Rapid Communication

A diagrammatic categorification of q-boson and q-fermion algebras*

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(Received 13 September 2011; revised manuscript received 22 September 2011)

In this paper, we study the diagrammatic categorification of q-boson algebra and also q-fermion algebra. We construct a graphical category corresponding to q-boson algebra. q-Fock states correspond to some kind of 1-morphisms, and the graded dimension of the graded vector space of 2-morphisms is exactly the inner product of the corresponding q-Fock states. We also find that this graphical category can be used to categorify q-fermion algebra.

Keywords: categorification, q-boson algebra, q-Fock state, q-fermion algebra

PACS: 02.10.Hh, 03.65.Ca, 03.65.Fd **DOI:** 10.1088/1674-1056/20/21/2/020201

1. Introduction

Categorification is a process of replacing set-theoretic theorems by category-theoretic analogues. It replaces sets with categories, functions with functors, and equations between functions by natural transformations between functors, which in turn should satisfy certain equations of their own called the coherence law.[1] In recent years, there has been much interest in the study of categorification of some algebras in mathematical physics.[2−8] In physics, many people have introduced category theory into studies of fundamental physical theories.[9−15] Since most existing physical theories are built from set-theoretic mathematical foundations, and there are still many unsolved problems in fundamental physics, some researchers have made efforts to replace set-theoretic theorems in physical theories by category-theoretic analogues. These procedures may extend the mathematical structures of existing theories and help us to solve the remaining problems in fundamental physics. The categorification of physical theories can also help us to better understand the physical essence.

A harmonic oscillator is the simplest and most fundamental model in quantum physics, and Heisenberg algebra (or boson algebra) lies at the heart of the theory of a quantum harmonic oscillator. Recently, Khovanov has constructed a categorification of Heisenberg algebra based on a graphical category that can act naturally on the category of representations of all symmetric groups,[4] Licata *et al.* have also carried out related work.[5,6] q-boson algebra is the natural extension of normal boson algebra, and it has been fully investigated in physics and mathematics in the past two decades, as well as q-fermion algebra and quon algebra.[16−19]

In fact, Khovanov has already used nilCoxeter algebra to study categorification of quantum Weyl algebra which is similar to q-boson algebra.[2] In this presnt work, we use the diagrammatic methods used in Refs. [3], [4], and [6] to study the categorification of q-boson algebra. We first construct a graphical category corresponding to q-boson algebra. We find that this graphical category is similar to the categories introduced in Refs. [4] and [6], except that we introduce the degree of the planar diagrams. q-Fock states correspond to some kind of 1-morphisms, and the graded dimension of the graded vector space of 2-morphisms is exactly the inner product of the corresponding q-Fock states. We also find that the graphical category can be used to categorify q-fermion algebra. Our graphical category can also act naturally on the module categories constructed in Refs. [4] and [6].

This paper is organized as follows. In the next section, we will briefly review q-boson algebra and q-Fock states in normal quantum mechanics. In Section

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10975102, 10871135, 11031005, and 11075014).

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3, we construct a graphical category that can be used to categorify the q-boson algebra, and investigate the properties of this category. We also use this graphical category to study the categorification of the q-fermion algebra in Section 4. In Section 5, we briefly review the categories of modules for symmetric group algebras and bimodule maps constructed by Khovanov. Some discussions are given in Section 6.

2. q-boson oscillator and q-Fock states

The q-boson oscillator is described by the Hamiltonian[18]

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}), \tag{1}$$

where ω is the oscillator frequency, and $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators satisfying the so-called q-boson algebraic relation

$$[\hat{a}, \hat{a}^\dagger]_q := \hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = 1 \tag{2}$$

with the deformation parameter q being some real number, $0 < q \leqslant 1$.

In what follows, we use the system of units where $\hbar = 1$ and $\omega = 1$. Let $|n\rangle = (\hat{a}^\dagger)^n|0\rangle$, where $|0\rangle$ is the vacuum state and is destroyed by the annihilation operator $\hat{a}|0\rangle = 0$. $|n\rangle$ are the eigenvectors of the Hermitian operator $\hat{N} = \hat{a}^\dagger\hat{a}$,

$$\hat{N}|n\rangle = [n]|n\rangle, \tag{3}$$

and these eigenvectors $|n\rangle$ are orthogonal,

$$\langle n|n'\rangle = [n]!\,\delta_{n,n'}, \quad (n, n' \in \mathbb{N}), \tag{4}$$

where $[n] = (1-q^n)/(1-q)$ is the q-integer, and $[n]! = [n][n-1]\cdots[2][1]$.

We may also define the orthonormal eigenstates $|n)$

$$\begin{aligned} |n) &= \frac{1}{\sqrt{[n]!}}|n\rangle = \frac{1}{\sqrt{[n]!}}(\hat{a}^\dagger)^n|0\rangle, \\ (n|n') &= \delta_{n,n'}, \end{aligned} \tag{5}$$

and the operators $\hat{a}$ and $\hat{a}^\dagger$ act as ladder operators

$$\begin{aligned} \hat{a}|n) &= \sqrt{[n]}|n-1), \\ \hat{a}^\dagger|n) &= \sqrt{[n+1]}|n+1). \end{aligned} \tag{6}$$

$|n)$ are the eigenvectors of the Hamiltonian

$$\hat{H}|n) = E_n|n), \tag{7}$$

and we have

$$E_n = [n] + \frac{q^n}{2} = \frac{1}{2}([n] + [n+1]). \tag{8}$$

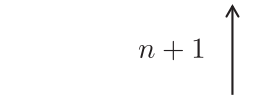
3. Categorification of the q-boson algebra

From the previous section, we know that the q-boson oscillator is very similar to the standard harmonic oscillator, so we may categorify the q-boson algebra by borrowing ideas from the categorification of the normal boson algebra.[4,6,20]

We may construct an additive, $\Bbbk$-linear, $\mathbb{Z}$-graded 2-category $\mathcal{Q}$ corresponding to the q-boson algebra, where $\Bbbk$ is some commutative ring. The objects of $\mathcal{Q}$ are non-negative integers $n \in \mathbb{N}$, and the set of the 1-morphisms in $\mathcal{Q}$ is generated by 1-morphisms $Q_+\{i\} : n \to n+1$ and $Q_-\{i\} : n \to n-1$ for $i \in \mathbb{Z}$, where $\{i\}$ means shifting the grading by i. An arbitrary 1-morphism of $\mathcal{Q}$ is a finite compositions of $Q_\varepsilon\{i\} := Q_{\varepsilon_1}\{i_1\} \circ \cdots \circ Q_{\varepsilon_n}\{i_n\}$, where $\varepsilon = \varepsilon_1 \ldots \varepsilon_n$ is a finite sequence of $+$ and $-$ signs, and we have $i = i_1 + \cdots + i_n$. The identity 1-morphism is $\mathbf{1} = Q_\emptyset\{0\}$. The positive and negative grading shifts in the category $\mathcal{Q}$ correspond to the multiplication by q and q^{-1} on the Grothendieck group $K_0(\mathcal{Q})$, $q^{\pm1}[Q_\varepsilon] := [Q_\varepsilon\{\pm1\}]$, and obviously, $K_0(\mathcal{Q})$ is a $\mathbb{Z}[q, q^{-1}]$-module. In the following, we will just consider the 1-morphisms $Q_\varepsilon := Q_\varepsilon\{0\}$, and all the results can be easily extended to the case of $Q_\varepsilon\{i\}$ for $i \neq 0$.

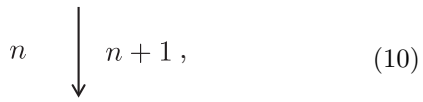
We may use some planar diagrams to represent the 2-category $\mathcal{Q}$. The objects are represented by the regions of the planar diagram labelled by non-negative integers, and we use the orientations of the lines to distinguish two different types of 1-morphism Q_+ and Q_-. The 2-morphisms $\mathrm{Hom}_\mathcal{Q}(Q_\varepsilon, Q_{\varepsilon'})$ are represented by some kinds of planar diagrams, and the endpoints of the strands of the planar diagrams must agree with the signs in the sequences ε and ε', and triple intersections are not allowed. The bottoms of the diagrams correspond to Q_ε, and the tops of the diagrams correspond to $Q_{\varepsilon'}$.

For example, the identity 2-morphism of $\mathrm{Hom}_\mathcal{Q}(Q_+, Q_+)$ corresponds to the following diagram



(9)

and the identity 2-morphism of $\mathrm{Hom}_{\mathcal{Q}}(Q_-, Q_-)$ corresponds to the following diagram



(10)

and the diagram



(11)

is one of the 2-morphisms from Q_{-+} to Q_{+-}.

A diagram equals 0 if it has a region labelled by a negative number.

The space of 2-morphisms $\mathrm{Hom}_{\mathcal{Q}}(Q_\varepsilon, Q_{\varepsilon'})$ is the $\mathbb{Z}$-graded $\Bbbk$-module generated by planar diagrams. Obviously, for the 1-morphisms $Q_\varepsilon : n_1 \to n_2$ and $Q_{\varepsilon'} : m_1 \to m_2$, the 2-morphisms $\mathrm{Hom}_{\mathcal{Q}}(Q_\varepsilon, Q_{\varepsilon'})$ are nonzero only if $m_1 = n_1$ and $m_2 = n_2$. Also, if for some k, the last k terms of ε contain at least $(n+1)$ more $-$ signs than $+$ signs, then the 1-morphism $Q_\varepsilon : n \to m$ will be set to zero, because in this case, there will be some regions in the planar diagram labelled by negative integers, and this is not allowed.

Since one must encode the q-boson algebraic relation into the 2-category $\mathcal{Q}$, we must introduce some constraint relations of the planar diagrams which correspond to the 2-morphisms of $\mathcal{Q}$. Similar to Ref. [3], we may appeal to the inner products of the q-Fock states to understand the properties of 2-morphisms of $\mathcal{Q}$.

The 1-morphisms Q_+, Q_- correspond to the operators $\hat{a}^\dagger$, $\hat{a}$ respectively, we may introduce the notation $\hat{A}_\varepsilon := \hat{a}_{\varepsilon_1} \cdots \hat{a}_{\varepsilon_n}$, where $\varepsilon = \varepsilon_1 \ldots \varepsilon_n$ is a finite sequence of $+$ and $-$ signs, and $\hat{a}_+ := \hat{a}^\dagger$, $\hat{a}_- := \hat{a}$. Similar to Ref. [20], a q-Fock state $|\hat{A}_\varepsilon\rangle := \hat{A}_\varepsilon|0\rangle = \lambda|n\rangle$ for $\lambda \in \mathbb{R}$ corresponds to some 1-morphism $Q_\varepsilon : 0 \to n$. The inner product $\langle \hat{A}_{\varepsilon'}|\hat{A}_\varepsilon\rangle$ corresponds to the graded dimension of the graded vector space $\mathrm{Hom}_{\mathcal{Q}}(Q_\varepsilon, Q_{\varepsilon'})$ for the corresponding 1-morphisms Q_ε and $Q_{\varepsilon'}$,

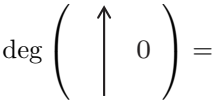
$$\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_\varepsilon, Q_{\varepsilon'})) = \langle \hat{A}_{\varepsilon'}|\hat{A}_\varepsilon\rangle, \tag{12}$$

each term aq^t appearing in the inner product will be interpreted as the dimension of the a-dimensional homogeneous space of 2-morphisms in degree t.

In the following, we will only consider the 1-morphisms $Q_\varepsilon : 0 \to n$ for some $0 < n \in \mathbb{N}$. Let us first consider the simplest case,

$$\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_+, Q_+))$$
$$= \langle \hat{a}^\dagger|\hat{a}^\dagger\rangle = \langle 0|\hat{a}\hat{a}^\dagger|0\rangle = [1] = 1, \tag{13}$$

so we have

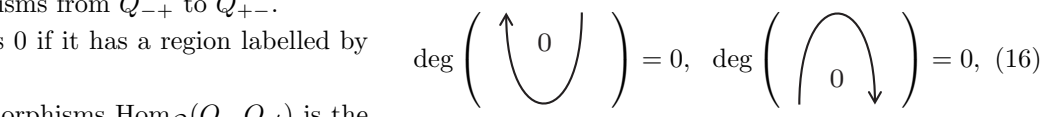


(14)

Similarly, we have

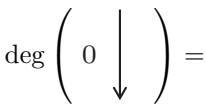
(15)

so we also have



(16)

and

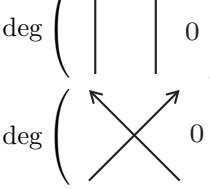


(17)

For the case of Q_{++}, we have

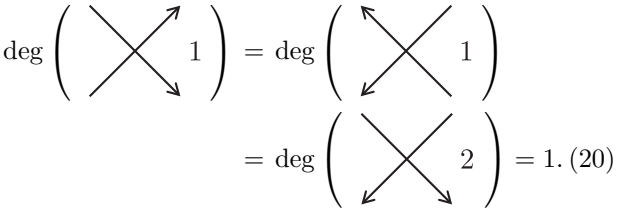
$$\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{++}, Q_{++}))$$
$$= \langle (\hat{a}^\dagger)^2|(\hat{a}^\dagger)^2\rangle = \langle 0|\hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger|0\rangle = [2]! = 1 + q. \tag{18}$$

Obviously, there are



(19)

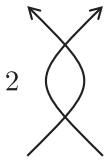
So we also have



(20)

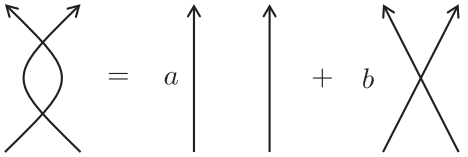
Furthermore, it is easy to check that the degree of these fundamental diagrams do not depend on the labels of the regions.

For a slightly more complicated diagram



(21)

since there is no term λq^2, so we should introduce the additional relation



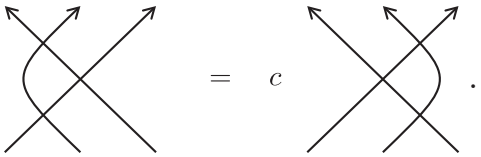
(22)

where the coefficients a, b should satisfy some conditions.

Similarly, for the case of Q_{+++}, we have

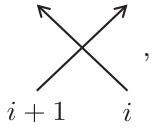
$$\begin{aligned}&\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{+++},Q_{+++}))\\&=\langle(\hat{a}^{\dagger})^3|(\hat{a}^{\dagger})^3\rangle=\langle 0|\hat{a}\hat{a}\hat{a}\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}^{\dagger}|0\rangle=[3]!\,,\end{aligned}\tag{23}$$

it is easy to see that, there should be the following relation



(24)

If we label the strands by some $i\in\mathbb{N}$, for example



, (25)

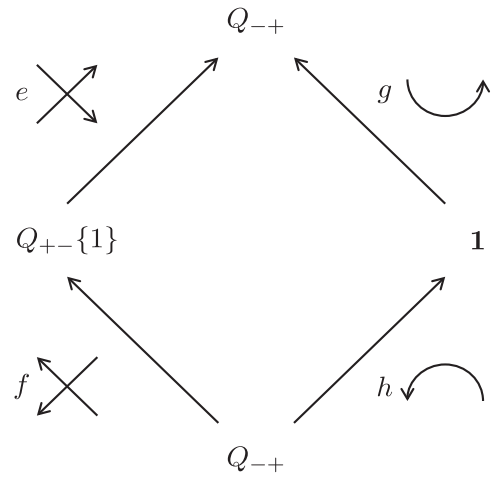
we find that the crossing corresponds to $s_i\in S_n$ (or the generator of nilCoxeter algebra or Hecke algebra). Usually, a 2-morphism of $\mathrm{Hom}_{\mathcal{Q}}(Q_{+^n},Q_{+^n})$ corresponds to $x=s_is_j\cdots s_k\in S_n$, and the degree of this 2-morphism is equal to the standard length of x in the symmetric group S_n.

The composition of 2-morphisms $\alpha\in\mathrm{Hom}_{\mathcal{Q}}(Q_{\varepsilon},Q_{\varepsilon'})$ and $\beta\in\mathrm{Hom}_{\mathcal{Q}}(Q_{\varepsilon'},Q_{\varepsilon''})$ is achieved by stacking the corresponding diagrams on top of each other. Note that the relation (22) does not preserve the degree of the diagrams, and usually, the degree of the composition of two 2-morphisms does not equal the sum of the degree of each 2-morphism. So the "effective degree" of a crossing maybe 1, 0 or -1. This is quite different from the categorification of quantum $\mathfrak{sl}_2$ in Ref. [3]. Usually we cannot obtain the correct degree of a 2-morphism by simply counting the crossings of the corresponding diagram, so we need some additional rules to simplify the diagram and guarantee that there is good correspondence between the degree and the crossings, we will introduce such rules as follows.

In the 2-category $\mathcal{Q}$, we should have the following isomorphic relation

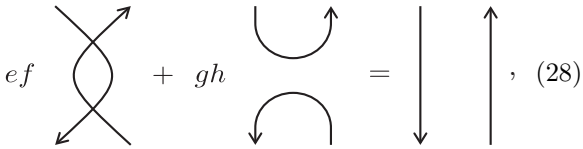
$$Q_{-+}\cong Q_{+-}\{1\}\oplus\mathbf{1},\tag{26}$$

this corresponds to $[Q_-][Q_+]=q[Q_+][Q_-]+1$ in the Grothendieck group $K_0(\mathcal{Q})$. We have the following 2-morphisms between Q_{-+} and $Q_{+-}\{1\}\oplus\mathbf{1}$,

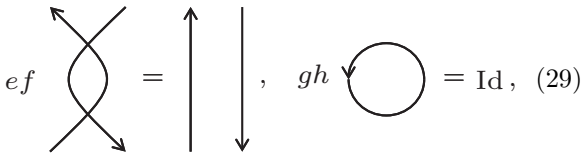


(27)

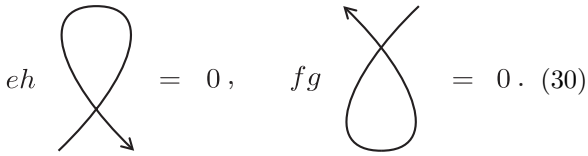
so there will be



(28)

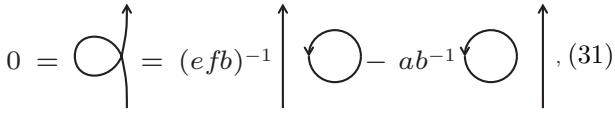


(29)



(30)

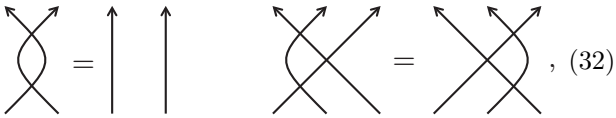
In the relations (22), (24), (28), (29), and (30), we may naturally choose $c=gh=1$. From relations (22) and (30), if $b\neq 0$ we have



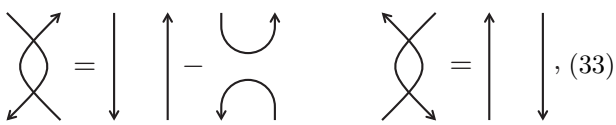
(31)

so we should have $a=(ef)^{-1}$, and obviously, $a\neq 0$.

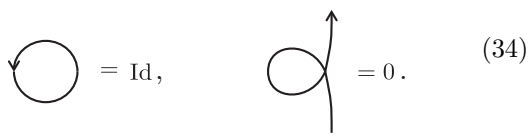
The most natural choice is $a=c=ef=gh=1$, $b=0$, and then we have



(32)



(33)

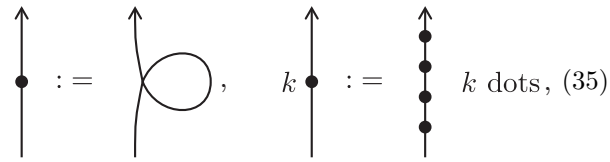


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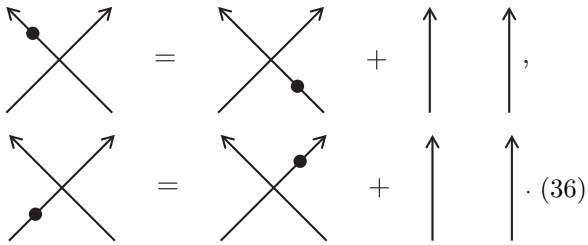
If we choose $c = gh = 1$, $a = (ef)^{-1} = t$, and $b = t-1$ for some deformation parameter t, we will obtain the local relations used in Ref. [6].

The above relations are the same as those in Ref. [4], so we have the same graphical calculational relations as those in Ref. [4].

Similarly, we may introduce the dot to denote the right curl



and the degree of a dot is 1. We also have



From Ref. [20] we know that, a basis of the graded $\Bbbk$-vector space $\mathrm{Hom}_{\mathcal{Q}}(Q_{\varepsilon}, Q_{\varepsilon'})$ consists of the planar diagrams which satisfy the following rules:

(i) Any two strands intersect once, at most, and no triple intersections are allowed.

(ii) For the strands that connect the bottom and top, only the upward strands are allowed.

(iii) For the strands that connect two points at the bottom, only the left-turn strands are allowed.

(iv) For the strands that connect two points at the top, only the right-turn strands are allowed.

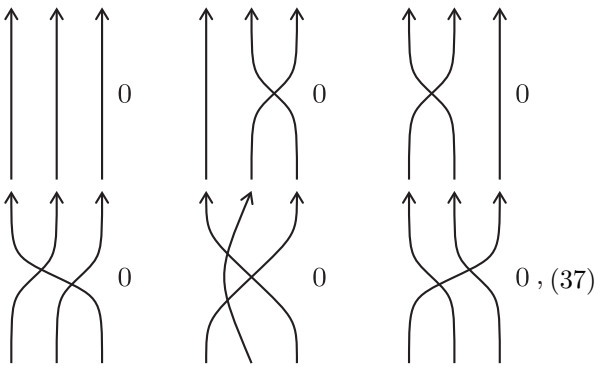
(v) Circles and strands with dots are not allowed.

The graded dimension (gdim) of the graded vector space $\mathrm{Hom}_{\mathcal{Q}}(Q_{\varepsilon}, Q_{\varepsilon'})$ equals the inner product of the corresponding states $|\hat{A}_{\varepsilon}\rangle$ and $|\hat{A}_{\varepsilon'}\rangle$.

From a physical point of view, these rules mean that we must create one particle before we destroy it, and the diagrams which are allowed just correspond to the possible ways to create and then destroy the particles.

One may find that the relations (22), (28), and (29) do not preserve the degree of the diagrams, but after we introduce the above rules, there is no arbitrariness of the degree of the diagrams. So we can uniquely determine the graded dimension of the graded vector space $\mathrm{Hom}_{\mathcal{Q}}(Q_{\varepsilon}, Q_{\varepsilon'})$ and the inner product $\langle\hat{A}_{\varepsilon'}|\hat{A}_{\varepsilon}\rangle$.

For example, for the 2-morphisms $\mathrm{Hom}_{\mathcal{Q}}(Q_{+++}, Q_{+++})$, we have the following different kinds of planar diagrams



so we have

$$\begin{aligned}&\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{+++}, Q_{+++}))\\ &= 1 + 2q + 2q^2 + q^3\\ &= (1+q+q^2)(1+q) = [3]!\,,\end{aligned} \tag{38}$$

this coincides exactly with $\langle\hat{A}_{+++}|\hat{A}_{+++}\rangle$.

In general, for the case of Q_{+^n}, we have

$$\begin{aligned}&\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{+^n}, Q_{+^n}))\\ &= \langle\hat{A}_{+^n}|\hat{A}_{+^n}\rangle = [n]! = 1 + \sum_{i=1}^{\frac{n(n-1)}{2}} a_i q^i,\end{aligned} \tag{39}$$

where the coefficient a_i is the number of different kinds of planar diagrams with degree i, and we have the following formula

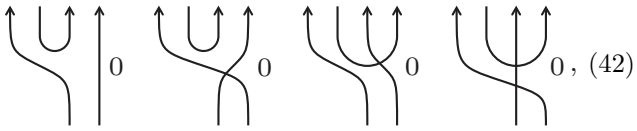
$$a_i = \sum_{\lambda_j^{(i)}} \prod_{k=1}^{l} C^{m_k}_{n-\lambda_{j,k}^{(i)}-\sum_{r=1}^{k-1} m_r}, \tag{40}$$

where $C_n^k := n!/(k!(n-k)!)$, and the first summation is over all possible partitions of i, $\lambda_j^{(i)} = \left(\left(\lambda_{j,1}^{(i)}\right)^{m_1}, \left(\lambda_{j,2}^{(i)}\right)^{m_2}, \ldots, \left(\lambda_{j,l}^{(i)}\right)^{m_l}\right)$ for $\lambda_{j,1}^{(i)} > \lambda_{j,2}^{(i)} > \cdots > \lambda_{j,l}^{(i)} > 0$ is a partition of i, $m_1\lambda_{j,1}^{(i)} + m_2\lambda_{j,2}^{(i)} + \cdots + m_l\lambda_{j,l}^{(i)} = i$, and the length of $\lambda_j^{(i)}$ equals $(m_1 + m_2 + \cdots + m_l)$. Obviously, a_i is also the number of equivalent classes with the standard length i in the symmetric group S_n. Furthermore, we may obtain the following formula of $n!$

$$n! = 1 + \sum_{i=1}^{n(n-1)/2} \sum_{\lambda_j^{(i)}} \prod_{k=1}^{l} C^{m_k}_{n-\lambda_{j,k}^{(i)}-\sum_{r=1}^{k-1} m_r}. \tag{41}$$

If there are n points at the top of the planar diagram and n points at the bottom, we may use n strands to connect the points at the top and bottom respectively, and we may draw $n!$ kinds of different diagrams modulo the local relations introduced above, so there will be a_i different diagrams which have i crossings.

For the case of 2-morphisms $\mathrm{Hom}_{\mathcal{Q}}(Q_{++}, Q_{+-++})$, we have



(42)

so

$$\begin{aligned}&\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{++}, Q_{+-++}))\\ &= 1 + 2q + q^2 = [2]^2,\end{aligned} \tag{43}$$

this equals $\langle \hat{A}_{+-++} | \hat{A}_{++} \rangle$.

4. Categorification of the q-fermion algebra

Although we have only discussed the case of $0 < q \leqslant 1$ in the previous sections, our result can be easily extended to the more general case of $-1 \leqslant q \leqslant 1$, this is the so-called quon algebra.[16] When $q = -1$, we will obtain the normal fermion algebra. It is easy to check that our diagrammatic categorification method also works for the fermion algebra.

For example, for the case of Q_{++}, we have the following 2-morphisms,



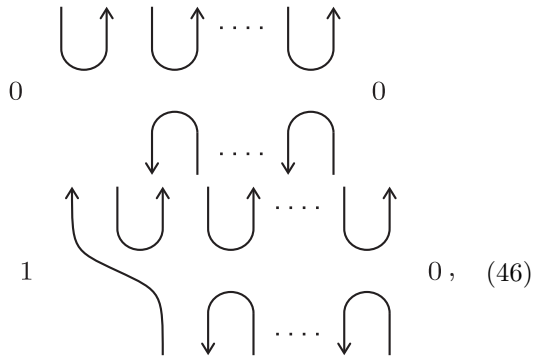
(44)

so we have

$$\mathrm{gdim}(\mathrm{Hom}_{\mathcal{Q}}(Q_{++}, Q_{++})) = 1 + q = 0, \tag{45}$$

this corresponds to $\langle \hat{A}_{++} | \hat{A}_{++} \rangle = \left| \hat{A}_{++} |0\rangle \right|^2 = 0$, or $\hat{A}_{++} = (\hat{a}^{\dagger})^2 = 0$.

It is easy to see that, with respect to the inner products of the corresponding states, there are only two kinds of nonzero 1-morphisms $Q_{-+-+\cdots-+}$, $Q_{+-+-+\cdots-+}$ corresponding to the states $|0\rangle$, $|1\rangle$ respectively, and two kinds of non-zero 2-morphisms

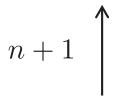


(46)

5. Action on modules for symmetric group algebras

Since we have the similar graphical category as that in Ref. [4], we may similarly construct the same modules for symmetric group algebras and some bimodule maps, and the graphical category can naturally act on the category of modules for symmetric group algebras. We will briefly review the main result of Ref. [4], and we refer readers to Ref. [4] for more details.

For the symmetric group S_n, we have the induction and restriction functors for inclusions $S_n \subset S_{n+1}$. We will use the following notations for bimodules for $\Bbbk[S_n]$, for instance, ${}_n(n+1)_n(n+2)$ stands for $\Bbbk[S_{n+1}] \otimes_{\Bbbk[S_n]} \Bbbk[S_{n+2}]$, viewed as a $(\Bbbk[S_n], \Bbbk[S_{n+2}])$-bimodule. An upward oriented line separating regions labelled n and $n+1$



(47)

denotes the identity endomorphism of the induction functor

$$\mathrm{Ind}_n^{n+1} \ : \ \Bbbk[S_n]\text{−mod} \longrightarrow \Bbbk[S_{n+1}]\text{−mod}. \tag{48}$$

This is the functor of tensoring with the bimodule $(n+1)_n$. A downward oriented line separating regions $n+1$ and n



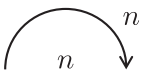
(49)

denotes the identity endomorphism of the restriction functor

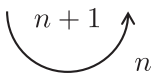
$$\mathrm{Res}_{n+1}^{n} \ : \ \Bbbk[S_{n+1}]\text{−mod} \longrightarrow \Bbbk[S_n]\text{−mod}. \tag{50}$$

The bimodule for this functor is ${}_n(n+1)$.

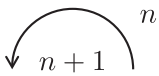
We may define the following bimodule maps whose diagrams correspond to the four U-turns:



$$(n+1)_n(n+1) \longrightarrow (n+1), \quad g \otimes h \longmapsto gh, \quad (g,h \in S_{n+1}), \tag{51}$$



$$(n) \longrightarrow {}_n(n+1)_n, \quad g \longmapsto g, \quad (g \in S_n), \tag{52}$$



$$p_n \ : \ {}_n(n+1)_n \longrightarrow (n), \quad p_n(g) = \begin{cases} g & \text{if } g \in S_n, \\ 0 & \text{otherwise,} \end{cases} \tag{53}$$



$$q_n \ : \ (n+1) \longrightarrow (n+1)_n(n+1), \tag{54}$$

where the bimodule map q_n is determined by the condition

$$q_n(1) = \sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i. \tag{55}$$

We also define the following bimodule maps whose diagrams correspond to the four crossings:



$$: (n+2)_n \to (n+2)_n, \quad g \mapsto g s_{n+1}, \quad (g \in S_{n+2}), \tag{56}$$



$$: {}_n(n+2) \to {}_n(n+2), \quad g \mapsto s_{n+1} g, \quad (g \in S_{n+2}), \tag{57}$$



$$: (n)_{n-1}(n) \to {}_n(n+1)_n, \quad g \otimes h \mapsto g s_n h, \quad (g,h \in S_n), \tag{58}$$



$$: {}_n(n+1)_n \to (n)_{n-1}(n), \quad g \mapsto 0, \ g s_n h \mapsto g \otimes h. \quad (g,h \in S_n). \tag{59}$$

Since the symmetric group S_n for $n < 0$ is meaningless, so naturally the bimodule map equals 0 if the corresponding diagram has a region labelled by a negative number.

The algebra $\Bbbk[S_n]$ is naturally graded, with each s_i in degree 1.

It is easy to verify that the local relations (32)–(34) hold when the diagrams are interpreted as maps of bimodules defined above. So the graphical category $\mathcal{Q}$ can act naturally on these categories of modules for symmetric group algebras.

Similar to Ref. [6], we may define a 2-category $\mathcal{B}$ as follows listed below.

(I) $\mathrm{Ob}\mathscr{B} = \mathbb{N}$.

(II) The 1-morphisms from n to m are functors from $\Bbbk[S_n]$-mod to $\Bbbk[S_m]$-mod that are direct summands of compositions of induction and restriction functors.

(III) The 2-morphisms are natural transformations of functors.

We may also define a 2-functor $\boldsymbol{F} : \mathscr{Q} \to \mathscr{B}$ in the obvious way.

I) For $n \in \mathbb{N} = \mathrm{Ob}\mathscr{Q}$, $\boldsymbol{F}(n) = n$.

II) On 1-morphisms, $\boldsymbol{F}$ maps $Q_\epsilon \in \mathrm{Hom}_{\mathscr{Q}}(n,m)$ to the tensor product of induction and restriction bimodules, where each $+$ corresponds to the induction bimodule and each $-$ corresponds to the restriction bimodule.

III) On 2-morphisms, $\boldsymbol{F}$ maps a planar diagram to the corresponding bimodule map (or, more precisely, to the natural transformation corresponding to this bimodule map) according to the definitions given in this section.

The functor $\boldsymbol{F}$ is a categorification of the q-Fock space representation of the q-boson algebra.

6. Summary

In this paper, based on the diagrammatic methods developed in Refs. [3], [4], and [6], we studied the categorification of q-boson algebra and also q-fermion algebra. We first constructed a graphical category corresponding to q-boson algebra. We find that this graphical category is similar to the categories constructed in Refs. [4] and [6] which were used to categorify normal Heisenberg algebra, except that we in-

troduce the degree of the planar diagrams. The q-Fock states correspond to some kind of 1-morphisms, and the graded dimension of the graded vector space of 2-morphisms is exactly the inner product of the corresponding q-Fock states. We also find that the graphical category we constructed can be used to categorify q-fermion algebra. The 2-category corresponding to fermion algebra is exactly the same as that corresponding to boson algebra, this is very interesting.

The main difference between the categorification of q-boson algebra and that of normal boson algebra is that we introduced the degree of the planar diagrams, and we have seen that the degree of the planar diagrams can effectively help us to study the structure of the planar diagrams and also the classification of the planar diagrams. Since the planar diagrams can be regarded as some physical processes which relate to the creation and annihilation of the particles, so we hope this method will help us in the study of some physical models.

Since we have a similar graphical category as that in Ref. [4], we may construct the same modules for symmetric group algebras and bimodule maps, and the graphical category can naturally act on the category of modules for symmetric group algebras. Since algebra $\Bbbk[S_n]$ is naturally graded, it is more natural to the modules for symmetric group algebras in the categorification of q-boson algebra than that of Heisenberg algebra. Of course, normal Heisenberg algebra is just a special kind of q-boson algebra. Since the boson and fermion oscillators are the simplest and most fundamental models in quantum physics, so we hope that our results will help us to study the categorification of other physical theories.

Acknowledgements

The authors grately acknowledge the support of the Morningside Center of Mathematics and Prof. Wang Shi-Kun.

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