Finite symmetry transformation group of the Konopelchenko–Dubrovsky equation from its Lax pair*

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Starting from a weak Lax pair, the general Lie point symmetry group of the Konopelchenko–Dubrovsky equation is obtained by using the general direct method. And the corresponding Lie algebra structure is proved to be a Kac–Moody–Virasoro type. Furthermore, a new multi-soliton solution for the Konopelchenko–Dubrovsky equation is also given from this symmetry group and a known solution.

Keywords: Lax pairs, symmetries, symmetry group, exact solution

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1. Introduction

The study of symmetries plays an important role in various branches of some natural sciences especially in integrable systems.[1−4] For a partial differential equation (PDE), if we find its symmetry, we can obtain the exact solution, corresponding conservation law and Hamiltonian structure. Therefore, to find some symmetry groups for a PDE is always an interesting topic.[5−9] There is a standard method depending on the famous first fundamental theorem of Lie[1,2] to find the Lie point symmetry group of some nonlinear systems. Although this standard method had been widely used to find Lie point symmetry algebras and groups for almost all known integrable systems, it is still quite complicated and difficult to find non-Lie point symmetry groups. Fortunately, Lou and Ma[10−13] had proposed a general direct method. By the general direct method, both the Lie point symmetry groups and the non-Lie symmetry ones can be obtained for some nonlinear PDEs. Moreover, the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches.

Note that there are lots of studies on the symmetries of integrable systems using the general direct method, while there is little application of this method on Lax pairs. In fact, the Lax pair of integrable models has been proved to be very important in the study of its various interesting properties. In particular, it is useful to obtain infinite numbers of symmetries.[14,15] Based on Lax pair, Lou[16] developed a simple method to find directly a finite symmetry transformation group.

In this paper, starting from a weak Lax pair of the Konopelchenko–Dubrovsky (KD) equation, the general Lie point symmetry group of the KD equation is obtained.

2. Symmetry transformation group and exact solution of the KD equation

The KD system, presented by Konopelchenko and Dubrovsky[17] in 1984, has the form

$$\begin{aligned} &u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0,\\ &u_y - v_x = 0, \end{aligned} \tag{1}$$

where

$$u = u(x, y, t),$$

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$$v = v(x, y, t),$$

the subscripts denote partial differential derivatives, and a and b are real parameters. Equation (1) is a nonlinear integrable evolution equation on two spatial dimensions and one temporal. It is just the compatibility conditions of the following system

$$\psi_y + \psi_{xx} + \left(au + \frac{2b}{a}\right)\psi_x = 0, \tag{2}$$

$$\psi_t - 4\psi_{xxx} - 6\left(au - \frac{2b}{a}\right)\psi_{xx} + \left(3av - \frac{3}{2}a^2u^2 + 6bu - 12\frac{b^2}{a^2} - 3au_x\right)\psi_x = 0. \tag{3}$$

That is to say, equations (2) and (3) are a Lax pair of system (1). For $u_y = 0$, equation (1) becomes the Gardner equation (combined KdV and modified KdV equations). For $a = 0$, equation (1) is the well-known Kadomtsev–Petviashvili (KP) equation. When $b = 0$, equation (1) is the modified KP equation. The KD equation is solved by the inverse scattering transformation,[18] and some soliton wave solutions and soliton-like solutions are given.[19−22] In Ref. [23], Li *et al.* calculated the generalized symmetries by the formal series method presented by Lou.[24−26] Using the standard truncated Painlevé analysis, Lin *et al.*[27] obtained some new types of multi-soliton solutions of the KD equation.

In order to obtain the finite symmetry transformation group of the KD equation, we let

$$\psi = \beta G(\xi, \eta, \tau), \tag{4}$$

where β, ξ, η, τ are functions of x, y, and t, and G satisfies the following equations

$$G(\xi,\eta,\tau)_\eta = -G(\xi,\eta,\tau)_{\xi\xi} - \left[aU(\xi,\eta,\tau) + \frac{2b}{a}\right]G(\xi,\eta,\tau)_\xi, \tag{5}$$

$$4G(\xi,\eta,\tau)_{\xi\xi\xi} = (3aV(\xi,\eta,\tau) - \frac{3}{2}a^2U(\xi,\eta,\tau)^2 - 6bU(\xi,\eta,\tau) - 12\frac{b^2}{a^2} - 3aU(\xi,\eta,\tau)_\xi)G(\xi,\eta,\tau)_\xi + G(\xi,\eta,\tau)_\tau - 6\left(aU(\xi,\eta,\tau) - \frac{2b}{a}\right)G(\xi,\eta,\tau)_{\xi\xi}. \tag{6}$$

Substituting Eq. (4) and Eqs. (5), (6) into Eq. (2), we obtain

$$-19\beta\eta_x^2U^2a^5G_{\xi\xi} + 8\beta\tau_x^2a^3G_{\tau\tau} + 8\beta\tau_ya^3G_\tau + \cdots = 0. \tag{7}$$

Obviously, β should not be zero and there exists no nontrivial solution for $\tau_x = 0$, $\tau_y = 0$, and $\eta_x = 0$, so

$$\tau_x = 0, \quad \tau_y = 0, \quad \eta_x = 0 \quad \text{i.e.} \quad \tau \equiv \tau(t), \quad \eta \equiv \eta(y,t). \tag{8}$$

Thus we have

$$8a^2(a\beta_y + 2b\beta_x + a^2u\beta_x + a\beta_{xx})G + 8a^3\beta_x(\xi_x^2 - \eta_y)G_{\xi\xi} + \cdots = 0, \tag{9}$$

$$(-24a^6u\beta_{xx} - 48b^2\beta_xa^3 + 12a^6v\beta_x + 48a^4b\beta_{xx} - 6a^7u^2\beta_x - 12a^6u_x\beta_x + 24ua^5\beta_x + 4a^5\beta_t - 16a^5\beta_{xxx})G + 4a^5\beta(\tau_t - \xi_x^3)G_{\xi\xi\xi} + \cdots = 0. \tag{10}$$

It is easy to see that

$$\beta = c, \quad \xi = \tau_t^{1/3}x + f(y,t), \quad \eta = \tau_t^{2/3}y + g(t). \tag{11}$$

Now the substitution of Eq. (11) with Eq. (8) into Eqs. (9) and (10) leads to

$$-8a^2\left(a^2\tau_t^{2/3}U - 2b\tau_t^{1/3} - af_y + 2\tau_t^{2/3}b - a^2u\tau_t^{1/3}\right)G_\xi = 0, \tag{12}$$

$$4a^4\left(-12\tau_tb + 12b\tau_t^{2/3} + 6a^2\tau_tU - \left(\frac{2}{3}y\tau_{tt}\tau_t^{-1/3} + g_t\right)a - 6a^2u\tau_t^{2/3}\right)G_{\xi\xi} + 2a^3\left(-6a^3\tau_tV - 6a^3u_x\tau_t^{1/3} - 4a(2y\tau_{tt}\tau_t^{-1/3} + g_t)b + 2a^2\left(\frac{1}{3}x\tau_{tt}\tau_t^{-1/3} + f_t\right) - 2a^3(2y\tau_{tt}\tau_t^{-1/3} + g_t)U + 3a^4\tau_tU^2 - 3a^4u^2\tau_t^{1/3} + 12a^2\tau_tbU + 12a^2bu\tau_t^{1/3} + 6a^3v\tau_t^{1/3} - 24b^2\tau_t^{1/3} + 6a^3\tau_tU + 24\tau_tb^2\right)G_\xi = 0. \tag{13}$$

By vanishing the coefficients of the polynomials of G and its derivatives, we obtain the determining equations. With the help of Maple software, an infinite-dimensional symmetry algebra, depending on three

arbitrary functions of the variable t, is derived. The result in detail is given below.

Theorem 1 If ($U = U(x,y,t)$ and $V = V(x,y,t)$) is a solution to the KD equation, then (u,v) is given by

$$\begin{aligned}
u =& U(\xi,\ \eta,\ \tau)\tau_t^{1/3} + \frac{2b}{a^2}(1-\tau_t^{1/3}) \\
&- \frac{y}{9a}\tau_{tt}\tau_t^{-1} - \frac{g_t}{6a}\tau_t^{-2/3}, \\
v =& \tau_t^{2/3}V(\xi,\ \eta,\ \tau) + \left(\frac{\tau_{tt}^2}{54a}\tau_t^{-2} - \frac{\tau_{ttt}}{54a}\tau_t^{-1}\right)y^2 \\
&+ \left(\frac{g_t}{27a}\tau_{tt}\tau_t^{-5/3} - \frac{g_{tt}}{18a}\tau_t^{-2/3} - \frac{2b}{9a^2}\tau_{tt}\tau_t^{-2/3}\right)y \\
&+ \left(\frac{1}{9}\tau_{tt}y\tau_t^{-2/3} + \frac{1}{6}g_t\tau_t^{-1/3}\right)U(\xi,\ \eta,\ \tau) \\
&- \frac{1}{9a}\tau_{tt}x\tau_t^{-1} - \frac{1}{3a}h_t\tau_t^{-1/3} + \frac{1}{72a}g_t^2\tau_t^{-4/3} \\
&- \frac{b}{3a^2}g_t\tau_t^{-1/3} + \frac{2b^2}{a^3}(1-\tau_t^{2/3}),
\end{aligned}$$

with

$$\begin{aligned}
\xi &= \tau_t^{1/3}x + \frac{1}{18}y^2\tau_{tt}\tau_t^{-2/3} + \frac{1}{6}yg_t\tau_t^{-1/3} + h(t), \\
\eta &= \tau_t^{2/3}y + g(t), \quad \tau = \tau(t),
\end{aligned} \tag{14}$$

where $\tau(t)$, $g(t)$, $h(t)$ are all arbitrary functions of t.

Writing down a special selection of the arbitrary functions (τ, h, g) in the forms

$$\tau = t + \epsilon f(t), \qquad h = \epsilon k(t), \qquad g = \epsilon l(t),$$

then we can re-obtain the Lie point symmetry algebra with the general symmetry elements

$$\begin{aligned}
\sigma &= \sigma_1(f) + \sigma_2(l) + \sigma_3(k) \\
&= \begin{pmatrix} \left(\frac{1}{3}f_tx + \frac{1}{18}y^2f_{ttt}\right)u_x + \frac{2}{3}f_tyu_y + fu_t + \frac{2}{3}f_tu - \frac{1}{9a}f_{tt}y - \frac{2b}{3a^2}f_t \\ \left(\frac{1}{3}f_tx + \frac{1}{18}y^2f_{ttt}\right)v_x + \frac{2}{3}f_t(yv_y + v) + fv_t - \frac{1}{54a}f_{ttt}y^2 - \frac{4b}{3a^3}f_t + p(x,y,t) \end{pmatrix} \\
&\quad + \begin{pmatrix} \frac{1}{6}l_tyu_x + lu_y - \frac{1}{6a}l_t \\ \frac{1}{6}l_tyv_x + lv_y - \frac{1}{6a}l_t \end{pmatrix} + \begin{pmatrix} ku_x \\ kv_x - \frac{1}{3a}k_t \end{pmatrix},
\end{aligned} \tag{15}$$

where

$$p(x,y,t) = \left(\frac{1}{9}yu - \frac{1}{9a}x - \frac{2b}{a^2}y\right)f_{tt}.$$

The related Kac–Moody–Virasoro type symmetry algebra is as follows:

$$\begin{aligned}
&[\sigma_1(f_1), \sigma_1(f_2)] = \sigma_1(f_1f_{2t} - f_2f_{1t}), \\
&[\sigma_1(f), \sigma_2(l)] = \sigma_2\left(fl_t - \frac{2}{3}lf_t\right), \\
&[\sigma_1(f), \sigma_3(k)] = \sigma_3\left(\frac{1}{3}kf_t - fk_t\right), \\
&[\sigma_2(l_1), \sigma_2(l_2)] = \frac{1}{6}\sigma_3(l_1l_{2t} - l_2l_{1t}), \\
&[\sigma_2(l), \sigma_3(k)] = 0, \\
&[\sigma_3(k_1), \sigma_3(k_2)] = 0.
\end{aligned}$$

By means of Theorem 1, starting from a known multi-soliton solution, a new exact solution of the KD equation can be obtained. In Ref. [22], a new kind of multi-soliton solution is given:

$$\begin{aligned}
u =& \frac{2}{b(1+\sum_{i=1}^n(\cosh(H_i)\exp(E_i)))} \\
&\times \sum_{i=1}^n[(\sinh(H_i)K_i + \cosh(H_i)k_i)\exp(E_i)],
\end{aligned} \tag{16}$$

$$\begin{aligned}
v =& \frac{2}{b(1+\sum_{i=1}^n(\cosh(H_i)\exp(E_i)))} \\
&\times \sum_{i=1}^n\left[\frac{2}{b}(a-bk_i)K_i\sinh(H_i)\exp(E_i)\right. \\
&\left. + \left(K_i^2 + k_i^2 - \frac{2}{b}ak_i\right)\cosh(H_i)\exp(E_i)\right],
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
H_i =& K_ix + \frac{2}{b}(a-bk_i)K_iy \\
&+ \frac{4K_i}{b^2}(3a^2 - 6abk_i + 3b^2K_i^2 + b^2K_i^2)t + a_i, \\
E_i =& k_ix + \left(\frac{2}{b}ak_i - K_i^2 - k_i^2\right)y \\
&+ \frac{4}{b^2}(3b^2k_iK_1^2 - 3ab(K_i^2 + k_i^2) \\
&+ 3a^2k_i + b^2k_i^3)t + b_i,
\end{aligned}$$

and k_i, K_i, a_i, b_i are arbitrary constants.

Applying Theorem 1 to the solution, Eqs. (16) and (17), the group-invariant solution is

$$\begin{aligned} u =& \frac{2\tau_t^{1/3}}{b(1+\sum_{i=1}^{n}(\cosh(H_i)\exp(E_i)))} \\ &\times \sum_{i=1}^{n}[(\sinh(H_i)K_i+\cosh(H_i)k_i)\exp(E_i)] \\ &+ \frac{2b}{a^2}(1-\tau_t^{1/3}) - \frac{y}{9a}\tau_{tt}\tau_t^{-1} - \frac{g_t}{6a}\tau_t^{-2/3}, \end{aligned} \tag{18}$$

$$\begin{aligned} v =& \frac{2\tau_t^{2/3}}{b(1+\sum_{i=1}^{n}(\cosh(H_i)\exp(E_i)))} \\ &\times \sum_{i=1}^{n}\left[\frac{2}{b}(a-bk_i)K_i\sinh(H_i)\exp(E_i)\right. \\ &\left.+\left(K_i^2+k_i^2-\frac{2}{b}ak_i\right)\cosh(H_i)\exp(E_i)\right] \\ &+ \frac{4\tau_{tt}y\tau_t^{-2/3}+6g_t\tau_t^{-1/3}}{18b(1+\sum_{i=1}^{n}(\cosh(H_i)\exp(E_i)))} \\ &\times \sum_{i=1}^{n}[(\sinh(H_i)K_i+\cosh(H_i)k_i)\exp(E_i)] \\ &+ \left(\frac{g_t}{27a}\tau_{tt}\tau_t^{-5/3} - \frac{g_{tt}}{18a}\tau_t^{-2/3} - \frac{2b}{9a^2}\tau_{tt}\tau_t^{-2/3}\right)y \\ &+ \left(\frac{\tau_{tt}^2}{54a}\tau_t^{-2} - \frac{\tau_{ttt}}{54a}\tau_t^{-1}\right)y^2 - \frac{1}{9a}\tau_{tt}x\tau_t^{-1} \\ &- \frac{1}{3a}h_t\tau_t^{-1/3} + \frac{1}{72a}g_t^2\tau_t^{-4/3} - \frac{b}{3a^2}g_t\tau_t^{-1/3} \\ &+ \frac{2b^2}{a^3}(1-\tau_t^{2/3}), \end{aligned} \tag{19}$$

where

$$\begin{aligned} H_i =& K_i\xi + \frac{2}{b}(a-bk_i)K_i\eta \\ &+ \frac{4K_i}{b^2}(3a^2-6abk_i+3b^2K_i^2+b^2K_i^2)\tau + a_i, \\ E_i =& k_i\xi + \left(\frac{2}{b}ak_i - K_i^2 - k_i^2\right)\eta \\ &+ \frac{4}{b^2}(3b^2k_iK_1^2 - 3ab(K_i^2+k_i^2) \\ &+ 3a^2k_i + b^2k_i^3)\tau + b_i, \end{aligned}$$

and ξ, η, τ are determined by Eq. (14), k_i, K_i, a_i, b_i are arbitrary constants. By selecting the arbitrary functions $h(t)$, $g(t)$, introduced by the transformation group, one can obtain complicated and abundant structures of the KD equation. And when we take $\tau(t) = t$, $g(t) = 0$, $h(t) = 0$, then the solutions, equations (18) and (19), are reduced to Eqs. (16) and (17) (the result shown in Ref. [22]), respectively.

Figure 1 shows the structure of a four-soliton solution of the field u at $t = 0$ for Eq. (18) with $k_1 = 1/2$, $k_2 = 1$, $K_1 = 3/2$, $K_2 = -1/2$, $a = -1$, $b = 1$, $s_1 = 0$, $s_2 = 0$, $r_1 = 0$, $r_2 = 8$, $\xi = x + \sin(t)$, $\eta = y$, $\tau = t$.

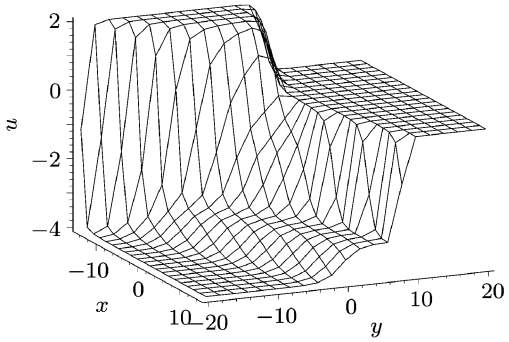


Fig. 1. Four-soliton solution for the field u at $t = 0$ in Eq. (18) with $k_1 = 1/2$, $k_2 = 1$, $K_1 = 3/2$, $K_2 = -1/2$, $a = -1$, $b = 1$, $s_1 = 0$, $s_2 = 0$, $r_1 = 0$, $r_2 = 8$, $\xi = x + \sin(t)$, $\eta = y$, and $\tau = t$.

3. Summary and discussion

In summary, starting from the Lax pair expression of the KD equation, the finite symmetry transformation group theorem is obtained in a very simple way. According to Theorem 1, the related Lie point symmetries which have a Kac–Moody–Virasoro structure can be obtained simply by restricting the arbitrary functions in infinitesimal forms. It is necessary to point out that we can obtain all group-invariant solutions from known solutions by Theorem 1. This method is valid for other types of known (2+1)-dimensional integrable models such as the Kadomtsev–Petviashvili equation and the Davey–Stewartson system, Broer–Kaup equation, and even (3+1)-dimensional Jimbo–Miwa equation.

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