Parameter design of a liquid-filled sloshing system*

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The nonlinear governing equations of the liquid sloshing modals in a cylindrical storage tank are established. Through analytical analysis, the analytical expressions of the solutions of this kind of system are obtained. With different parameters, the dynamical behaviors of the solutions are different from the trivial ones. To prevent system instability, two selection principles that the stiffness equations are positive-definite and the nonlinear terms of the system are not regenerative elements are given. Meanwhile, numerical simulations are also given, which confirm the analytical results.

Keywords: liquid-filled system, analytical analysis, selection principle, instability

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1. Introduction

The sloshing dynamics and control of a liquid-storage tank coupling system is an important research field in the hydrodynamics and control and its interdiscipline. It is widely used in civil engineering, automobile engineering, ship dynamics, and aerospace industries. For instance, in the aerospace field, the sloshing problem of a liquid–solid coupling system and its optimal control are always core issues in the overall design of aerospace vehicles.[1−3] They are widely studied by aerospace mechanics scholars.[4,5] The damage caused by liquid sloshing leads to the issue needing more research. However, nonlinear factors make theoretical studies on this problem very complicated.

This kind of system is mostly studied using experimental methods.[6,7] It is necessary to establish an effective mathematical model to study the dynamics behaviors of the liquid-filled coupling system using an analytical method. Yin *et al.*[8] and Ma *et al.*[9] expanded the velocity potential function at the free surface of the liquid and obtained a group of generalized modal coordinate nonlinear equations by the Lagrange function to make a theoretical analysis. Using this method and considering different sloshing modals, a group of nonlinear governing equations of the liquid-filled coupling system are established. Furthermore, through investigations on the system, the abundant nonlinear dynamical behaviors of the coupling system are obtained. In addition, more and more studies[10,11] show that the parameter estimation of the system is important. So, through analytical analysis, the principles of selecting the parameters of the system to make the system stable are given in the present paper, which can provide a theoretical basis for investigating the stability of the liquid-filled system used in engineering.

2. Nonlinear governing equations of the coupling system

The system model is shown in Fig. 1, where a is the radius of the tank; h is the height of the liquid; ρ is the density of the liquid. Let $OXYZ$ be an inertial rectangular coordinate system with its origin at the centre of the static liquid surface and let $O\theta rz$ be a cylindrical coordinates system; η is the height difference between the free-surface of the liquid and the static one, which is also defined as a wave height function.

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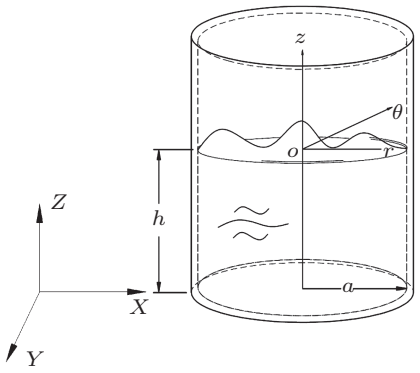


Fig. 1. Physical model.

Assume that the liquid is uniform, inviscid, irrotational and incompressible ideal fluid. We define the region composed of $0 < r < a$, $0 < \theta \leq 2\pi$, and $-h < z < \eta\,(r, \theta, t)$ as the liquid motion region, $\varphi(r, \theta, z, t)$ as the absolute velocity potential function and $\phi(r, \theta, z, t)$ as the relative velocity potential function. From $\nabla^2\phi = 0$, the kinematics equation and the dynamics equation are obtained,[12] respectively, as follows:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}\frac{\partial \eta}{\partial \theta} - \frac{\partial \phi}{\partial z} = 0, \tag{1}$$

$$\frac{\partial \varphi}{\partial t} + g\eta + \frac{1}{2}\left(\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta} + \frac{\partial \phi}{\partial z}\right)^2 - \frac{\sigma}{\rho}\left(\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}\right) = 0. \tag{2}$$

The boundary conditions of the coupling system are

$$\begin{aligned} &\frac{\partial \varphi}{\partial z} = 0, \quad z = -h, \\ &\frac{\partial \varphi}{\partial r} = 0, \quad r = a. \end{aligned} \tag{3}$$

By the method of separation of variables, the velocity potential function and the wave height function could be, respectively, expanded into[7,8]

$$\varphi(r, \theta, z, t) = \sum_{n=1}^{N} \frac{\cosh[k_n(z+h)]}{\cosh k_n h}\phi_n(r, \theta)\varphi_n(t), \tag{4}$$

$$\eta(r, \theta, t) = f(r) + \sum_{n=1}^{N} \xi_n(r, \theta)\eta_n(t), \tag{5}$$

where, $\varphi_n(t)$, $(n = 1, 2, \ldots)$ and $\eta_n(t)$, $(n = 1, 2, \ldots)$ are the generalized modal displacements of the velocity potential function and the wave height function, respectively; $\phi_n(r, \theta) = A_n \mathrm{J}_n(k_n r)\cos(n\theta)$, $(n = 1, 2, \ldots)$ are the modals of the velocity potential functions; $\xi_n(r, \theta) = B_n \mathrm{J}_n(k_n r)\cos(n\theta)$, $(n = 1, 2, \ldots)$ are the modals of the wave height functions; A_n, $(n = 1, 2, \ldots)$ and B_n, $(n = 1, 2, \ldots)$ are the modal orthonormal coefficients; J is the Bessel function; k_n, $(n = 1, 2, \ldots)$ are the solutions of $\partial \mathrm{J}_n(k_n r)/\partial r|_{r=a} = 0$, $(n = 1, 2, \ldots)$ and $f(r)$ is the static liquid level.

The first-order sloshing modal of the liquid caused by the vibration of the tank in the X direction is shown in Fig. 2. For convenience, we take the first-order of the liquid sloshing modal and the displacement in the X direction of the tank into consideration. The truncation error of higher modals is around 10%, which is generally acceptable in the qualitative analysis.

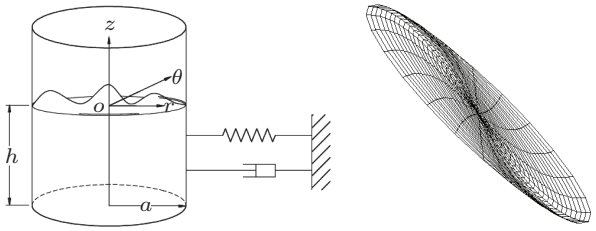


Fig. 2. Sloshing modal.

We substitute Eqs. (4) and (5) into Eqs. (1) and (2) for the Galerkin discretization.[10] As for cylindrical vessel, using modal orthogonality and considering the effect of the dynamic inertial force from liquid on the tank, the nonlinear governing equations can be obtained as follows:

$$\begin{aligned} &\ddot{\eta}_1 + K_{11}\eta_1 + K_{12}X = \varepsilon\left(K_{13}\dot{X} + K_{14}\dot{\eta} + K_{15}\eta_1^3\right), \\ &\ddot{X} + K_{21}\eta_1 + K_{22}X = \varepsilon\left(K_{23}\dot{X} + K_{24}\dot{\eta} + K_{25}\eta_1^3\right), \end{aligned} \tag{6}$$

where η_1 and X are generalized coordinates of the first-order modal of the liquid and the displacement of the tank in the X direction, respectively; K_{11}, K_{12}, K_{13}, K_{14}, K_{15}, K_{21}, K_{22}, K_{23}, K_{24}, and K_{25} are consolidation coefficients, respectively, of which the expressions may be found in Appendix A, and $\varepsilon > 0$ is a small parameter.

3. Analytical analysis

There exist stiffness coupling terms and nonlinear coupling terms in Eq. (6). With different system parameters, besides the trivial solution, the solutions of the system would be periodic, aperiodic, and even unbounded.[13]

Using a multi-scale method,[14] we seek a first-order solution in the following form:

$$\begin{cases} \eta_1 = u_{11}(T_0, T_1) + \varepsilon u_{12}(T_0, T_1) + \cdots, \\ X = u_{21}(T_0, T_1) + \varepsilon u_{22}(T_0, T_1) + \cdots, \end{cases} \tag{7}$$

with order ε^0

$$\varepsilon^0 : \begin{cases} D_0^2 u_{11} + K_{11} u_{11} + K_{12} u_{21} = 0, \\ D_0^2 u_{21} + K_{21} u_{11} + K_{22} u_{21} = 0, \end{cases} \tag{8}$$

and order ε^1

$$\varepsilon^1 : \begin{cases} D_0^2 u_{12} + K_{11} u_{12} + K_{12} u_{12} \\ = -2D_1 D_0 u_{11} + K_{13} D_0 u_{21} \\ \quad + K_{14} D_0 u_{11} + K_{15} u_{11}^3, \\ D_0^2 u_{22} + K_{21} u_{12} + K_{22} u_{22} \\ = -2D_1 D_0 u_{21} + K_{23} D_0 u_{21} \\ \quad + K_{24} D_0 u_{11} + K_{25} u_{11}^3, \end{cases} \tag{9}$$

where $D_n = \partial / \partial T_n$.

The solutions of Eq. (8) can be expressed as

$$\begin{aligned} u_{11} &= A_1(T_1)\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + A_2(T_1)\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + cc, \\ u_{21} &= \Gamma_1 A_1(T_1)\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \Gamma_2 A_2(T_1)\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + cc, \end{aligned} \tag{10}$$

where the ω_n^2, $n = 1, 2$ are the roots of

$$\begin{aligned} &\omega^4 - (K_{11} + K_{22})\omega^2 + K_{11}K_{22} - K_{12}K_{21} = 0; \\ &\Gamma_l = -\frac{K_{11} - \omega_l^2}{K_{12}} = -\frac{K_{21}}{K_{22} - \omega_l^2}, \quad \text{for } l = 1 \text{ and } 2; \end{aligned}$$

and the ω_n are assumed to be distinct.

Substituting Eq. (10) into Eq. (9) yields

$$\begin{aligned} & D_0^2 u_{12} + K_{11} u_{12} + K_{12} u_{12} \\ = & -\mathrm{i}2A_1'\omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{14} A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + \mathrm{i}K_{13}\Gamma_1 A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + 3K_{15} A_2 \bar{A}_1^2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0 + \mathrm{i}\sigma T_1} + 3K_{15} A_1^2 \bar{A}_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + 6K_{15} A_1 A_2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} - 2\mathrm{i}A_2'\omega_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + \mathrm{i}K_{14} A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \mathrm{i}K_{13}\Gamma_2 A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + K_{15} A_1^3\, \mathrm{e}^{\mathrm{i}\omega_2 T_0 - \mathrm{i}\sigma T_1} + 3K_{15} A_2^2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + 6K_{15} A_1 \bar{A}_1 A_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + K_{15} A_2^3\, \mathrm{e}^{\mathrm{i}3\omega_2 T_0} + 3K_{15} A_1 A_2^2\, \mathrm{e}^{\mathrm{i}(\omega_1 + 2\omega_2)T_0} \\ & + 3K_{15} A_1^2 A_2\, \mathrm{e}^{\mathrm{i}(2\omega_1 + \omega_2)T_0} \\ & + 3K_{15} A_1 \bar{A}_2^2\, \mathrm{e}^{\mathrm{i}(\omega_1 - 2\omega_2)T_0} + cc, \end{aligned} \tag{11}$$

$$\begin{aligned} & D_0^2 u_{22} + K_{21} u_{12} + K_{22} u_{22} \\ = & -\mathrm{i}2\Gamma_1 A_1'\omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + \mathrm{i}K_{24} A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{23}\Gamma_1 A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & + 3K_{25} A_2 \bar{A}_1^2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0 + \mathrm{i}\sigma T_1} \\ & + 3K_{25} A_1^2 \bar{A}_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + 6K_{25} A_1 A_2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} \\ & - 2\mathrm{i}\Gamma_2 A_2'\omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \mathrm{i}K_{24} A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + \mathrm{i}K_{23}\Gamma_2 A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + K_{25} A_1^3\, \mathrm{e}^{\mathrm{i}\omega_2 T_0 - \mathrm{i}\sigma T_1} + 3K_{25} A_2^2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + 6K_{25} A_1 \bar{A}_1 A_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} \\ & + K_{25} A_2^3\, \mathrm{e}^{\mathrm{i}3\omega_2 T_0} + 3K_{25} A_1 A_2^2\, \mathrm{e}^{\mathrm{i}(\omega_1 + 2\omega_2)T_0} \\ & + 3K_{25} A_1^2 A_2\, \mathrm{e}^{\mathrm{i}(2\omega_1 + \omega_2)T_0} \\ & + 3K_{25} A_1 \bar{A}_2^2\, \mathrm{e}^{\mathrm{i}(\omega_1 - 2\omega_2)T_0} + cc, \end{aligned} \tag{12}$$

where primes denote differentiation with respect to T_1.

In the nonresonant case, the solvability conditions of Eqs. (11) and (12) yield $A_n' = 0$, $n = 1, 2$. The stationary solutions can be obtained as

$$A_n = 0, \quad n = 1, 2.$$

In the resonant case, we introduce a detuning parameter σ defined by

$$\omega_2 = 3\omega_1 + \varepsilon\sigma.$$

The solvability conditions of Eqs. (11) and (12) can be obtained from

$$Q_{1l} + \Gamma_l Q_{2l} = 0, \quad l = 1, 2,$$

in which

$$\begin{aligned} Q_{11} = & -\mathrm{i}2A_1'\omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{14} A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{13}\Gamma_1 A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + 3K_{15} A_2 \bar{A}_1^2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0 + \mathrm{i}\sigma T_1} \\ & + 3K_{15} A_1^2 \bar{A}_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + 6K_{15} A_1 A_2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0}, \\ Q_{12} = & -\mathrm{i}2A_2'\omega_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{14} A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \mathrm{i}K_{13}\Gamma_2 A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + K_{15} A_1^3\, \mathrm{e}^{\mathrm{i}\omega_2 T_0 - \mathrm{i}\sigma T_1} \\ & + 3K_{15} A_2^2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + 6K_{15} A_1 \bar{A}_1 A_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0}, \\ Q_{21} = & -\mathrm{i}2\Gamma_1 A_1'\omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{24} A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + \mathrm{i}K_{23}\Gamma_1 A_1 \omega_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + 3K_{25} A_2 \bar{A}_1^2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0 + \mathrm{i}\sigma T_1} \\ & + 3K_{25} A_1^2 \bar{A}_1\, \mathrm{e}^{\mathrm{i}\omega_1 T_0} + 6K_{25} A_1 A_2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_1 T_0}, \\ Q_{22} = & -\mathrm{i}2\Gamma_2 A_2'\omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \mathrm{i}K_{24} A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + \mathrm{i}K_{23}\Gamma_2 A_2 \omega_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + K_{25} A_1^3\, \mathrm{e}^{\mathrm{i}\omega_2 T_0 - \mathrm{i}\sigma T_1} \\ & + 3K_{25} A_2^2 \bar{A}_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0} + 6K_{25} A_1 \bar{A}_1 A_2\, \mathrm{e}^{\mathrm{i}\omega_2 T_0}. \end{aligned}$$

Ordering the above conditions, we can obtain

$$\begin{aligned}&-\mathrm{i}2\omega_1 B_{11}A_1' + \mathrm{i}\omega_1 B_{21}A_1 + 3B_{31}A_1^2\bar{A}_1\\&+6B_{31}A_1A_2\bar{A}_2 + 3B_{31}A_2\bar{A}_1^2\,\mathrm{e}^{\mathrm{i}\sigma T_1} = 0,\\&-\mathrm{i}2\omega_2 B_{12}A_2' + \mathrm{i}\omega_2 B_{22}A_2 + 3B_{32}A_2^2\bar{A}_2\\&+6B_{32}A_1\bar{A}_1A_2 + 3B_{32}A_1^3\,\mathrm{e}^{-\mathrm{i}\sigma T_1} = 0,\end{aligned}\tag{13}$$

where

$$\begin{aligned}B_{11} &= 1+\Gamma_1^2,\\B_{12} &= 1+\Gamma_2^2,\\B_{21} &= K_{14}+K_{24}\Gamma_1+K_{13}\Gamma_1+K_{23}\Gamma_1^2,\\B_{22} &= K_{14}+K_{24}\Gamma_2+K_{13}\Gamma_2+K_{23}\Gamma_2^2,\\B_{31} &= K_{15}+K_{25}\Gamma_1,\\B_{32} &= K_{15}+K_{25}\Gamma_2.\end{aligned}$$

Let

$$A_1 = \frac{1}{2}a_1\,\mathrm{e}^{\mathrm{i}\beta_1},\quad A_2 = \frac{1}{2}a_2\,\mathrm{e}^{\mathrm{i}\beta_2},$$

Then, four equations can be obtained by the separation of the real and imaginary parts of A_n with $n=1,2$,

$$-\omega_1 B_{11}a_1' + \frac{1}{2}\omega_1 B_{21}a_1 + \frac{3}{8}B_{31}a_1^2a_2\sin\gamma = 0,\tag{14}$$

$$-\omega_2 B_{12}a_2' + \frac{1}{2}\omega_2 B_{22}a_2 - \frac{3}{8}B_{32}a_1^3\sin\gamma = 0,\tag{15}$$

$$\begin{aligned}&\omega_1 B_{11}a_1\beta_1' + \left(\frac{3}{8}B_{32}a_1^2 + \frac{6}{8}B_{31}a_2^2\right)a_1\\&+\frac{3}{8}B_{31}a_1^2a_2\cos\gamma = 0,\end{aligned}\tag{16}$$

$$\begin{aligned}&\omega_2 B_{12}a_2\beta_2' + \left(\frac{3}{8}B_{32}a_2^2 + \frac{6}{8}B_{32}a_1^2\right)a_2\\&+\frac{3}{8}B_{32}a_1^3\cos\gamma = 0,\end{aligned}\tag{17}$$

in which

$$\gamma = 3\beta_2 - \beta_1 + \sigma T_1.\tag{18}$$

Eliminating β_1 and β_2 from Eqs. (16) and (17) through Eq. (18) gives

$$\begin{aligned}a_2\gamma' = {}& a_2\sigma + \left(\frac{9B_{31}}{4\omega_1 B_{11}} - \frac{3B_{32}}{8\omega_2 B_{12}}\right)a_2^3\\&+\left(\frac{9B_{31}}{8\omega_1 B_{11}} - \frac{3B_{32}}{4\omega_2 B_{12}}\right)a_1^2a_2\\&+\left(\frac{9B_{31}}{8\omega_1 B_{11}}a_1a_2^2 - \frac{3B_{32}}{8\omega_2 B_{12}}a_1^3\right)\cos\gamma.\end{aligned}\tag{19}$$

Eliminating γ in Eqs. (14) and (15) yields

$$\begin{aligned}&B_{31}a_2\left(\omega_2 B_{12}a_2' - \frac{1}{2}\omega_2 B_{22}a_2\right)\\&= -B_{32}a_1\left(\omega_1 B_{11}a_1' - \frac{1}{2}\omega_1 B_{21}a_1\right).\end{aligned}\tag{20}$$

Simplifying Eq. (20) yields

$$a_1a_1' + va_2a_2' = \frac{1}{2}\frac{B_{21}}{B_{11}}a_1^2 + \frac{1}{2}\frac{B_{22}}{B_{12}}va_2^2,\tag{21}$$

where

$$v = \frac{B_{31}\omega_2 B_{12}}{B_{32}\omega_1 B_{11}}.$$

When $v>0$, Eq. (21) can be integrated if $B_{21} = B_{22} = \varepsilon^2\mu > 0$. The result is

$$a_1^2 + va_2^2 = E\,\mathrm{e}^{-\varepsilon^2\mu t},$$

where E is a constant of integration proportional to the initial energy of the system. Thus, as $t\to\infty$, $a_1^2 + va_2^2 \to 0$. That is, the energy in the system decays exponentially with time.

If $B_{21} = B_{22} = 0$, Eq. (21) can be integrated and the result is

$$a_1^2 + va_2^2 = E.\tag{22}$$

Then a second integral of the motion can be found.

Changing the independent variable in Eq. (19) from T_1 to a_2, we obtain

$$\begin{aligned}&a_1^3a_2\frac{\mathrm{d}\gamma}{\mathrm{d}a_2}\sin\gamma\\&= \frac{8\omega_2 B_{12}\sigma}{3B_{32}}a_2 + C_{11}a_2^3 + C_{12}a_1^2a_2\\&\quad+\left(\frac{3B_{31}\omega_2 B_{12}}{B_{32}\omega_1 B_{11}}a_1a_2^2 - a_1^3\right)\cos\gamma,\end{aligned}\tag{23}$$

where

$$\begin{aligned}C_{11} &= \left(\frac{9B_{31}}{4\omega_1 B_{11}} - \frac{3B_{32}}{8\omega_2 B_{12}}\right)\frac{8\omega_2 B_{12}}{3B_{32}},\\C_{12} &= \left(\frac{9B_{31}}{8\omega_1 B_{11}} - \frac{3B_{32}}{4\omega_2 B_{12}}\right)\frac{8\omega_2 B_{12}}{3B_{32}}.\end{aligned}$$

Using Eq. (22), we can integrate Eq. (23) to obtain

$$\begin{aligned}&a_1^3a_2\cos\gamma - \frac{1}{2}\left(\frac{8\omega_2 B_{12}}{3B_{32}} + C_{12}E\right)a_2^2\\&+\frac{1}{4}(vC_{12} - C_{11})a_2^4 = L,\end{aligned}\tag{24}$$

where L is a constant of integration.

Let $a_1^2 = E\xi$, then we will have $a_2^2 = Ev^{-1}(1-\xi)$. From Eqs. (22) and (24), we obtain

$$\frac{16\omega_1^2 B_{11}^2 v}{9E^2 B_{31}^2}(\xi')^2 = F^2(\xi) - G^2(\xi),\tag{25}$$

where

$$F = \pm\sqrt{\xi^3(1-\xi)},$$

$$G = \frac{\sqrt{v}}{E^2}\left\{L + \frac{E}{2v}\left(\frac{8\omega_2 B_{12}\sigma}{3B_{22}} + C_{12}E\right)(1-\xi) - \frac{E^2}{4v^2}(vC_{12} - C_{11})(1-\xi)^2\right\}.$$

When $v < 0$, using $-v$ instead of v, it can be obtained as follows:

$$\frac{16\omega_1^2 B_{11}^2 |v|}{9E^2 B_{31}^2}(\xi')^2 = F^2(\xi) - G^2(\xi), \tag{26}$$

where

$$F = \pm\sqrt{\xi^3(1+\xi)},$$
$$G = \frac{\sqrt{-v}}{E^2}\left\{L - \frac{E}{2v}\left(\frac{8\omega_2 B_{12}\sigma}{3B_{22}} + C_{12}E\right)(1+\xi) - \frac{E^2}{4v^2}(vC_{12} - C_{11})(1+\xi)^2\right\},$$

Using numerical method to solve Eqs. (25) and Eq. (26), functions $F(\xi)$ and $G(\xi)$ are shown schematically in Fig. 3. A point such as P_1 in Fig. 3(a) where G_1 touches F represents a stationary solution for ξ and hence a_1 and a_2. Consequently the motion corresponding to such a point is periodic, and the effect of the nonlinearity is to modulate the phase in such a way that the nonlinear frequencies are commensurable.

A curve such as G_2 in Fig. 3(a) which meets F at two different points corresponds to a periodic solution for ξ and hence a_1 and a_2. Consequently the motion is aperiodic. It shows that in the absence of damping, the energy is continuously exchanged between the two modes. The energy in a given mode can be more effectively damped if that mode is coupled via an internal resonance to another mode having a large coefficient of damping.

A curve such as G_3 in Fig. 3(b), which meets F at one point only, represents an unbounded motion.

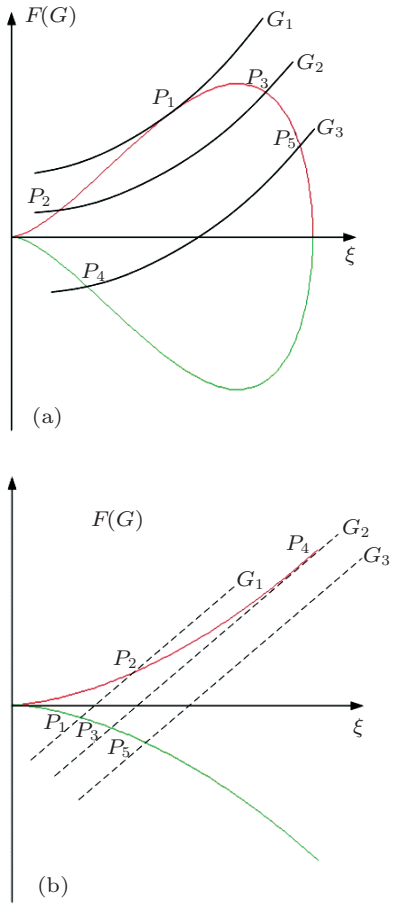


Fig. 3. (color online) Schematic diagrams of F and G for (a) $v > 0$ and (b) $v < 0$.

4. Selection principles of the parameters

To sum up, in order to prevent system instability, the parameters should accord with the following conditions.

(i) Stiffness equations are positive-definite.

Calculating

$$\omega^4 - (K_{11} + K_{22})\omega^2 + K_{11}K_{22} - K_{12}K_{21} = 0,$$
$$\Gamma_l = -\frac{K_{11} - \omega_l^2}{K_{12}} = -\frac{K_{21}}{K_{22} - \omega_l^2}, \quad \text{for } l = 1 \text{ and } 2,$$

yields

$$\omega_{1,2}^2 = \frac{(K_{11} + K_{12}) \pm \sqrt{(K_{11} + K_{22})^2 - 4(K_{11}K_{22} - K_{12}K_{21})}}{2}.$$

When the stiffness equations are positive, there are

$$\omega_{1,2}^2 = \frac{(K_{11} + K_{12}) \pm \sqrt{(K_{11} - K_{12})^2 + K_{12}K_{21}}}{2},$$

thus

$$\omega_2^2 > K_{11}; \quad \omega_1^2 > K_{12}.$$

So Γ_l $(l = 1, 2)$ are greater than zero, that is, B_{2l} $(l = 1, 2)$ are greater than zero, which means that the equivalent damping is greater than zero.

(ii) The nonlinear terms of the system are not regenerative elements.[11]

If the coupling nonlinear elements are not regenerative, B_{3l} $(l = 1, 2)$ have the same signs. Integrating

$$B_{31}a_2\omega_2 B_{12}a_2' + B_{32}\omega_1 B_{11}a_1 a_1' = \frac{1}{2}\omega_1 B_{21}B_{32}a_1^2 + \frac{1}{2}\omega_2 B_{22}B_{31}a_2^2$$

yields

$$a_1^2 + va_2^2 = E\,\mathrm{e}^{-\varepsilon^2\mu t},$$

which can illustrate that the system has trivial solution.

Table 1. Parameters of the system.

Diameter $d = 2a$/m	Depth h/m	Gravity	Connect antenna hysteretic constant
0.3	0.5	1.0	0

Frequency ratio υ_1	Mass ratio μ	Liquid damping ζ	Spring damper ζ_x
0.9	0.16	0.0348	0.05

The parameters of the system are selected using the principles given above and shown in Table 1. The numerical results are given in Fig. 4, showing that the system is stable.

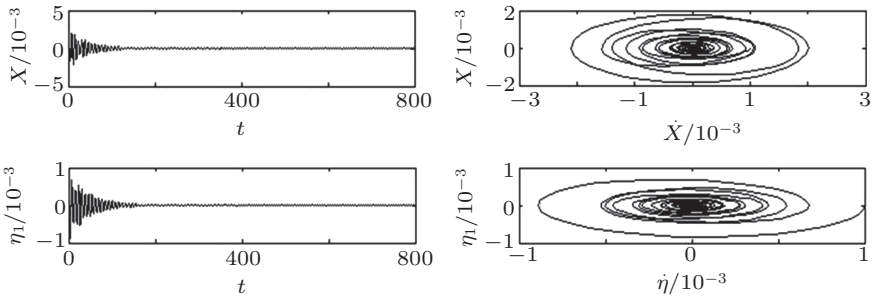


Fig. 4. Time-history curves and phase diagrams.

5. Conclusions

In this paper, the coupling sloshing equations are established. Using an analytical method, the solutions of the coupling equations with different parameters are obtained, which show that with the different parameters, the solutions of the system could be periodic, aperiodic, and even unbounded. These cases should be prevented in engineering. So, from the analytical analysis, the selection principles of the parameters are given to prevent system instability. The principles are that the stiffness equations are positive-definite and the nonlinear terms of the system are not regenerative elements, which will provide a theoretical basis for the optimal design and control of liquid-filled system parameters. It is effective to control the sloshing by choosing the system parameters according to the principle given above.

Appendix A

$$K_{11} = \mu_{xq}\mu_{11}, \quad K_{12} = -\mu_{xq}\mu_{11}\alpha_1\upsilon_1^2,$$
$$K_{13} = -2\mu_{xq}\mu_{11}\zeta_x/\varepsilon, \quad K_{14} = 2\mu_{xq}\mu_{11}\alpha_1\zeta_1\upsilon_1/\varepsilon,$$
$$K_{15} = \mu_{xq}\alpha_1\sigma_{1111}/\varepsilon, \quad K_{21} = \mu_{xq}\mu_{11}\upsilon_1^2(1+\mu),$$
$$K_{22} = -\mu_{xq}\alpha_1, \quad K_{23} = 2\mu_{xq}\alpha_1\zeta_x/\varepsilon,$$
$$K_{24} = -2\mu_{xq}\mu_{11}\upsilon_1\zeta_1(1+\mu)/\varepsilon,$$
$$K_{25} = -\mu_{xq}(1+\mu)\sigma_{1111}/\varepsilon,$$

where

$$\mu = m/M, \quad \mu_{xq} = 1/\mu_{11} + \mu_{11}\mu - \alpha_1^2,$$
$$\alpha_1 = \frac{\tilde{d}_1}{m}, \quad \tilde{d}_1 = \rho\iint_S \left[1 + \frac{1}{2}f'^2(r)\right]\xi_1 r\cos\theta \mathrm{d}S,$$
$$\mu_{11} = \frac{2h_{11}}{m}, \quad h_{nn} = \frac{1}{2}\rho_l S l_{nn} \quad (n = 1, 2, \ldots),$$

$$\zeta_1 = \zeta, \quad \upsilon_1 = \frac{\sqrt{\rho g}}{\omega_0}\sqrt{\frac{S + \dfrac{2a^2}{B_0}S_{11}}{2h_{11}}},$$

$$\sigma_{1111} = \frac{\rho g d^4 S_{1111}}{m\omega_0^2 B_0}, \quad l_{jj} = d_0 A_{jj} \quad (j = 1, 2, \ldots),$$

$$d_0 = \frac{1}{D_0}, \quad A_{jj} = \prod_{\substack{i=1 \\ i\neq j}}^{N} Q_{ii}, \quad j = 1, 2, \ldots,$$

$$D_0 = \prod_{i=1}^{N} Q_{ii},$$

$$Q_{jn} = \iint_S \xi_j \left[k_n \tanh(k_n h)\phi_n - f'(r)\frac{\partial \phi_n}{\partial r}\right] \mathrm{d}S,$$

$$j = 1, 2, \ldots; \ n = 1, 2, \ldots,$$

$$\begin{aligned} S_{nmji} = & -\frac{1}{8}\iint_S \left\{ \frac{4f'^2(r) - 1}{[1 + f'^2(r)]^{7/2}} \left(\frac{\partial \xi_n}{\partial r}\frac{\partial \xi_m}{\partial r}\frac{\partial \xi_j}{\partial r}\frac{\partial \xi_i}{\partial r} \right) \right. \\ & + \frac{2f'^2(r) - 1}{[1 + f'^2(r)]^{5/2}} \frac{1}{r^2}\frac{\partial \xi_n}{\partial r}\frac{\partial \xi_m}{\partial r}\frac{\partial \xi_j}{\partial \theta}\frac{\partial \xi_i}{\partial \theta} \\ & \left. - \frac{1}{[1 + f'^2(r)]^2}\frac{1}{r^4}\frac{\partial \xi_n}{\partial \theta}\frac{\partial \xi_m}{\partial \theta}\frac{\partial \xi_j}{\partial \theta}\frac{\partial \xi_i}{\partial \theta} \right\} \mathrm{d}S, \\ & i = 1, 2, \ldots; \ j = 1, 2, \ldots; \\ & m = 1, 2, \ldots; \ n = 1, 2, \ldots. \end{aligned}$$

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