A multidimensional subdiffusion model: An arbitrage-free market*

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To capture the subdiffusive characteristics of financial markets, the subordinated process, directed by the inverse α-stale subordinator $S_\alpha(t)$ for $0 < \alpha < 1$, has been employed as the model of asset prices. In this article, we introduce a multidimensional subdiffusion model that has a bond and K correlated stocks. The stock price process is a multidimensional subdiffusion process directed by the inverse α-stable subordinator. This model describes the period of stagnation for each stock and the behavior of the dependency between multiple stocks. Moreover, we derive the multidimensional fractional backward Kolmogorov equation for the subordinated process using the Laplace transform technique. Finally, using a martingale approach, we prove that the multidimensional subdiffusion model is arbitrage-free, and also gives an arbitrage-free pricing rule for contingent claims associated with the martingale measure.

Keywords: subordination, arbitrage-free, contingent claim valuation, fractional backward Kolmogorov equation

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1. Introduction

In finance the most popular model of the market for option pricing is the Black–Scholes (BS) model.[1,2] In this model the market consists of a riskless security (i.e., a bond) and a risky security (i.e., a stock). The bond price verifies the equation $\mathrm{d}B_t = rB_t\mathrm{d}t$, with $r \in R$ being the riskless interest rate. The price of the risky security is a geometric Brownian motion (GBM), satisfying the equation $\mathrm{d}S_t = \mu S_t\mathrm{d}t + \lambda S_t\mathrm{d}B(t)$, where $\mu \in R$ is the average rate of return and $\lambda \in R$ is the volatility.[3,4]

The idealized market described above is still a theoretical adoption and not necessarily consistent with many empirical properties, such as: non-independence, heavy-tailed and skewed marginal distributions, lack of scale invariance, and especially periods of stagnation.[1,2,5–7]

It is worth noticing that analogous behaviors are observed in complex physical systems.[8] As shown in Refs. [7], [9], and [10], the periods of stagnation, namely the characteristic periods of time when the economic process stays motionless, correspond to the so-called trapping events: the random periods of time when the subdiffusive test particle gets immobilized in a trap.

In statistical physics, subdiffusion is characterized through the power law form $\langle x^2\rangle \sim t^\alpha$ of the mean square displacement for $0 < \alpha < 1$.[11–14] A usual description of subdiffusion is in terms of the fractional Kolmogorov equation (FKE) derived from the power law waiting time continuous-time random walk (CTRW).[15,16] The solution of the FKE to describe the subdiffusive behavior is equal to the probability density function (PDF) of the subordinated process $X(S_\alpha(t))$, where the parent process $X(\tau)$ is the normal diffusion represented by an ordinary Itô stochastic differential process driven by the standard Brownian motion $B(\tau)$, and the directing process $S_\alpha(t)$, assumed to be independent of $B(\tau)$, is a hitting time process called the inverse α-stale subordinator defined by

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{1}$$

where $U_\alpha(\tau)$ denotes a strictly increasing α-stable Lévy motion with tempered α-stable waiting times between successive jumps.[13,15,17,18] The PDF of the process $S_\alpha(t)$ takes the form $g(\tau,t) = t^{-\alpha}F_\alpha(\tau t^\alpha)$, here the function $F_\alpha(x)$ can be represented as [3]

$$F_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma(1-\alpha(1+k))}, \tag{2}$$

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and in the particular case for $\alpha = 1$, $F_\alpha(x)$ corresponds to the Dirac δ-function, i.e., $F_1(x) = \delta(x-1)$.[18–20]

In recent years, many scholars have started applying the subordinated process to generalize the classical BS model.[5,7,10,20–22] The BS model under subordination was firstly studied by A. A. Stanislavsky in Ref. [20]. The evolution of the option price in this model is governed by the subordinated process $r_t = R(S_\alpha(t))$, where the parent process is Bachelier's ordinary Brownian motion obeying

$$\mathrm{d}R(\tau) = f(R(\tau))\mathrm{d}\tau + g(R(\tau))\mathrm{d}B(\tau), \tag{3}$$

where $B(\tau)$ represents a standard Brownian motion. An improved BS model in the subdiffusive regime was investigated by M. Magdziarz in 2007,[7] using the subordinated process $Z_\alpha(t) = Z(S_\alpha(t))$ as the model of stock price movement, where the parent process is a geometric Brownian motion (GBM) instead of Bachelier's ordinary Brownian motion. He argued that this improved BS model can capture the empirical property of constant price periods mentioned above. There are some other subdiffusion models proposed for various types of markets,[5,10,21,22] for instance, a composite-diffusive geometric Brownian motion is discussed as the model of stock price (see Ref. [5]).

Most of the subdiffusion models on this topic are based on the single-factor model involving a single stock driven by a Brownian motion. In practice, more complex equity products, such as quantos, depend on the behavior of several separate securities, and the prices of these securities will not evolve independently.[4,23,24] Thus, in the present paper we shall propose a multidimensional subdiffusion (MS) model, where the parent process is a multidimensional diffusion that has a bond and K-correlated stocks driven by K independent Brownian motions, and the directing process is the inverse α-stable subordinator $S_\alpha(t)$. The MS model captures the characteristic period of stagnation for each stock, and represents the interaction and dependency between multiple stocks. Moreover, we will apply the martingale approach to investigate the structural features of this new model, especially whether the model is arbitrage-free.

The outline of the paper is as follows. In the next section we introduce the multidimensional subdiffusion model, and find the corresponding multidimensional time fractional backward Kolmogorov equation (BKE). In Section 3, we prove that the multidimensional subdiffusion model is arbitrage-free by constructing the martingale measure, and also give the corresponding arbitrage-free pricing rule for contingent claims. We sum up and discuss the problems in Section 4.

2. The multidimensional subdiffusion model

We start by recalling the multidimensional diffusion model with a bond and K correlated stocks introduced in Refs. [4], [25], and [26]. Let $(\Omega, \mathscr{F}, P)$ be a fixed probability space, and let $T < \infty$ be a time horizon. The bond price process is $Y^0(\tau) = 1$, meaning that the riskless interest rate is zero, and each individual stock price process $Y^k(\tau)$ with $Y^k(0) = 1$ for $k = 1, \ldots, K$ is a geometric Brownian motion (GBM), defined as

$$\mathrm{d}Y^k(\tau) = Y^k(\tau)\left[\mu^k \mathrm{d}\tau + \sum_{j=1}^{K} \lambda_{kj}\mathrm{d}B^j(\tau)\right], \tag{4}$$

or equivalently written in the form

$$Y^k(\tau) = \exp\left[\sum_{j=1}^{K} \lambda_{kj}B^j(\tau) + \left(\mu^k - \frac{1}{2}a_{kk}\right)\tau\right], \tag{5}$$

where $B^1, \ldots, B^K$ are independent standard Brownian motions with $B_0^1 = \cdots = B_0^K = 0$, defined on the probability space $(\Omega, \mathscr{F}, P)$, the process $Y^k(\tau)$ represents the time τ value of the k-th security, the drift μ^k and the volatility λ_{kj} are constants, and $\Lambda = (\lambda_{ij})$ is a non-singular $K \times K$ matrix, the covariance matrix $A = (a_{ij})$ is symmetric and positive by setting $A = \Lambda\Lambda'$, meaning that

$$a_{ij} = \sum_{k=1}^{K} \lambda_{ik}\lambda_{jk}$$

for $i, j = 1, \ldots, K$.

As was shown in Refs. [7] and [20], the stochastic time arrow can be applied to the GBMs. Then we find the subdiffusive processes $Y_\alpha^k(t) = Y^k(S_\alpha(t))$ by randomizing the time clock τ of $Y^k(\tau)$ for $k = 1, \ldots, K$ by means of the inverse α-stable subordinator $S_\alpha(t)$ defined in Eq. (1), where the directing process $S_\alpha(t)$ is assumed to be independent of the parent processes $Y^k(\tau)$ for $k = 1, \ldots, K$. We call the subordination $Y_\alpha(t) = (Y_\alpha^1(t), \ldots, Y_\alpha^K(t))$ a multidimensional subdiffusion (MS) process.

We introduce now a multidimensional subdiffusion (MS) model which is a generalization of the multidimensional diffusion model defined on $(\Omega, \mathscr{F}, P)$.

This model has a bond and K correlated stocks. The bond price process is $Y^0(t) = 1$, and the stock price process is the MS process $Y_\alpha(t) = (Y_\alpha^1(t), \ldots, Y_\alpha^K(t))$, whose component process $Y_\alpha^k(t)$ represents the time t value of the k-th security, and satisfies the stochastic differential equation (SDE)

$$\begin{aligned} &\mathrm{d}Y_\alpha^k(t) \\ &= Y_\alpha^k(t)\left[\mu^k \mathrm{d}(S_\alpha(t)) + \sum_{j=1}^K \lambda_{kj}\,\mathrm{d}B^j(S_\alpha(t))\right], \end{aligned} \tag{6}$$

or the equivalent stochastic exponential equation (SEE)

$$\begin{aligned} Y_\alpha^k(t) = \exp\Bigg[&\sum_{j=1}^K \lambda_{kj} B^j(S_\alpha(t)) \\ &+ \left(\mu^k - \frac{1}{2}a_{kk}\right) S_\alpha(t)\Bigg], \end{aligned} \tag{7}$$

here we assume $Y_\alpha^k(0) = 1$ for $k = 1, \ldots, K$. Upon setting $\beta = 1/Y^0$, the discounted price process $Z_\alpha(t) = (Z_\alpha^1(t), \ldots, Z_\alpha^K(t))$ is defined by setting $Z_\alpha^k = \beta Y_\alpha^k$ for $k = 1, \ldots, K$. Recall the assumption that $Y^0 = 1$, so $\beta = 1$. This means that there is no distinction between the stock price process $Y_\alpha(t)$ and the discounted price process $Z_\alpha(t) = \beta Y_\alpha(t)$, so we shall be free to use the symbols Y and Z throughout. When $K = 1$, the MS model is the BS model in the subdifussive regime proposed by M. Magdziarz.[7]

Next, we shall investigate some structural features of the MS model. As we know, the backward Kolmogorov equations (BKE) are the usual tool used in finance to analyze asset price dynamics. We shall first derive the BKE associated with our MS process $Y_\alpha(t)$.

Let $p_\alpha(y, t; y_0, t_0)$ denote the conditional probability density function (PDF) for the stock price process $Y_\alpha(t)$ taking the value y at the time t if at time $t_0 \le t$ it takes the value y_0, where $y = (y^1, \ldots, y^K)$ and $y_0 = (y_0^1, \ldots, y_0^K)$ are K-dimensional constant vectors. In order to derive the BKE in which the operators depends on the initial condition, hereafter we shall treat the PDF $p_\alpha(y, t; y_0, t_0)$ as the function of variables of y_0, t_0 and parameters of y, t. Moreover, $p(y, \tau; y_0, \tau_0)$ denotes the transition PDF of the multidimensional GBM $Y(\tau) = (Y^1(\tau), \ldots, Y^K(\tau))$ with variables y_0, τ_0 and parameters y, τ, where each $Y^k(\tau)$ for $k = 1, \ldots, K$ is defined in Eq. (4). The relationship between $p_\alpha(y, t; y_0, t_0)$ and $p(y, \tau; y_0, \tau_0)$ in Laplace space is established in Appendix A, namely that

$$\hat{p}_\alpha(y, t; y_0, s) = s^{\alpha-1}\int_0^\infty \hat{p}(y, \tau; y_0, s^\alpha) g(t, \tau)\,\mathrm{d}\tau. \tag{8}$$

Recall that the PDF of the multidimensional GBM $Y(\tau)$ obeys the multidimensional BKE

$$\begin{aligned} &\frac{\partial p(y, \tau; y_0, \tau_0)}{\partial \tau_0} \\ &= \left[-\sum_{k=1}^K \mu^k \frac{\partial}{\partial y_0^k} - \frac{1}{2}\sum_{i,j=1}^K a_{ij}\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right] \\ &\times p(y, \tau; y_0, \tau_0). \end{aligned} \tag{9}$$

A Laplace transform from τ_0 to s of Eq. (9) yields

$$\begin{aligned} &s\hat{p}(y, \tau; y_0, s) - p(y, \tau; y_0, 0) \\ &= \left[-\sum_{k=1}^K \mu^k \frac{\partial}{\partial y_0^k} - \frac{1}{2}\sum_{i,j=1}^K a_{ij}\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right] \\ &\times p(y, \tau; y_0, s). \end{aligned} \tag{10}$$

Changing variables $s \to s^\alpha$ of Eq. (10), one comes by

$$\begin{aligned} &s^\alpha \hat{p}(y, \tau; y_0, s^\alpha) - p(y, \tau; y_0, 0) \\ &= \left[-\sum_{k=1}^K \mu^k \frac{\partial}{\partial y_0^k} - \frac{1}{2}\sum_{i,j=1}^K a_{ij}\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right] \\ &\times p(y, \tau; y_0, s^\alpha). \end{aligned} \tag{11}$$

Now by integrating the above formula multiplied by $g(t, \tau)$ over τ, in combination with Eq. (8) we can obtain

$$\begin{aligned} &s\hat{p}_\alpha(y, t; y_0, s) - p_\alpha(y, t; y_0, 0) \\ &= s^{1-\alpha}\left[-\sum_{k=1}^K \mu^k \frac{\partial}{\partial y_0^k} - \frac{1}{2}\sum_{i,j=1}^K a_{ij}\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right] \\ &\times p_\alpha(y, t; y_0, s). \end{aligned} \tag{12}$$

Inverting to the time domain $s \to t_0$, we find the multidimensional fractional BKE

$$\begin{aligned} &\frac{\partial p_\alpha(y, t; y_0, t_0)}{\partial t_0} \\ &= \left[-\sum_{k=1}^K \mu^k \frac{\partial}{\partial y_0^k} - \frac{1}{2}\sum_{i,j=1}^K a_{ij}\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right]_0 \\ &\times D_{t_0}^{1-\alpha} p_\alpha(y, t; y_0, t_0). \end{aligned} \tag{13}$$

Here, the operator ${}_0D_{t_0}^{1-\alpha}$, $0 < \alpha < 1$, is the fractional derivative of the Riemann–Liouville type,[27] equal in Laplace $t_0 \to s$ space to $s^{1-\alpha}$.[19] This operator introduces memory effects in financial system.

Remark 1 When $\alpha = 1$, the multidimensional fractional BKE equation (13) reduces to the ordinary

multidimensional BKE, while for $K = 1$, we find the fractional BKE derived recently in Ref. [28].

Remark 2 In the general case that the coefficient functions μ and $\Lambda = (\lambda_{ij})$ of the subdiffusive process $Y_\alpha(t)$ are modified to be dependent on price y_0 as $\mu(y_0)$ and $\Lambda(y_0) = (\lambda_{ij}(y_0))$ respectively, the associated multidimensional fractional BKE can be found in the analogous form

$$\frac{\partial p_\alpha(y,t;y_0,t_0)}{\partial t_0} = \left[-\sum_{k=1}^{K}\mu^k(y_0)\frac{\partial}{\partial y_0} - \frac{1}{2}\sum_{i,j=1}^{K}a_{ij}(y_0)\frac{\partial^2}{\partial y_0^i \partial y_0^j}\right]_0 \times D_{t_0}^{1-\alpha}p_\alpha(y,t;y_0,t_0), \quad (14)$$

where $a_{ij}(y_0) = \sum_{k=1}^{K}\lambda_{ik}(y_0)\lambda_{jk}(y_0)$ for $i,j = 1,\ldots,K$.

3. The martingale measure

We introduce the following filtration $\mathscr{G}_t = \mathscr{H}_{S_\alpha(t)}$ on the probability space (Ω, F, P) where

$$\mathscr{H}_{S_\alpha(t)} = \bigcap_{u>S_\alpha(t)}\{\sigma[B^1(\tau),...,B^K(\tau) : 0 \le \tau \le u] \vee \sigma[S_\alpha(t) : t \ge 0]\}. \quad (15)$$

The filtration $\mathscr{G}_t$ is then interpreted as the information of the subordination $Z_\alpha(t)$ revealed to the observer at time t, which increases with time.

Recall that the matrix Λ is non-singular by assumption. Then there exists a unique K-vector γ satisfying

$$\Lambda\gamma = \mu. \quad (16)$$

Now define a stochastic exponential

$$M_t = \exp\left[-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t)) - \frac{1}{2}\sum_{k=1}^{K}(\gamma^k)^2 S_\alpha(t)\right]. \quad (17)$$

Here, the symbol $B^k(S_\alpha(t))$ represents the subdiffusion process obtained by randomizing the time τ of the process $B^k(\tau)$ by the inverse α-stale subordinator $S_\alpha(t)$ for $0 < \alpha < 1$, see Ref. [29]. The stochastic exponential M_t has the following important property.

Theorem 1 M_t is a P martingale with respect to the filtration $\mathscr{G}_t$, which means that $M_s = E^P(M_t|\mathscr{G}_s)$ for $0 \le s \le t$, where E^P represents expectation with respect to the probability measure P.

Proof Let $\mathscr{G}_t^k = H_{S_\alpha(t)}^k$ be a filtration where

$$H_{S_\alpha(t)}^k = \bigcap_{u>S_\alpha(t)}\{\sigma[B^k(\tau) : 0 \le \tau \le u] \vee \sigma[S_\alpha(t) : t \ge 0]\}. \quad (18)$$

Then for each k, the subdiffusion process $B^k(S_\alpha(t))$ is a P martingale with respect to this filtration $\mathscr{G}_t^k$ (see Ref. [29]). Moreover, the filtration $\mathscr{G}_t$ can be obtained by expanding $\mathscr{G}_t^k$ with $k-1$ σ-algebras $\sigma[B^i(\tau) : 0 \le \tau \le u]$ for all other i. By the assumption that $B^1(\tau),\ldots,B^K(\tau), S_\alpha(t)$ are independent of one another, we obtain that $B^k(S_\alpha(t))$ remains a P martingale under $\mathscr{G}_t$.[30] Consequently, using the martingale properties, we have $-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))$ is a $(\mathscr{G}_t, P)$ martingale. Now, from Proposition 3.4, Chapter 4 in Ref. [31], we obtain that the process

$$\exp\left\{-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t)) - \frac{(-1)^2}{2}\left[\sum_{k=1}^{K}(\gamma^k)^2 S_\alpha(t), \sum_{k=1}^{K}(\gamma^k)^2 S_\alpha(t)\right]\right\}$$

is a P local martingale under $\mathscr{G}_t$. Since

$$\exp\left\{-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t)) - \frac{1}{2}\left[\sum_{k=1}^{K}(\gamma^k)^2(S_\alpha(t)), \sum_{k=1}^{K}(\gamma^k)^2 S_\alpha(t)\right]\right\} = \exp\left[-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t)) - \frac{1}{2}\sum_{k=1}^{K}(\gamma^k)^2 S_\alpha(t)\right], \quad (19)$$

where we used the fact that $[B^k(S_\alpha(t)), B^k(S_\alpha(t))] = S_\alpha(t)$,[29] and the independence of $B^1(\tau),\ldots,B^K(\tau)$, we find that M_t is at least a P local martingale in $\mathscr{G}(t)$.

Let $M_t^* = \sup_{0\le s\le t} M_t$. Then $M_t^* \le \sup_{0\le s\le t}\exp[-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))]$. Note that the process $\exp[-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))]$ is a positive submartingale. Therefore, from Doob's maximal inequality,[30] we have

$$\exp\left[-\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))\right] \le 4E\left(\exp\left[-2\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))\right]\right). \quad (20)$$

Using the assumption that $B^1(\tau),\ldots,B^K(\tau), S_\alpha(t)$ are independent and conditioning on $\sigma(S_\alpha(t) : t \ge 0)$, we obtain

$$E^p\left(\exp\left[-2\sum_{k=1}^{K}\gamma^k B^k(S_\alpha(t))\right]\right)$$

$$= E^p(\exp[-2\gamma^1 B^1(S_\alpha(t))])\cdots E^p(\exp[-2\gamma^K B^K(S_\alpha(t))]) = \sum_{n=0}^{\infty}\frac{(2t^\alpha(\lambda^1)^2)^n}{\Gamma(n\alpha+1)}\cdots\sum_{n=0}^{\infty}\frac{(2t^\alpha(\lambda^K)^2)^n}{\Gamma(n\alpha+1)}. \quad (21)$$

In the last equation we used the fact[29] that

$$E^P(\exp[-2\gamma^k B^k(S_\alpha(t))]) = \sum_{n=0}^{\infty}\frac{(2t^\alpha(\lambda^k)^2)^n}{\Gamma(n\alpha+1)}. \quad (22)$$

Thus, $E^P(M_t^*) < \infty$ and then M_t is a P martingale with respect to the filtration $\mathscr{G}_t$.

As a vital point, let us investigate whether the MS model, defined by (Ω, F, P) and $Z_\alpha(t)$ equipped with the filtration $\mathscr{G}_t$, has arbitrage opportunities. Intuitively, an arbitrage opportunity is the possibility of making a financial operation without any investment, but leading to profit without any risk of a loss.[3] A security market containing arbitrage opportunities cannot be one in which an economic equilibrium exists.[25] In the language of martingale theory, the market model is arbitrage-free if and only if there exists a probability measure P^* equivalent to P such that the discounted assets are martingales with respect to P^* (see Ref. [23]). This P^* is called the martingale measure. Thus, in order to prove the absence of arbitrage in our model from the martingale theory, it may be necessary to find first a martingale measure P^* for the discounted stock price process $Z_\alpha(t)$.

Let us define a measure P^* on $(\Omega, \mathscr{G}_t)$ by

$$\mathrm{d}P^* = M_T\,\mathrm{d}P. \quad (23)$$

Note that M_T is a strictly positive martingale with $E^P(M_T) = 1$, so P^* is a probability measure equivalent to P. Moreover, the following result will show that we can take it as our martingale measure for $Z_\alpha(t)$.

Theorem 2 The processes $Z_\alpha^1, \ldots, Z_\alpha^K$ are $\mathscr{G}_t$ martingales under P^*.

Proof It follows from Theorem 1 that for $0 \le t \le T$

$$M_t = E^P(M_T|\mathscr{G}_t). \quad (24)$$

It is convenient to write $M_t = \prod_{k=1}^K M_t^k$ where $M_t^k = \exp[-\gamma^k B^k(S_\alpha(t)) - (\gamma^k)^2(S_\alpha(t))/2]$. Since[7]

$$\mathrm{d}M_t^k = -\gamma^k M_t^k\,\mathrm{d}B^k(S_\alpha(t)), \quad (25)$$

we have

$$\mathrm{d}M_t = -M_t\sum_{k=1}^K \gamma^k\,\mathrm{d}B^k(S_\alpha(t)) \quad (26)$$

by repeated application of the product rule.[4,24] Now let us define X_t^k by

$$X_t^k = B^k(S_\alpha(t)) + \gamma^k S_\alpha(t) \quad (27)$$

for $k = 1, \ldots, K$. Then applying the product rule, we obtain

$$\mathrm{d}X_t^k M_t = \left(-X_t^k M_t\sum_{k=1}^K \gamma^k + L_t\right)\mathrm{d}B^k(S_\alpha(t)). \quad (28)$$

Note that $B^k(S_\alpha(t))$ is a $\mathscr{G}_t$ martingale under P. Therefore, by Theorem 29 of Chapter 3 in Ref. [30], we find that $X_t^k M_t$ is a P local martingale in $\mathscr{G}_t$ too, from which we deduce that X_t^k is a $(\mathscr{G}_t, P^*)$ local martingale, which means that $E^{P^*}(X_t^k|\mathscr{G}_s) = X_s^k$ for $0 \le s \le t$, where E^{P^*} denotes expectation under P^*. Moreover, we have that the subdiffusive process $Z_\alpha^k(t)$ satisfies

$$\mathrm{d}Z_\alpha^k(t) = Z_\alpha^k(t)\sum_{j=1}^K \lambda_{kj}\,\mathrm{d}X_t^j. \quad (29)$$

By Theorem 29 of Chapter 3 in Ref. [30], the above formula yields $E^{P^*}(Z_\alpha^k(t)|\mathscr{G}(s)) = Z_\alpha^k(s)$. Finally, since

$$Z_\alpha^k(0) = 1, \quad (30)$$

the processes $Z_\alpha^k(t)$ are $\mathscr{G}(t)$ martingales under P^* for $k = 1, \ldots, K$.

The second proof of Theorem 2 Let

$$H_\alpha^k(t) = \sum_{j=1}^K \lambda_{kj}B^j(S_\alpha(t)) + \mu^k S_\alpha(t). \quad (31)$$

By the quadratic variation property, in combination with the fact that $[B^j(S_\alpha(t)), B^j(S_\alpha(t))] = S_\alpha(t)$,[29] we have

$$[H_\alpha^k(t), H_\alpha^k(t)] = \sum_{j=1}^K \lambda_{kj}^2 S_\alpha(t). \quad (32)$$

Then the process $Z_\alpha^k(t)$ can be written as

$$Z_\alpha^k(t) = \exp\left[H_\alpha^k(t) - \frac{1}{2}\sum_{j=1}^K \lambda_{kj}^2 S_\alpha(t)\right] = \exp\left\{H_\alpha^k(t) - \frac{1}{2}[H_\alpha^k(t), H_\alpha^k(t)]\right\}, \quad (33)$$

where in the first series we used the fact $a_{kk} = \sum_{j=1}^K \lambda_{kj}^2$. In the same way as in the first proof of Theorem 2, we obtain that the martingale $M_t = E^P(M_T|\mathscr{G}_t)$ satisfies Eq. (26). Recall that $B^k(S_\alpha(t))$ is a P martingale relative to $\mathscr{G}_t$ (see Theorem 1). It

follows that the process $\sum_{j=1}^{K}\lambda_{kj}B^j(S_\alpha(t))$ is also a $(\mathscr{G}_t,P)$ martingale. Now, using Eq. (26), the fact that

$$\sum_{j=1}^{K}\lambda_{kj}\gamma^j=\mu^k, \tag{34}$$

and the Girsanov–Meyer theorem,[30] we find that the process

$$\begin{aligned}&\sum_{j=1}^{K}\lambda_{kj}B^j(S_\alpha(t))\\&-\int_0^t\frac{1}{M_t}\mathrm{d}\left[M_t,\sum_{j=1}^{K}\lambda_{kj}B^j(S_\alpha(t))\right]\\&=\sum_{j=1}^{K}\lambda_{kj}B^j(S_\alpha(t))+\mu^kS_\alpha(t)=H_\alpha^k(t)\end{aligned} \tag{35}$$

is a $(\mathscr{G}_t,P)$ local martingale. Then by Proposition 3.4, Chapter 4 in Ref. [31], we obtain $E^{P^*}(Z_\alpha^k(t)|\mathscr{G}_s)=Z_\alpha^k(s)$. Finally, since $Z_\alpha^k(0)=1$, the processes $Z_\alpha^k(t)$ are $\mathscr{G}_t$ martingales under P^* for $k=1,\ldots,K$.

Theorem 2 means that the BS model, in which the asset price follows the MS process $Z_\alpha(t)$, is arbitrage-free.

Recall that in a market model, every martingale measure gives a possible arbitrage-free pricing rule for contingent claims.[23] As a final point, we now turn to the contingent claim valuation with respect to the martingale measure P^* in our model.

Let $C_T\in\mathscr{G}(t)$ be a contingent claim with maturity at time T, and let P^* be the fixed martingale measure. Recall that $Z_\alpha(t)$ is a $(\mathscr{G}_t,P^*)$ martingale. Then one can find a fair price of the derivative at time t as (For the proof see Appendix B)

$$V_\alpha(t)=E^{P^*}[C_T|\mathscr{G}_t]. \tag{36}$$

Remark 3 When $\alpha\to1$, equation (36) is consistent with the usual derivative pricing formula for the multidimensional diffusion model (see Appendix C).

4. Conclusion

As an approach to recover the exact empirical properties in a financial market, sudiffusion offers an advantage through a subordination of one random process by another as it does in physics. In this paper, a multidimensional subdiffusion (MS) model with a bond and K correlated stocks is proposed, where the stock price process is the subordination $Y_\alpha(t)=(Y_\alpha^1(t),\ldots,Y_\alpha^K(t))$, the component process $Y_\alpha^k(t)=Y^k(S_\alpha(t))$ representing the time t value of the k-th security, is a subdiffusive geometric Brownian motion obeying the SDE (6). This model describes the characteristic periods of stagnation and the interaction between K stocks. For the stock price process $Y_\alpha(t)$, the multidimensional time fractional backward Kolmogorov equation is derived using the Laplace transform technique in the form of Eq. (13). Finally, we prove that the MS model is free of arbitrage by constructing the martingale measure P^* satisfying $\mathrm{d}P^*=M_T\,\mathrm{d}P$, where

$$M_T=\exp\left[-\sum_{k=1}^{K}\gamma^kB^k(S_\alpha(T))-\frac{1}{2}\sum_{k=1}^{K}(\gamma^k)^2S_\alpha(T)\right],$$

and obtain the fair option pricing formula $V_\alpha(t)=E^{P^*}[C_T|\mathscr{G}_t]$ associated with the martingale measure P^*. There are problems such as that one does not know if the fair price obtained above is unique because the completeness of the MS model is still unknown.

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Appendix A: Proof of Eq. (8)

Using the total probability formula and the assumption that $S_\alpha(t)$ is independent of $Z(\tau)$, we obtain that

$$\begin{aligned}p_\alpha(y,t;y_0,t_0)=&\int_0^\infty\int_0^\infty p(y,\tau;y_0,\tau_0)g(t,\tau)\\&\times\ g(t_0,\tau_0)\mathrm{d}\tau\,\mathrm{d}\tau_0.\end{aligned} \tag{A1}$$

Equivalently, in the Laplace space with respect to time, $t_0\to s$, the above formula yields

$$\begin{aligned}\hat{p}_\alpha(y,t;y_0,s)=&\int_0^\infty\int_0^\infty p(y,\tau;y_0,\tau_0)\\&\times\ g(t,\tau)\hat{g}(s,\tau_0)\mathrm{d}\tau\,\mathrm{d}\tau_0,\end{aligned} \tag{A2}$$

where $\hat{p}_\alpha(y,t;y_0,s)$ is the Laplace transform $t_0\to s$ of the PDF $p_\alpha(y,t;y_0,t_0)$ and $\hat{g}(s,\tau_0)$ is the Laplace transform $t_0\to s$ of the PDF $g(t_0,\tau_0)$. Using the fact[17] that $\hat{g}(s,\tau)=s^{\alpha-1}\,\mathrm{e}^{-\tau s^\alpha}$ in combination with Eq. (A2), we find

$$\hat{p}_\alpha(y,t;y_0,s)=\int_0^\infty\int_0^\infty p(y,\tau;y_0,\tau_0)g(t,\tau)s^{\alpha-1}$$

$$\times \ \mathrm{e}^{-\tau_0 s^\alpha}\,\mathrm{d}\tau\,\mathrm{d}\tau_0$$
$$= s^{\alpha-1}\int_0^\infty \hat{p}(y,\tau;y_0,s^\alpha)g(t,\tau)\,\mathrm{d}\tau. \tag{A3}$$

Here, $\hat{p}(x,\tau;x_0,s^\alpha)$ is obtained by Laplace transforming $\tau_0 \to s$ of $p(x,\tau;x_0,\tau_0)$ and changing the variable $s \to s^\alpha$.

Appendix B: Proof of Eq. (36)

From the martingale representation theorem, there exists a $\mathscr{G}_t$-previsible process $\phi_s = (\phi_t^1,\ldots,\phi_t^K)$ such that

$$E_t = E_0 + \sum_{k=1}^K \int_0^t \phi_s\,\mathrm{d}Z_\alpha(s), \tag{B1}$$

where $E_t = E^{P^*}[C_T|\mathscr{G}_t]$ is a $(\mathscr{G}_t, P^*)$ martingale. Now take ϕ_t to be our stock portfolio holding at time t and set the bond portfolio holding ψ_t to be $\psi_t = V_t - \phi_t Z_\alpha(t)$. Then the value of the portfolio is $V_t = \psi_t Y^0 + \phi_t Y_\alpha(t) = E_t$. It also follows that the portfolio is self-financing in that

$$V_t = \psi_t\,\mathrm{d}Y^0 + \phi_t\,\mathrm{d}Y_\alpha(t). \tag{B2}$$

Then one can find a fair option price formula at time t as

$$V_\alpha(t) = E^{P^*}[C_T|\mathscr{G}_t]. \tag{B3}$$

Appendix C

It follows from Eq. (36) that

$$V_\alpha(t) = E^P\left\{\exp\left[-\sum_{k=1}^K \gamma^k B^k(S_\alpha(T)) - \frac{1}{2}\sum_{k=1}^K (\gamma^k)^2 S_\alpha(T)\right] C_T|\mathscr{G}_t\right\}. \tag{C1}$$

Now, conditioning on $S_\alpha(T)$, we obtain

$$V_\alpha(t) = \int V(\tau)g(t,\tau)\,\mathrm{d}\tau, \tag{C2}$$

where $V(\tau) = E^P(C_\mathscr{T}|\mathscr{F}_\tau)$ is the derivative pricing formula for the multidimensional diffusion model,[24] where $\mathscr{F}_\tau = \sigma\{B^1(\varsigma),...,B^1(\varsigma) : 0 \le \varsigma \le \tau\}$ is the natural filtration, and $g(t,\tau)$ represents the PDF of the process $S_\alpha(t)$. When $\alpha \to 1$, we have $g(t,\tau) = t^{-1}\delta(\tau/t-1)$, and hence $V_\alpha(t) = V(\tau)$.

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