp(rt) \\ & + \frac{\varepsilon C_1^3 \exp(3rt)}{2r(B\sigma-1)}. \end{aligned} \tag{13}$$

Further, the second-order approximate analytic solution is given by

$$\begin{aligned} T_2(t) = {} & -\frac{\varepsilon C_1^3}{B\sigma-1}\left[\frac{1}{2r} - \frac{3\varepsilon C_1^2}{8r^2(B\sigma-1)} + \frac{\varepsilon^2 C_1^4}{8r^3(B\sigma-1)^2} - \frac{\varepsilon^3 C_1^6}{64r^4(B\sigma-1)^3}\right]\exp(rt) \\ & + \frac{\varepsilon}{B\sigma-1}\left(\frac{C_1^3}{2r} - \frac{3\varepsilon C_1^5}{4r^2(B\sigma-1)} + \frac{3\varepsilon^2 C_1^7}{8r^3(B\sigma-1)^2} - \frac{\varepsilon^3 C_1^9}{16r^4(B\sigma-1)^3}\right)\exp(3rt) \\ & + \frac{\varepsilon}{B\sigma-1}\left(\frac{3\varepsilon C_1^5}{8r^2(B\sigma-1)} - \frac{3\varepsilon^2 C_1^7}{8r^3(B\sigma-1)^2} + \frac{3\varepsilon^3 C_1^9}{32r^4(B\sigma-1)^3}\right)\exp(5rt) \\ & + \frac{\varepsilon}{B\sigma-1}\left[\frac{\varepsilon^2 C_1^7}{8r^3(B\sigma-1)^2} - \frac{\varepsilon^3 C_1^9}{16r^4(B\sigma-1)^3}\right]\exp(7rt) \\ & + \frac{\varepsilon^4 C_1^9}{64r^4(B\sigma-1)^4}\exp(9rt) + C_1\exp(rt). \end{aligned} \tag{14}$$

By means of the MVIM and the formula (10), we can obtain a higher order approximate solution of the ENSO air–sea oscillator model (2).

4. Illustrative example

In this section, in order to verify the accuracy of the above results, we consider a special case. Let $B = 0$ in the ENSO model (1) or (2), then a nonlinear differential equation is given as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = AT - \varepsilon T^3, \tag{15}$$

and we can easily obtain the exact solution of SST[19] as

$$T(t) = \left[\frac{\varepsilon}{A} + \left(\frac{1}{(T(0))^2} - \frac{\varepsilon}{A}\right)\exp(-2At)\right]^{-1/2}, \tag{16}$$

where $T(0)$ is the initial value of SST. Making the assumptions that ε is a small enough but positive parameter, from Eq. (16) we have

$$\begin{aligned} T(t) = {} & T(0)\exp(At) - \frac{\varepsilon(T(0))^3}{2A}\exp(3At) \\ & \times [1 - \exp(-2At)] + O(\varepsilon^2). \end{aligned} \tag{17}$$

Note that $r = A$, when $B = 0$. So, in view of Eqs. (13) and (14), the first-order approximate analytical solution by the MVIM can be simplified into the following formula:

$$T_1(t) = C_1 \left[1 + \frac{\varepsilon C_1^2}{2A}\right] \exp(At) - \frac{\varepsilon C_1^3 \exp(3At)}{2A}.$$

Incorporating the initial condition into $T_1(t)$, we obtain $C_1 = T(0)$. So, the first-order approximate analytical solution of nonlinear equation (15) by the MVIM can be further written in the following expression:

$$T_1(t) = T(0)\exp(At) + \frac{\varepsilon T^3(0)}{2A}\exp(At) - \frac{\varepsilon T^3(0)\exp(3At)}{2A}. \quad (18)$$

By comparing Eqs. (17) and (18), we can observe that the solution $T_1(t)$ by the MVIM is the same as the first-order component of the exact series solution expanded in view of small parameter ε. So we can conclude that the MVIM is of high accuracy. Similarly, when $B = 0$ the second-order approximate analytic solution by the MVIM can be simplified to the following formula:

$$\begin{aligned} & T_2(t) \\ &= \left(\frac{\varepsilon C_1^3}{2A} + \frac{3\varepsilon^2 C_1^5}{8A^2} + \frac{\varepsilon^3 C_1^7}{8A^3} + \frac{\varepsilon^4 C_1^9}{64A^4}\right)\exp(At) \\ &+ C_1 \exp(At) \\ &- \left(\frac{\varepsilon C_1^3}{2A} + \frac{3\varepsilon^2 C_1^5}{4A^2} + \frac{3\varepsilon^3 C_1^7}{8A^3} + \frac{\varepsilon^4 C_1^9}{16A^4}\right)\exp(3At) \\ &+ \left(\frac{3\varepsilon^2 C_1^5}{8A^2} + \frac{3\varepsilon^3 C_1^7}{8A^3} + \frac{3\varepsilon^4 C_1^9}{32A^4}\right)\exp(5At) \\ &- \left(\frac{\varepsilon^3 C_1^7}{8A^3} + \frac{\varepsilon^4 C_1^9}{16A^4}\right)\exp(7At) \\ &+ \frac{\varepsilon^4 C_1^9}{64A^4}\exp(9At), \end{aligned} \quad (19)$$

where the unknown parameter can be readily determined to be $C_1 = T(0)$ by using the initial condition.

In order to illustrate the validity and correctness of the MVIM, the exact, first, and second order approximate solutions of nonlinear equation (15) in the time interval $[0, 5]$ are shown together in Fig. 1, where $A = 0.5$, $\varepsilon = 0.01$, and $T(0) = 0.5$. The exact solution and first and second order MVIM ones are denoted by black solid, dashed, and dotted lines respectively. From the figure we can see that the three lines are almost completely overlapping when $0 < t < 4$. So, good consistency is verified between an exact solution and an approximate one by the MVIM. When $4 < t < 5$, there is a certain difference between the first order solution and the exact one, but the second order solution is still consistent with the exact solution. Therefore, a conclusion can be made that the higher order approximate solution by the MVIM is superior to the lower order one and the MVIM series solution will converge to the exact solution with the increase of the order.

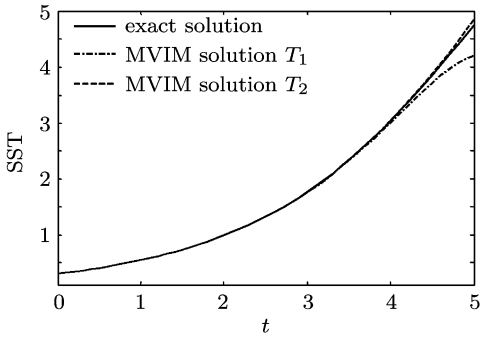


Fig. 1. Exact solution, first and second order approximate solutions by the MVIM of the nonlinear equation (15).

Table 1 shows the specific numerical values of the exact, first and second order approximate solutions by the MVIM at different times. As can be seen from the table, the second order approximate solution is significantly better than the first-order one and is closer to the exact solution. In the time interval $0 < t < 5$, the relative error of the second order approximate solution is less than 2.5% by comparing with the exact one. Therefore, the numerical results demonstrate that the modified variational iteration method is effective and convenient for the ENSO delayed oscillator model.

Table 1. Comparison of exact solution, T_1 and T_2 by the MVIM at different times.

t	0.5	1.0	2.0	3.0	4.0	4.5	5.0
Exact solution	0.40471	0.54569	0.98863	1.76830	3.04331	3.86982	4.75829
T_1 by MVIM	0.40471	0.54568	0.98855	1.76644	3.00806	3.72601	4.20699
T_2 by MVIM	0.40471	0.54569	0.98863	1.76835	3.04620	3.88965	4.87719

5. Conclusion

In this paper, the MVIM[34] proposed by He is applied to an ENSO delayed oscillator and the approximate analytic solutions are obtained successfully. First, a correction functional is introduced and the Lagrange multiplier is identified optimally via variational theory. Second, the improved iteration formula of the MVIM is determined, which is superior to the formula of the VIM. Finally, the approximate analytic solution of the ENSO delayed oscillator is obtained by the MVIM, and the correctness of the MVIM is demonstrated by comparing with an exact solution in a special case. The numerical results illustrate that the MVIM can effectively reduce the cost of computation and memory and speed the rate of convergence. In addition, since the MVIM is an approximate analytical method, so the obtained approximate solution can be used to further study the qualitative and quantitative aspects of sea surface temperature anomalies in the equatorial eastern Pacific.

References

[1] Battisti D S and Hirst A C 1989 *J. Atmos. Sci.* **46** 1687

[2] Bjerknes J 1969 *Mon. Wea. Rev.* **97** 163

[3] Cane M A and Sarachik E S 1981 *J. Mar. Res.* **39** 651

[4] Chao Y and Philander S G H 1993 *J. Climate* **6** 450

[5] Hirst A C 1988 *J. Atmos. Sci.* **45** 830

[6] Jin F F 1997 *J. Atmos. Sci.* **54** 811

[7] Jin F F 1997b *J. Atmos. Sci.* **54** 830

[8] Lin W T, Ji Z Z, Li S L and Yang X Z 2000 *Prog. Nat. Sci.* **10** 936

[9] Lin W T, Ji Z Z, Wang B and Yang X Z 2000 *Chin. Sci. Bull.* **45** 1358

[10] Lin W T, Ji Z Z and Wang B 2002 *Adv. Atmos. Sci.* **19** 699

[11] Lin W T and Mo J Q 2004 *Chin. Sci. Bull.* **48** 5

[12] Feng G L, Dong W J, Jia X J and Cao H X 2002 *Acta Phys. Sin.* **51** 1181 (in Chinese)

[13] Feng G L, Dai X G, Wang A H and Chou J F 2001 *Acta Phys. Sin.* **50** 606 (in Chinese)

[14] Feng G L and Dong W J 2005 *Acta Meteo. Sin.* **63** 864 (in Chinese)

[15] Mo J Q, Wang H, Lin W T and Lin Y H 2006 *Acta Phys. Sin.* **55** 6 (in Chinese)

[16] Mo J Q, Wang H and Lin W T 2006 *Acta Phys. Sin.* **55** 3229 (in Chinese)

[17] Mo J Q, Lin W T and Zhu J 2006 *Adv. Math.* **35** 232

[18] Mo J Q and Lin W T 2005 *Chin. Phys.* **14** 875

[19] Mo J Q, Lin Y H and Wang H 2005 *Chin. Phys.* **14** 2387

[20] Mo J Q, Wang H, Lin W T and Lin Y H 2006 *Chin. Phys.* **15** 671

[21] Mo J Q, Lin W T and Wang H 2007 *Chin. Phys.* **16** 951

[22] Mo J Q and Lin W T 2008 *Chin. Phys.* B **17** 370

[23] Mo J Q and Lin W T 2008 *Chin. Phys.* B **17** 743

[24] Zhu M, Lin W T, Lin Y H and Mo J Q 2011 *Acta Phys. Sin.* **60** 030204 (in Chinese)

[25] Mo J Q 2011 *Acta Phys. Sin.* **60** 030203 (in Chinese)

[26] Mo J Q, Cheng R J and Ge H X 2011 *Acta Phys. Sin.* **60** 040203 (in Chinese)

[27] Wu Q K 2011 *Acta Phys. Sin.* **60** 068802 (in Chinese)

[28] He J H 2002 *Approximate Analytical Methods in Engineering and Sciences* (Zhengzhou: Henan Science and Technology Publisher) (in Chinese)

[29] He J H 1997 *Commun. Nonlinear Sci. Numer. Simul.* **2** 235

[30] He J H 1998 *Comp. Meth. Appl. Mech. Eng.* **167** 57

[31] He J H 1999 *Int. J. Nonlinear Mech.* **34** 699

[32] He J H 1999 *Comp. Meth. Appl. Mech. Eng.* **178** 257

[33] He J H 2000 *Int. J. Nonlinear Mech.* **35** 37

[34] He J H 2007 *Comput. Math. Appl.* **54** 881

[35] Tamer A A 2007 *Comput. Math. Appl.* **54** 955

[36] Ramos J I 2008 *Appl. Math. Comput.* **199** 39

[37] Zhou X C, Lin Y H, Lin W T and Mo J Q 2009 *Chin. Phys.* B **18** 4603

[38] Wang C 2001 *Adv. Atmos. Sci.* **18** 674

[39] Feng M C 2005 *Acta Sc. Nat. Univer. Sunyatseni* **44** 152

[40] Boutle I, Richard H S and Rudolf A R 2007 *Am. J. Phys.* **75** 15