Modeling effective viscosity reduction behaviour of solid suspensions*

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Under a simple shearing flow, the effective viscosity of solid suspensions can be reduced by controlling the inclusion particle size or the number of inclusion particles in a unit volume. Based on the Stokes equation, the transformation field method is used to model the reduction behaviour of effective viscosity of solid suspensions theoretically by enlarging the particle size at a given high concentration of particles. With a lot of samples of random cubic particles in a unit cell, our statistical results show that at the same higher concentration, the effective viscosity of solid suspensions can be reduced by increasing the particle size or reducing the number of inclusion particles in a unit volume. This work discloses the viscosity reduction mechanism of increasing particle size, which is observed experimentally.

Keywords: effective viscosity, solid suspensions, transformation field method

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1. Introduction

Effective properties of fluid composites have been extensively investigated because of the technological application as well as the academic interest.[1−6] In particular, the effective viscosity of solid suspensions has important applications to engineering, such as solid particles transportation and rheological behavior of semisolid slurries.[7−10] To discuss the effective viscosity of solid–liquid composites, it is shown that the effective viscosity of the composite is related not only to viscosity of inclusions, volume fraction, and solid particle shapes, but also to particle size and spatial contribution of inclusion particles. For example, a long particle-chain may enhance the effective viscosity of solid particle suspensions, and the particle aggregation leads to the viscosity reduction of solid suspensions. Generally, viscosity reduction can be performed by increasing suspended particle size, aggregating particles into a streamline cluster and increasing the suspension polydispersity.[11,12] In natural crude oil, the formation of larger asphalt (or paraffin) agglomerates always occurs under an external electric or magnetic field due to dynamic interactions between solid particles, and the agglomerates having complex shapes may also be broken up and produce a lot of smaller asphalt (or paraffin) particles under particle collision forces.[7,8] The changes of particle size and inclusion shapes can directly affect effective viscosity of solid suspensions, and the effect of particle size clearly appears at higher volume fraction. However, there are few theoretical investigations into the viscosity reduction (or enhancing) behaviour with the first principles method. The difficulty results from the lack of a valid theoretical method of estimating the effective viscosity of solid suspensions at a high concentration of inclusions having complex shapes on the basis of fluid governing equations even if there are a lot of formulas in the dilute limit. Therefore, theoretically, we in this paper use the transformation field method to model the effective viscosity reduction behaviour due to the change of particle size having complex shapes at higher concentration.

To exactly deal with the random solid suspensions of high inclusion concentrations, the developed transformation field method has an advantage in calculating the effective properties of fluid composites having the complex shapes of inclusions. This method was first proposed by Eshelby[13] to solve the elastic problem of solid composites and further used by

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Nemat–Nasser and Taya[14] to investigate the effective modulus of elastic composites. Subsequently, Gu and Tao[15,16] and Gu *et al.*[17,18] developed this method for estimating the effective properties of electric conductive, piezoelectric, and photonic periodic composites having complex shapes and high concentration. Recently, Gu and Tao[19] and Gu and Yu[20] used the method to calculate the effective viscosity of lattice fluid composites on the basis of the Stokes equation. Here, we use the method to calculate the effective viscosity of random solid suspensions having complex shapes and higher concentration, and discuss the viscosity behaviour by changing the particle sizes. If solid cubic particles are randomly embedded in an isotropic infinite fluid matrix, the effective viscosity of solid suspensions is estimated by regarding a random sample of suspensions as a periodic cell sample at higher concentration. Furthermore, the effective viscosity reduction behaviour is revealed by comparing the effective viscosity of random solid suspensions having small particles with that of solid suspensions of large particles.

The rest of this paper is arranged as follows. In Section 2, based on the Stokes equation, the transformation field method is derived for estimating the effective viscosity of random solid suspensions using a cubic periodic model. The transformation velocity field is introduced, and integration equations of velocity strain tensors are given. Furthermore, a set of closed linear algebraic equations is built to determine the transformation field, which is expressed in terms of power series with three-dimensional spatial variables. In Section 3, the effective viscosity formulas are derived using the transformation field and the Stokes equation. Meanwhile, as an example, the effective viscosity reduction behaviour is modeled by calculating a lot of samples of random solid suspensions, where the shape of solid inclusion is assumed to be of a small cube. A brief conclusion is given in Section 4.

The notes in explanation of symbols used in the transformation field method are listed, where the superscripts 0, h, d, and t refer to the external, host fluid region, solid particle region, and the unified quantities, respectively, and the subscripts α, β, γ, and η represent one component of coordinate (x, y, z). v^0_α the α-component of external flow velocity field; p^0 the external pressure force of fluid; $v^0_{\gamma,\eta}$ the partial derivative of external γ-component velocity with respect to η variable; $E^0_{\gamma\eta}$ the external velocity strain tensor; $\mu^d_{\alpha\beta\gamma\eta}$ the viscosity tensor of inclusion particle; $\sigma^d_{\alpha\beta}$ the viscous stress tensor of inclusion particle in α plane along β-direction; v^h_α the α-component of fluid velocity field; $v^h_{\gamma,\eta}$ the partial derivative of fluid γ-component velocity with respect to η variable; $v^h_{\alpha,\alpha}$ the Einstein sum of the partial derivative of host fluid velocity v^h_α with respect to α variable; $\mu^h_{\alpha\beta\gamma\eta}$ the fluid viscosity tensor; μ^h the isotropic fluid viscosity parameter; p^h the pressure force of host fluid; $E^h_{\gamma\eta}$ the velocity strain tensor of host fluid; $\Pi^h_{\alpha\beta}$ the sum of fluid pressure force tensor $-p^h\delta_{\alpha\beta}$ and viscosity stress tensor $\mu^h_{\alpha\beta\gamma\eta}E^h_{\gamma\eta}$; $\Pi^h_{\alpha\beta,\alpha}$ the Einstein sum of stress tensor $\Pi^h_{\alpha\beta}$ partial derivative with respect to α variable; $\sigma^h_{\alpha\beta}$ the fluid viscous stress tensor; v_α the α component of perturbation velocity field induced by solid particles; $E_{\gamma\eta}$ the perturbation strain tensor induced by solid particles; p the perturbation pressure force induced by solid particles; $E^*_{\gamma\eta}$ the transformation velocity strain tensor; $\sigma^t_{\alpha\beta}$ the unified expression of viscous stress tensor in both host fluid and solid regions; $\Pi^t_{\alpha\beta}$ the unified expression of stress tensors in both host and inclusion regions; v^t_α the unified expression of α-component of velocity in both host and inclusion regions; $\mu^e_{\alpha\beta\gamma\eta}$ the effective viscosity tensor of whole composites; $\sigma^e_{\alpha\beta}$ the effective viscous stress tensor of whole composites; α^e the effective isotropic viscosity parameter calculated by $-\langle E^*_{\alpha\alpha}(\boldsymbol{x})\rangle_i/E^0_{\alpha\alpha}$; β^e the effective isotropic viscosity parameter calculated by $-\langle E^*_{\alpha\beta}(\boldsymbol{x})\rangle_i/E^0_{\alpha\beta}$; $c^{ijk}_{\alpha\beta}$ the coefficient of power series $(x/l_x)^i(y/l_y)^j(z/l_z)^k$ of a transformation field $E^*_{\alpha\beta}$, where i, j, and k are the powers of x, y, and z variables, respectively.

2. Transformation field method

Under a simple shearing flow v^0_α, we consider random solid inclusions of viscosity tensor $\mu^d_{\alpha\beta\gamma\eta}$ surrounded by incompressible isotropic Newtonian fluid with a constant viscosity tensor $\mu^h_{\alpha\beta\gamma\eta} = \mu^h(\delta_{\alpha\gamma}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma} - \frac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\eta})$ and pressure force p^h, the fluid velocity field v^h_α and stress tensor $\Pi^h_{\alpha\beta}$ of host fluid region $\Omega_{\rm h}$ which satisfy the Stokes equation,[19,20]

$$v^h_{\alpha,\alpha} = 0, \tag{1}$$

$$\Pi^h_{\alpha\beta,\alpha} = 0, \quad (\beta = 1, 2, 3), \tag{2}$$

where $\Pi^h_{\alpha\beta} = -p^h\delta_{\alpha\beta} + \mu^h_{\alpha\beta\gamma\eta}E^h_{\gamma\eta}$. The velocity strain tensor $E^h_{\gamma\eta} = (v^h_{\gamma,\eta} + v^h_{\eta,\gamma})/2$, and $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, otherwise $\delta_{\alpha\beta} = 0$. The subscripts α, γ, β, and $\eta = 1, 2, 3$. In Cartesian coordinates, x, y, and z are represented by 1, 2, and 3, respectively. The fluid viscous stress tensor is $\sigma^h_{\alpha\beta} = \mu^h_{\alpha\beta\gamma\eta}E^h_{\gamma\eta}$ in host fluid $\Omega_{\rm h}$. In fact, the velocity field v^h_α, the strain tensor $E^h_{\gamma\eta}$,

and the pressure force p^h have included their perturbation quantities (i.e. the velocity perturbation field v_α, the perturbation strain tensor $E_{\gamma\eta}$ and the pressure perturbation p) caused by solid particles and the quantities (i.e., the strain tensor $E^0_{\gamma\eta}$ and the pressure force p^0) caused by the external shearing flow v^0_α without solid inclusions in the host fluid. Thus, under a simple shearing flow v^0_α, the velocity field v^h_α, the strain tensor $E^h_{\gamma\eta}$, and the pressure force p^h can be rewritten as follows:

$$\begin{aligned} &v^h_\alpha = v_\alpha + v^0_\alpha, \quad E^h_{\gamma\eta} = E_{\gamma\eta} + E^0_{\gamma\eta}, \\ &p^h = p + p^0 \ \text{ in } \Omega_{\rm h}, \end{aligned} \tag{3}$$

where $E^0_{\gamma\eta}$ and p^0 are the velocity strain tensor and the fluid pressure force caused by the external simple shearing flow v^0_α, respectively.

In inclusion regions $\Omega_{\rm i}$, its viscous stress tensor is

$$\sigma^d_{\alpha\beta} = \mu^d_{\alpha\beta\gamma\eta}(E^0_{\gamma\eta} + E_{\gamma\eta}) \ \text{ in } \Omega_{\rm i}. \tag{4}$$

To avoid matching boundary conditions in complex interfaces of different phases, we introduce the transformation velocity strain tensor $E^*_{\gamma\eta}$ (where $E^*_{11} + E^*_{22} + E^*_{33} = 0$ due to incompressible behavior) so that the strain tensor $E^*_{\gamma\eta}$ satisfies Eqs. (5) and (6)[17,18]

$$E^*_{\gamma\eta} = 0, \ \text{ in } \Omega_{\rm h}, \tag{5}$$

$$\begin{aligned} &\mu^h_{\alpha\beta}(E^0_{\gamma\eta} + E_{\gamma\eta} - E^*_{\gamma\eta}) \\ &= \mu^d_{\alpha\beta\gamma\eta}(E^0_{\gamma\eta} + E_{\gamma\eta}), \ \text{ in } \Omega_{\rm i}. \end{aligned} \tag{6}$$

In this case, the viscous stress tensor of the whole suspensions region $\Omega = \Omega_{\rm i} + \Omega_{\rm h}$ can be rewritten as the unified formula $\sigma^t_{\alpha\beta} = \mu^h_{\alpha\beta\gamma\eta}(E^0_{\gamma\eta} + E_{\gamma\eta} - E^*_{\gamma\eta})$, and the stress tensor of the whole region Ω is the unified $\Pi^t_{\alpha\beta} = -(p^0 + p)\delta_{\alpha\beta} + \sigma^t_{\alpha\beta}$, and the unified velocity is $v^t_\alpha = v^0_\alpha + v_\alpha$. Because solid suspensions are governed by Stokes equation, $v^t_{\alpha,\alpha} = 0$, $\Pi^t_{\alpha\beta,\alpha} = 0$, and the external simple shearing flow v^0_α, the strain tensor $E^0_{\gamma\eta}$ and the pressure force p^0 satisfy the Stokes equation, $v^0_{\alpha,\alpha} = 0$, and $\partial_\alpha(-p^0\delta_{\alpha\beta} + \mu^h_{\alpha\beta\gamma\eta}E^0_{\gamma\eta}) = 0$, we obtain a set of equations about perturbation quantities and the transformation strain tensor in the region of solid suspensions,[20]

$$v_{\alpha,\alpha} = 0, \tag{7}$$

$$\Pi_{\alpha\beta,\alpha} = 0, \quad (\beta = 1, 2, 3), \tag{8}$$

where $\Pi_{\alpha\beta} = -p\delta_{\alpha\beta} + \mu^h_{\alpha\beta\gamma\eta}(E_{\gamma\eta} - E^*_{\gamma\eta})$.

If the system of solid suspensions is a cubic periodic cell, where the unit cubic cell consists of random solid inclusions of arbitrary geometric shapes in a host fluid and it infinitely extends in three-dimensional space periodically, the quantities in Eqs. (7) and (8) will be periodic functions of cubic cells. Therefore, the perturbation velocity field, the transformation strain tensor, and the pressure force can be expressed by the Fourier transformation components as

$$v_\alpha(\boldsymbol{x}) = \sum_{|n|\neq 0}^{\infty} v_\alpha(\boldsymbol{\xi}^n)\exp(\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x}), \tag{9}$$

$$E^*_{\alpha\beta}(\boldsymbol{x}) = \sum_{|n|\neq 0}^{\infty} E^*_{\alpha\beta}(\boldsymbol{\xi}^n)\exp(\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x}), \tag{10}$$

$$p(\boldsymbol{x}) = \sum_{|n|\neq 0}^{\infty} p(\boldsymbol{\xi}^n)\exp(\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x}), \tag{11}$$

where $\boldsymbol{\xi}^n = (\xi^n_1, \xi^n_2, \xi^n_3) = 2\pi(n_x/l_x, n_y/l_y, n_z/l_z)$, $|n| = (n_x^2 + n_y^2 + n_z^2)^{1/2}$, and l_x, l_y, and l_z are the lengths of the unit cell along x, y, and z directions, respectively.

$$\begin{aligned} v_\alpha(\boldsymbol{\xi}^n) &= \frac{1}{V}\int_\Omega v_\alpha(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \\ E^*_{\alpha\beta}(\boldsymbol{\xi}^n) &= \frac{1}{V}\int_\Omega E^*_{\alpha\beta}(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \\ p(\boldsymbol{\xi}^n) &= \frac{1}{V}\int_\Omega p(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

With the formula $E_{\gamma\eta}(\boldsymbol{x}) = [v_{\gamma,\eta}(\boldsymbol{x}) + v_{\eta,\gamma}(\boldsymbol{x})]/2$, $E_{\alpha\beta}(\boldsymbol{x})$ and $E_{\alpha\beta}(\boldsymbol{\xi}^n)$ can be given as follows:

$$E_{\alpha\beta}(\boldsymbol{x}) = \sum_{|n|\neq 0}^{\infty} E_{\alpha\beta}(\boldsymbol{\xi}^n)\exp(\mathrm{i}\boldsymbol{\xi}^n\cdot\boldsymbol{x}), \tag{12}$$

$$E_{\alpha\beta}(\boldsymbol{\xi}^n) = \frac{\mathrm{i}}{2}[v_\alpha(\boldsymbol{\xi}^n)\xi^n_\beta + v_\beta(\boldsymbol{\xi}^n)\xi^n_\alpha]. \tag{13}$$

Substituting Eqs. (9)–(12) into Eqs. (7) and (8), we have

$$\mathrm{i}\xi^n_\alpha v_\alpha(\boldsymbol{\xi}^n) = 0, \tag{14}$$

$$\begin{aligned} -\mathrm{i}p(\boldsymbol{\xi}^n)\xi^n_\beta - \frac{1}{2}\xi^n_\alpha\mu^h_{\alpha\beta\gamma\eta}[v_\gamma(\boldsymbol{\xi}^n)\xi^n_\eta \\ + v_\eta(\boldsymbol{\xi}^n)\xi^n_\gamma + \mathrm{i}E^*_{\gamma\eta}(\boldsymbol{\xi}^n)] = 0. \end{aligned} \tag{15}$$

For isotropic Newtonian fluid, solving Eqs. (14) and (15), we have,

$$p(\boldsymbol{\xi}^n) = -2\mu^h\xi^n_\alpha\xi^n_\beta E^*_{\alpha\beta}(\boldsymbol{\xi}^n)/\xi^2, \tag{16}$$

$$\begin{aligned} v_\alpha(\boldsymbol{\xi}^n) = 2\mathrm{i}[&\xi^n_\alpha\xi^n_\gamma\xi^n_\beta E^*_{\gamma\beta}(\boldsymbol{\xi}^n)/\xi^2 \\ &- E^*_{\alpha\beta}(\boldsymbol{\xi}^n)\xi^n_\beta]/\xi^2, \end{aligned} \tag{17}$$

$$\begin{aligned} E_{\alpha\beta}(\boldsymbol{\xi}^n) = \xi^n_\gamma[&\xi^n_\beta E^*_{\alpha\gamma}(\boldsymbol{\xi}^n) + \xi^n_\alpha E^*_{\beta\gamma}(\boldsymbol{\xi}^n)]/\xi^2 \\ &- 2\xi^n_\alpha\xi^n_\beta\xi^n_\gamma\xi^n_\eta E^*_{\gamma\eta}(\boldsymbol{\xi}^n)/\xi^4, \end{aligned} \tag{18}$$

where $\xi^2 = (\xi^n_1)^2 + (\xi^n_2)^2 + (\xi^n_3)^2$. In the derivation procedure, it is noted that we have used the formula

$E^*_{11}(\boldsymbol{\xi}^n)+E^*_{22}(\boldsymbol{\xi}^n)+E^*_{33}(\boldsymbol{\xi}^n)=0$ because of the incompressible assumption of fluid. Furthermore, substitute Eq. (18) into Eq. (6) and let viscosity $\mu^d_{\alpha\beta\gamma\eta}\to\infty$ for solid particles, then we will obtain a set of integral equations of the transformation velocity strain tensor in inclusion regions,

$$\begin{aligned}&-E^0_{\alpha\beta}\\&=V^{-1}\sum_{|n|\neq 0}^{\infty}\int_{\Omega_{\rm i}}\{\xi^n_\gamma[E^*_{\gamma\alpha}(\boldsymbol{x}')\xi^n_\beta+E^*_{\beta\gamma}(\boldsymbol{x}')\xi^n_\alpha]/\xi^2\\&\quad-2\xi^n_\alpha\xi^n_\beta\xi^n_\gamma\xi^n_\eta E^*_{\gamma\eta}(\boldsymbol{x}')/\xi^4\}\exp[{\rm i}\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{x}')]\,{\rm d}\boldsymbol{x}',\end{aligned}\tag{19}$$

where $\alpha,\ \beta=1,2,3$ and V is the whole volume of unit cubic cell. Clearly, for a uniform external velocity strain $E^0_{\alpha\beta}$, there exist six equations having five unknown transformation strain components $E^*_{\alpha\beta}(\boldsymbol{x})$ in Eq. (19) due to symmetry property of $E^*_{\alpha\beta}(\boldsymbol{x})=E^*_{\beta\alpha}(\boldsymbol{x})$. From the above derivation, it is found that equation (19) is suitable for arbitrary shape and spatial distribution of inclusions.

To solve Eq. (19), the transformation strain tensor can be expressed by power series of $(x/l_x)^i(y/l_y)^j(z/l_z)^k$, where i, j, and k denote the powers of series

$$E^*_{\alpha\beta}(\boldsymbol{x})=\sum_{i,j,k=0}^{\infty}c^{ijk}_{\alpha\beta}(x/l_x)^i(y/l_y)^j(z/l_z)^k,\tag{20}$$

where the unknown coefficients $c^{ijk}_{\alpha\beta}$ will be determined. If we multiply both sides of Eq. (19) by power function $(x/l_x)^l(y/l_y)^q(z/l_z)^r$, and integrate the equation over the inclusion regions, a set of linear algebraically closed equations is derived for determining unknown coefficients,

$$\begin{aligned}&-E^0_{\alpha\beta}A^{lqr}\\&=V^{-1}\sum_{i,j,k=0}^{\infty}\sum_{|n|\neq 0}^{\infty}|\xi|^{-2}\,G^{ijk}(-\boldsymbol{\xi}^n)G^{lqr}(\boldsymbol{\xi}^n)\\&\quad\times[\xi^n_\gamma(\xi^n_\beta c^{ijk}_{\alpha\gamma}+\xi^n_\alpha c^{ijk}_{\beta\gamma})-2\xi^n_\alpha\xi^n_\beta\xi^n_\gamma\xi^n_\eta c^{ijk}_{\gamma\eta}/\xi^2],\end{aligned}\tag{21}$$

where $\alpha,\ \beta=1,2,3$ (or $x,\ y,\ z$), $|\xi|^{-2}=[(\xi^n_1)^2+(\xi^n_2)^2+(\xi^n_3)^2]^{-1}$,

$$\begin{aligned}A^{lqr}&=\int_{\Omega_{\rm i}}(x/l_x)^l(y/l_y)^q(z/l_z)^r\,{\rm d}V,\\G^{ijk}(\boldsymbol{\xi}^n)&=\int_{\Omega_{\rm i}}(x/l_x)^i(y/l_y)^j(z/l_z)^k\\&\quad\times\exp({\rm i}\boldsymbol{\xi}^n\cdot\boldsymbol{x})\,{\rm d}V.\end{aligned}$$

Equation (21) can be approximately solved within an accuracy if we choose suitable values of M and N to close the above equations, where superscripts $i,j,k=0,1,2,3,\ldots,N$, $l,q,r=0,1,2,3,\ldots,M$ and N called the approximation order of transformation strain tensor. Therefore, the transformation velocity strain tensor is obtained using Eq. (21). Here, note that the symmetry property of $c^{ijk}_{\alpha\beta}=c^{ijk}_{\beta\alpha}$ should be considered when solving Eq. (21). In Refs. [19] and [20] the symmetrical properties of the effective viscosity tensor derived with the transformation field method have been demonstrated.

3. Effective viscosity reduction behaviour

As an effective viscous composite, the effective viscosity tensor $\mu^e_{\alpha\beta\gamma\eta}$ can be calculated by an ensemble average and definition of effective viscous stress tensor $\sigma^e_{\alpha\beta}$ of whole composites,

$$\sigma^e_{\alpha\beta}=\mu^e_{\alpha\beta\gamma\eta}\langle E^0_{\gamma\eta}+E_{\gamma\eta}(\boldsymbol{x})\rangle,\tag{22}$$

where $\langle A\rangle=V^{-1}\int_\Omega A\,{\rm d}V$. Furthermore, considering the viscous stress tensor of the whole suspension region Ω, $\sigma^t_{\alpha\beta}=\mu^h_{\alpha\beta\gamma\eta}[E^0_{\gamma\eta}+E_{\gamma\eta}(\boldsymbol{x})-E^*_{\gamma\eta}(\boldsymbol{x})]$, we have $\sigma^e_{\alpha\beta}=\langle\sigma^t_{\alpha\beta}\rangle$. Thus, we obtain the following effective viscosity tensor formula:[19,20]

$$\begin{aligned}&\mu^e_{\alpha\beta\gamma\eta}\langle E^0_{\gamma\eta}+E_{\gamma\eta}(\boldsymbol{x})\rangle\\&=\mu^h_{\alpha\beta\gamma\eta}\langle E^0_{\gamma\eta}+E_{\gamma\eta}(\boldsymbol{x})-E^*_{\gamma\eta}(\boldsymbol{x})\rangle.\end{aligned}\tag{23}$$

Because of the periodic property of the perturbation velocity field v_α in the whole suspension region, we obtain $\langle E_{\alpha\beta}(\boldsymbol{x})\rangle=0$. Then, for a uniform external strain field $E^0_{\gamma\eta}$, from Eq. (23), the effective viscosity tensor can be estimated by the following formula due to $E^*_{\gamma\eta}(\boldsymbol{x})=0$ in host fluid region $\Omega_{\rm h}$:

$$\mu^e_{\alpha\beta\gamma\eta}E^0_{\gamma\eta}=\mu^h_{\alpha\beta\gamma\eta}[E^0_{\gamma\eta}-\langle E^*_{\gamma\eta}(\boldsymbol{x})\rangle_i],\tag{24}$$

where $\langle E^*_{\gamma\eta}(\boldsymbol{x})\rangle_i=V^{-1}\int_{\Omega_{\rm i}}E^*_{\gamma\eta}(\boldsymbol{x})\,{\rm d}V$. It is clear that based on the transformation strain tensor obtained in Section 2, the effective viscosity can be calculated using Eq. (24). Moreover, for a cubic periodic isotropic suspension system, we can further simplify Eq. (24) into the following equation:[3]

$$\begin{aligned}\mu^e_{\alpha\beta\gamma\eta}&=\mu^h(1+\beta^e)(\delta_{\alpha\gamma}\delta_{\beta\eta}+\delta_{\alpha\eta}\delta_{\beta\gamma}-\tfrac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\eta})/2\\&\quad+\mu^h(\alpha^e-\beta^e)(\delta_{\alpha\beta\gamma\eta}-\delta_{\alpha\beta}\delta_{\gamma\eta}),\end{aligned}\tag{25}$$

where $\alpha^e=-\langle E^*_{\alpha\alpha}(\boldsymbol{x})\rangle_i/E^0_{\alpha\alpha}$, $\beta^e=-\langle E^*_{\alpha\beta}(\boldsymbol{x})\rangle_i/E^0_{\alpha\beta}$, and $\delta_{\alpha\beta\gamma\eta}=1$ if $\alpha=\beta=\gamma=\eta$, otherwise $\delta_{\alpha\beta\gamma\eta}=0$. Here, it is noted that in Refs. [17] and [18] equation (24) or equation (25) has been tested by comparing the results of solid or colloidal suspensions having a

simple central cubic periodic lattice model with those obtained from Einstein, Talylor, Nunan, and Keller formulas, showing that they are in good agreement in a larger range of concentrations of inclusions.[1,3,21] In addition, the zero- or lower-order approximation of this method is enough to estimate the effective viscosity of fluid composites in the dilute limit, and higher-order approximations are suitable to predict the effective properties at high concentrations because of the transformation velocity strain tensor including interactions of spatial random solid particles. Thus, we shall use high-order approximations of Eq. (24) or Eq. (25) to model the effective viscosity reduction behaviours of higher concentrations by changing the random particle size.

Motivated by Tao and Xu's[8] reducing effective viscosity experiments of fluid composites through enlarging particle size, we shall discuss the effect of particle size on the effective viscosity of solid suspensions. As a theoretical model, we choose the centre of the unit cubic periodic cell as the origin of Cartesian coordinates (x, y, z), and random solid cubic particles are imbedded in the suspension system (see Fig. 1) having unit viscosity coefficient of isotropic host fluid (i.e., $\mu^h = 1$). In order to investigate the effect of particle size on the effective viscosity at higher concentrations, we first investigate an example of the suspension system having a big cubic solid particle with edge length 0.7047 unit, i.e., the volume fraction is 0.35. For the third-order approximation, $N = 3$, under the external velocity strain $E^0_{21} = E^0_{12} \neq 0$ (the values of remaining $E^0_{\alpha\beta}$ are zero), the Einstein effective viscosity coefficient $\beta^e = 1.0945$. Next, we calculate Einstein effective viscosity coefficient β^e of ten small equal-volume cubic inclusions randomly embedded in the host fluid, where each small cubic inclusion has a volume fraction of 0.035 and an edge length of 0.3271 within the unit periodic cell. To calculate the statistical distribution of Einstein effective viscosity constant β^e of random samples, we randomly select ten small cubes (the volume fraction of each small cube is 0.035) fifty times. In the random sample process, there include random agglomerations consisting of a few small cubes of ten random small cubic inclusions. The random agglomeration will form one large particle of complex shape. Thus, our random results can be affected by the complex shapes of agglomerations. Figure 2 shows the statistical frequency histogram of the Einstein effective viscosity constant β^e in the random samples. The mean value of Einstein effective viscosity constant β^e is about 2.7980 and its mean square deviation is about 0.652. Comparing the effective viscosity coefficient β^e of ten small particles with that of one large particle, it is clear that the effective viscosity of solid suspensions having large size inclusions is less than that of small size inclusions at the same high concentrations, i.e., for one larger particle with cubic edge length 0.7047 unit, its effective viscosity coefficient is only about 1.0945. Furthermore, we estimate the effects of the number of cubic particles at the same volume fraction of total particles. At the total volume fraction 0.35 of particles, the number of cubic particles is selected from one to twenty-five. With random samples selected fifty times, the effective viscosity coefficient β^e is given in Fig. 3. It shows that with the number of particles increasing, the averaged effective viscosity coefficient β^e generally increases within a wide range of the number of inclusions. Here, note that although the effective viscosity of the large number of particles is less than that of the small one in a range of very small particle size (see Fig. 3, the effective viscosity from 15 to 25 particles), the effective viscosity remains a large value. Checking our random samples we find that there are some small particles connecting to each other and forming a large agglomeration of complex shape. This kind of large agglomeration gives rise to a reduction effect on effective viscosity of the sample system. Therefore, for smaller particles, a low effective viscosity appears since larger agglomerations are formed. If we exclude the large agglomeration and keep small particles off each other in each sample, it is possible to obtain a large steady effective viscosity of small particle composite system for the sufficient number of random samples at a given high concentration.

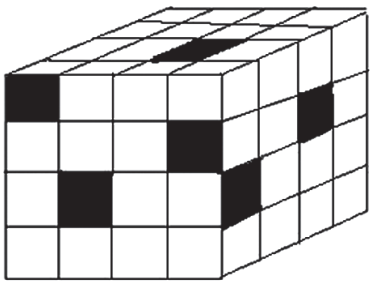


Fig. 1. One unit periodic cell of solid suspensions including the black small cubic random inclusions.

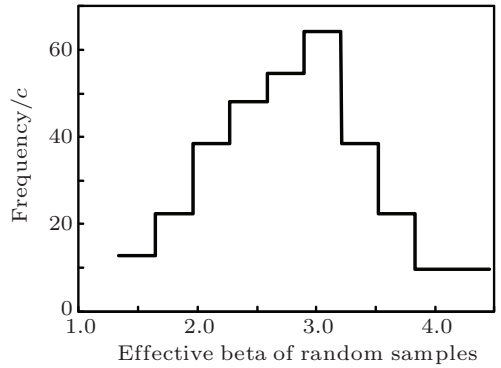


Fig. 2. Histogram shows the statistical distribution of Einstein effective viscosity constant β^e in random samples, where the total volume fraction of ten cubic inclusions in each random sample is 0.35 and the values of effective constant β^e are in the range [1.338, 4.455]. In the statistical analyses, the apparent frequency of effective constant β^e in each of the equally-spaced ranges is estimated with the frequency of β^e divided by the interval constant ($c = 0.3117$) of effective viscosity value β^e.

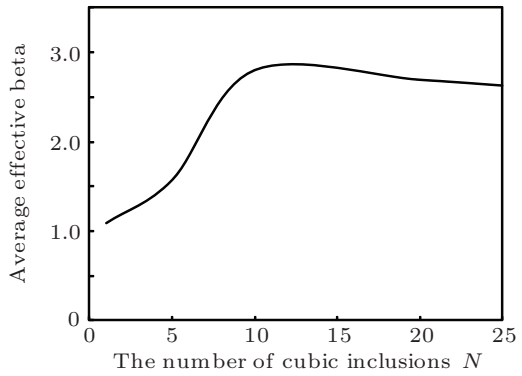


Fig. 3. The effective viscosity constant β^e changes with the number of particles at total volume fraction 0.35.

4. Conclusions

The transformation field method is used to estimate effective viscosity of random solid suspensions with a cubic periodic model having complex shapes of inclusions and the reduction behaviour of effective viscosity, which is caused by particle size of inclusions at a high concentration, and it is qualitatively modeled. With numerical calculations, we gave a frequency distribution histogram of effective viscosity coefficient β^e on the basis of many times selecting random samples of smaller cubic particle suspensions so that it can disclose the effective viscosity reduction mechanism. From our results, it is clearly seen that the larger size of particle inclusions can sharply reduce the effective viscosity of solid suspensions by comparing the results of smaller solid particle size. For the values of effective viscosity coefficient β^e of different samples of small particles, there exists a larger difference in effective viscosity coefficient between different samples, changing from 1.3380 to 4.4550. It also implies that the different spatial distributions or positions of particles give rise to an important effect on effective viscosity. Of course, when these small particles aggregate and produce only one large size particle, the effective viscosity will be reduced sharply. It indicates that large particles will induce less viscosity than the small ones. The transfer field method also provides a good way to quantitatively discuss the effects of random factors of solid inclusions on the effective viscosity, such as the particle shapes, spatial distribution or positions of inclusions. As a theoretical investigation, in the next work, we will further discuss the viscosity reducing or enhancing problem caused by other random factors of inclusions, for example, the particle shapes, spatial distribution or particle-chain structures, and find which random factor will play a key role in controlling the effective viscosity at high concentrations because it is very useful in the matter transportation.

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