Wronskian and Grammian solutions for the (3+1)-dimensional Jimbo–Miwa equation*

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In this paper, based on Hirota's bilinear method, the Wronskian and Grammian techniques, as well as several properties of the determinant, a broad set of sufficient conditions consisting of systems of linear partial differential equations are presented. They guarantee that the Wronskian determinant and the Grammian determinant solve the $(3+1)$-dimensional Jimbo–Miwa equation in the bilinear form. Then some special exact Wronskian and Grammian solutions are obtained by solving the differential conditions. At last, with the aid of Maple, some of these special exact solutions are shown graphically.

Keywords: (3+1)-dimensional Jimbo–Miwa equation, Wronskian determinant, Grammian determinant, exact solution

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1. Introduction

Nonlinear phenomena appear in many scientific and engineering fields, such as hydrodynamics, plasma physics, fluid dynamics, solid state physics, optics, acoustics, and so on. With the development of nonlinear dynamics, the study of nonlinear evolution equations (NEES) becomes increasingly important. Especially, it plays an important role in nonlinear dynamics for searching for exact solutions of NEES. The exact solutions of NEES can help us to analyze some physical phenomena and solve some relevant problems. In the past years, many methods such as Hirota's bilinear method,[1] the inverse scattering transformation,[2] the tanh–coth method,[3,4] the Wronskian technique,[5] and the Darboux transformation,[6] have been developed, and diverse classes of exact solutions have been obtained by using these methods. For instance, the soliton solutions of NEES were presented,[7] the traveling wave solutions were obtained by using the Hirota method,[8] the periodic wave solutions were given,[9] etc. Recently, Chen *et al.*[10] constructed new solutions of NEES by using the Hirota method and gave out a speculation of the general expression of N-soliton solutions. Zeng *et al.*[11] presented a method for finding N-soliton solutions by constructing a constraint flow. Furthermore, many new results have appeared for finding exact solutions of NEES. For example, a new method for finding new exact solutions and solitary wave-like solutions of the generalized NEES has been provided,[12] some new exact solutions of the Jacobian elliptic function for the generalized NEES have been given,[13] and some new exact solutions of the generalized NEES have been obtained by using the first integral method.[14] However, it is still necessary to search for exact solutions of some NEES. In this paper, based on the Wronskian and Grammian technique, the Wronskian and Grammian solutions of $(3+1)$-dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0 \quad (1)$$

is studied. A lot of research has been conducted about Eq. (1). The exact solutions of Eq. (1) have been presented by using a transformed rational function method.[15] The multiple soliton solutions of Eq. (1) have been obtained by using the Bäcklund transformation.[16] Exact and explicit generalized solitary solutions for the above equation have been given.[17] Some special similarity reductions of

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Eq. (1) have been obtained by Lou and Tang.[18] A set of Wronskian solutions of Eq. (1) have been provided by Tang.[19] In order to find more generalized exact solutions and further meet the demand in engineering fields, we will give a broad set of extended sufficient conditions and present the Wronskian and Grammian solutions of Eq. (1). Let us start with the following transformation. Firstly, we introduce the logarithmic transformation

$$u = u_0 + 2(\ln f)_x, \tag{2}$$

where u_0 is an arbitrary constant, therefore Eq. (1) is transformed into the following bilinear form

$$(3D_xD_z - 2D_yD_t - D_yD_x^3)f \bullet f = 0, \tag{3}$$

where the well-known Hirota bilinear operators D_x,D_y,D_z, and D_t, are defined by

$$\begin{aligned} & D_x^m D_y^n D_z^l D_t^k a \bullet b \\ = & (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n (\partial_z - \partial_{z'})^l \\ & \times (\partial_t - \partial_{t'})^k a(x,y,z,t) \\ & \times b(x,y,z,t)\mid_{x=x',y=y',z=z',t=t'} . \end{aligned} \tag{4}$$

From Eq. (2), we know that if we can get f, then Eq. (1) will be solved by $u = u_0 + 2(\ln f)_x$. Therefore, the main problem for us is how to obtain f from Eq. (3).

This paper is organized as follows. In Section 2, we will obtain the Wronskian solutions of Eq. (1) via the Wronskian technique and give some special exact solutions. In Section 3, we will present the Grammian solutions and get several special exact solutions. In Section 4, some results will be given.

2. Wronskian solutions

We first consider the following three lemmas before presenting the Wronskian solutions of Eq. (1).

Lemma 1 If M is an $N \times (N-2)$ matrix, and a, b, c, and d are N-order column vectors, then

$$\begin{aligned} & |M,a,b||M,c,d| - |M,a,c||M,b,d| \\ & + |M,a,d||M,b,c| = 0. \end{aligned} \tag{5}$$

Lemma 2 If α_j $(1 \le j \le N)$ are N-dimensional column vectors and $\gamma\alpha_j$ denotes $(\gamma_1\alpha_{1j}, \gamma_2\alpha_{2j}, \ldots, \gamma_N\alpha_{Nj})^{\mathrm{T}}$, then

$$\begin{aligned} & \sum_{j=1}^{n} |\alpha_1, \ldots, \alpha_{j-1}, \gamma\alpha_j, \alpha_{j+1}, \ldots, \alpha_N| \\ = & \sum_{j=1}^{n} \gamma |\alpha_1, \ldots, \alpha_N|. \end{aligned} \tag{6}$$

Lemma 3 Assume that $P = (p_{ij})$ is an $N \times N$ operator matrix and its entries p_{ij} are differential operators, $B = (b_{ij})$ is an $N \times N$ function matrix with column vector set b_i and row vector set b_i' $(i = 1, \ldots, N; j = 1, \ldots, N)$, $p_ib_i = (p_{1i}b_{1i}, p_{2i}b_{2i}, \ldots, p_{Ni}b_{Ni})^{\mathrm{T}}$, $p_i'b_i' = (p_{j1}b_{j1}, p_{j2}b_{j2}, \ldots, p_{jN}b_{jN})$, then

$$\sum_{i=1}^{N} |b_1, \ldots, p_ib_i, \ldots, b_{_N}| = \sum_{j=1}^{N} \begin{vmatrix} b_1' \\ \vdots \\ p_j'b_j' \\ \vdots \\ b_{_N}' \end{vmatrix}. \tag{7}$$

The above three identities (5)–(7) can be verified through a direct calculation. Next, we present the Wronskian solutions of Eq. (3) in detail.

Theorem 1 The solution of Eq. (3) can be expressed as the $N \times N$ Wronskian determinant

$$\begin{aligned} f_N &= Wr(\phi_1, \phi_2, \ldots, \phi_N) \\ &= \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \end{aligned} \tag{8}$$

where $\phi_j^{(m)}$ denotes $\partial^m\phi_j/\partial x^m$, and $\phi_j = \phi_j(x,y,z,t)$ $(j = 1, 2, \ldots, N)$ in $t \ge 0$, $-\infty < x, y, z < +\infty$ has a continuous derivative up to any order and satisfies the following conditions:

$$\begin{aligned} & \phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}\phi_j, \quad \phi_{i,y} = a\phi_{i,x}, \quad \phi_{i,t} = b\phi_{i,xxx}, \\ & \phi_{i,z} = c\phi_{i,xxx},\ 1 \le i \le N, \end{aligned} \tag{9}$$

with λ_{ij}, a, b, c being arbitrary real constants and satisfying

$$2ab - 3c + 4a = 0. \tag{10}$$

Proof For convenience, we adopt the simple notation $f_N = |\widehat{N-1}|$, the derivatives with respect to x, y, z, and t of f_N are obtained as follows:

$$
\begin{aligned}
&f_{N,x} = |\widehat{N-2}, N|, \quad f_{N,y} = a|\widehat{N-2}, N|,\\
&f_{N,z} = c[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|],\\
&f_{N,t} = b[|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|],\\
&f_{N,xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|,\\
&f_{N,xy} = b[|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|],\\
&f_{N,xz} = c[|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|],\\
&f_{N,yt} = ab[|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|],\\
&f_{N,xxx} = |\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+2|,\\
&f_{N,xxy} = a[|\widehat{N-4}, N-2, N-1, N| + 2|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+2|],\\
&f_{N,xxxy} = a[|\widehat{N-5}, N-3, N-2, N-1, N| + 3|\widehat{N-4}, N-2, N-1, N|\\
&\qquad + 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|].
\end{aligned}
$$

Substituting the above derivatives of f_N into Eq. (3), via a tedious calculation, we get the following formula:

$$
\begin{aligned}
&(3f_{xz} - 2f_{yt} - f_{xxxy})f - 3(f_z - f_{xxy})f_x + (2f_t + f_{xxx})f_y - 3f_{xy}f_{xx}\\
&= 3a[|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-4}, N-2, N-1, N+1|\\
&\quad - 2|\widehat{N-3}, N, N+1| - |\widehat{N-3}, N-1, N+2| + |\widehat{N-2}, N+3|]|\widehat{N-1}|\\
&\quad + 12a|\widehat{N-3}, N, N+1||\widehat{N-2}, N| - 3a[|\widehat{N-2}, N+1| + |\widehat{N-3}, N-1, N|]^2. \qquad (11)
\end{aligned}
$$

By virtue of the identity

$$
\begin{aligned}
&|\widehat{N-1}|\left\{\left(\sum_{j=1}^{N} k_j^2\right)\left[\left(\sum_{j=1}^{N} k_j^2\right)|\widehat{N-1}|\right]\right\}\\
&= \left[\left(\sum_{j=1}^{N} k_j^2\right)|\widehat{N-1}|\right]^2, \qquad (12)
\end{aligned}
$$

we can easily get

$$
\begin{aligned}
&\widehat{N-1}|(|\widehat{N-5}, N-3, N-2, N-1, N|\\
&+ 2|\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|\\
&- |\widehat{N-4}, N-2, N-1, N+1|\\
&- |\widehat{N-3}, N-1, N+2|)\\
&= (|\widehat{N-2}, N+1| - |\widehat{N-3}, N-1, N|)^2. \qquad (13)
\end{aligned}
$$

Substituting Eq. (13) into Eq. (11), through a simple calculation, we can reduce the right-hand side of Eq. (11) into

$$
\begin{aligned}
&- 12a[|\widehat{N-3}, N, N+1||\widehat{N-1}|\\
&- |\widehat{N-3}, N, N+1||\widehat{N-2}, N|\\
&+ |\widehat{N-2}, N+1||\widehat{N-3}, N-1, N|]. \qquad (14)
\end{aligned}
$$

Utilizing Lemma 1, we know expression (14) equals zero. Therefore, $f_N = |\widehat{N-1}|$ solves the bilinear equation (3), and then we prove $u = u_0 + 2\partial_x \ln f_N$ is the solution of Eq. (1).

In what follows, we would like to construct some special exact solutions to the Jimbo–Miwa equation. We know that the Jordan form of a real matrix has two types of blocks[20]

$$
\begin{pmatrix}
\lambda_i & & & 0\\
1 & \lambda_i & & \\
& \ddots & \ddots & \\
0 & & 1 & \lambda_i
\end{pmatrix}_{k_i \times k_i}, \qquad (15)
$$

$$
\begin{pmatrix}
A_i & & & 0\\
I_2 & A_i & & \\
& \ddots & \ddots & \\
0 & & I_2 & A_i
\end{pmatrix}_{k_i \times k_i}, \qquad (16)
$$

where $A_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and λ_i, α_i, $\beta_i > 0$ are all real constants. For the first type of block (15), we will give some special exact solutions as follows.[21–23]

2.1. Rational solutions

When $\lambda_i = 0$ and $k_i = N$, expression (15) becomes[24,25]

$$\begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}_{N\times N}. \tag{17}$$

Substituting expression (17) into expression (9), we get

$$\phi_{0,xx}=0,\quad \phi_{j+1,xx}=\phi_j,\quad \phi_{j,y}=a\phi_{j,x},$$
$$\phi_{j,t}=b\phi_{j,xxx},\quad \phi_{j,z}=c\phi_{j,xxx},\quad j\geq 0. \tag{18}$$

Solving Eq. (18) by uisng Maple, we get the following formulas:

$$\phi_0 = \frac{1}{b}C_1(x+ay)+C_4, \tag{19}$$
$$\phi_1 = \frac{1}{b}C_1\left(\frac{1}{6}x^3+\frac{1}{6}a^3y^3+\frac{1}{2}ax^2y+\frac{1}{2}a^2xy^2+cz\right) +\frac{1}{2}C_4a^2y^2+6C_4xy+C_3x +6C_3y+C_1t+C_2. \tag{20}$$

Let $C_1=1$, $C_2=C_3=C_4=0$, the following special rational solutions are obtained:

$$u_{1-\text{rational}} = u_0+2\partial_x\ln Wr(x+ay)=u_0+\frac{2}{x+ay}, \tag{21}$$
$$u_{2-\text{rational}} = u_0+2\partial_x\ln Wr(x+ay,\tfrac{1}{6}(x+ay)^3+(cz+bt)) = u_0+\frac{6x^2+6a^2y^2+12axy}{x^3+a^3y^3+3ax^2y+3a^2xy^2-cz-bt}. \tag{22}$$

2.2. Solitons

Substituting the special form of expression (15)

$$\begin{pmatrix} \eta_1^2 & & & 0 \\ & \eta_2^2 & & \\ & & \ddots & \\ 0 & & & \eta_N^2 \end{pmatrix}_{N\times N} \tag{23}$$

into Eq. (9), the following system of differential equations is obtained:[26]

$$\phi_{i,xx}=\eta_i^2\phi_i,\quad \phi_{i,y}=a\phi_{i,x},\quad \phi_{i,t}=b\phi_{i,xxx},$$
$$\phi_{i,z}=c\phi_{i,xxx},\quad 1\leq i\leq N. \tag{24}$$

By solving system (24), we get the n-soliton solution of Eq. (1)

$$u=u_0+2\partial_x\ln Wr(\phi_1,\phi_2,\ldots,\phi_n), \tag{25}$$

with ϕ_i being defined by

$$\phi_i=\cosh(\eta_ix+a\eta_iy+c\eta_i^3z+b\eta_i^3t),\quad i\quad \text{odd}, \tag{26a}$$
$$\phi_i=\sinh(\eta_ix+a\eta_iy+c\eta_i^3z+b\eta_i^3t),\quad i\quad \text{even}, \tag{26b}$$

where $0<\eta_1<\eta_2<\cdots<\eta_n$ are arbitrary constants. Let $u_0=0$, we present the 1-soliton and 2-soliton solutions

$$u_1 = 2\partial_x\ln(\cosh(\eta_1x+a\eta_1y+c\eta_1^3z+b\eta_1^3t))=2\eta_1\tanh(\eta_1x+a\eta_1y+c\eta_1^3z+b\eta_1^3t), \tag{27}$$
$$u_2 = 2\partial_x\ln Wr(\cosh(\eta_1x+a\eta_1y+c\eta_1^3z+b\eta_1^3t),\sinh(\eta_2x+a\eta_2y+c\eta_2^3z+b\eta_2^3t)) = \frac{2(\eta_2^2-\eta_1^2)}{\eta_2\coth(\eta_2x+a\eta_2y+c\eta_2^3z+b\eta_2^3t)-\eta_1\tanh(\eta_1x+a\eta_1y+c\eta_1^3z+b\eta_1^3t)}. \tag{28}$$

Similarly, we can obtain 3-soliton and 4-soliton solutions. These solutions are presented in Figs. 1–4.

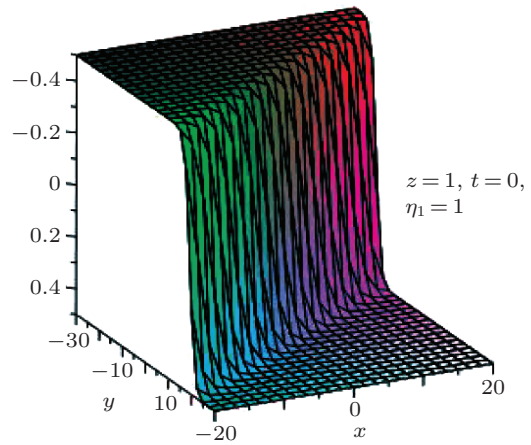


Fig. 1. (color online) 1-soliton solution for $a=1$, $b=1$, $c=2$.

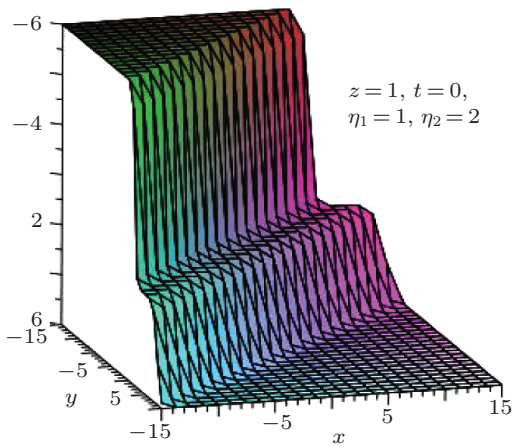


Fig. 2. (color online) 2-soliton solution for $a=1$, $b=1$, $c=2$.

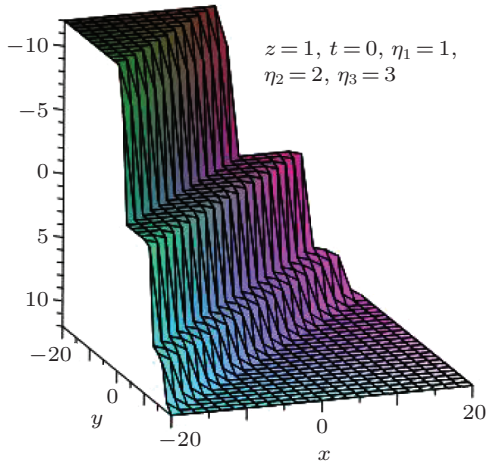


Fig. 3. (color online) 3-soliton solution for $a=1$, $b=1$, $c=2$.

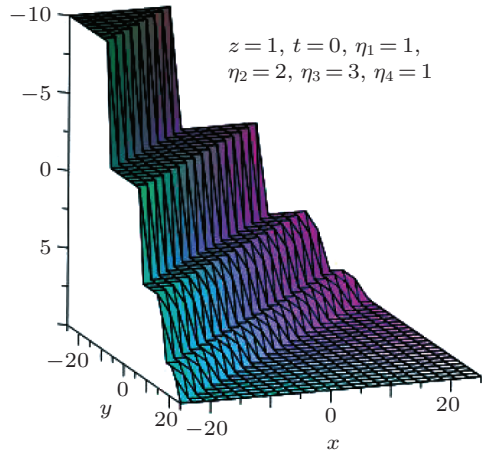


Fig. 4. (color online) 4-soliton solution for $a=1$, $b=1$, $c=2$.

From Figs. 1–4, we can easily observe the characteristics of the solitons. The different scalariform graphs represents different solitons at the same time $t=0$.

2.3. Negatons and positons

Similarly, for the Jordan block (15), substituting block (15) into Eq. (9) and solving the system of differential equations, we get the following two types of special negatons of order n:

$$u = u_0 + 2\partial_x \ln Wr(\phi, \partial_\lambda\phi, \ldots, \partial_\lambda^{n-1}\phi),$$
$$\phi = \cosh(\eta x + a\eta y + c\eta^3 z + b\eta^3 t), \tag{29}$$
$$u = u_0 + 2\partial_x \ln Wr(\phi, \partial_\lambda\phi, \ldots, \partial_\lambda^{n-1}\phi),$$
$$\phi = \sinh(\eta x + a\eta y + c\eta^3 z + b\eta^3 t), \tag{30}$$

where $\lambda<0$, $\eta=\sqrt{-\lambda}$. Equations (29) and (30) are equivalent to each other.

Similarly, we present two types of special positons of order n as follows:

$$u = u_0 + 2\partial_x \ln Wr(\phi, \partial_\lambda\phi, \ldots, \partial_\lambda^{n-1}\phi),$$
$$\phi = \cos(\eta x + a\eta y - c\eta^3 z - b\eta^3 t), \tag{31}$$
$$u = u_0 + 2\partial_x \ln Wr(\phi, \partial_\lambda\phi, \ldots, \partial_\lambda^{n-1}\phi),$$
$$\phi = \sin(\eta x + a\eta y - c\eta^3 z - b\eta^3 t), \tag{32}$$

where $\lambda>0$, $\eta=\sqrt{\lambda}$. Equations (31) and (32) are also equivalent to each other. By means of Eqs. (29)–(32), several exact solutions of lower-order are obtained as

$$\begin{aligned} u_{1-\text{negaton}} &= 2\partial_x \ln(\cosh(\eta x + a\eta y + c\eta^3 z + b\eta^3 t)) \\ &= \frac{-2\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x + a\sqrt{-\lambda}y + c\sqrt{-\lambda}^3 z + b\sqrt{-\lambda}^3 t)}{\cosh(\sqrt{-\lambda}x + a\sqrt{-\lambda}y + c\sqrt{-\lambda}^3 z + b\sqrt{-\lambda}^3 t)}, \end{aligned} \tag{33}$$

$$\begin{aligned} u_{1-\text{positon}} &= 2\partial_x \ln(\cos(\eta x + a\eta y - c\eta^3 z - b\eta^3 t)) \\ &= \frac{-2\sqrt{\lambda}\sin(\sqrt{\lambda}x + a\sqrt{\lambda}y - c\sqrt{\lambda}^3 z - b\sqrt{\lambda}^3 t)}{\cos(\sqrt{-\lambda}x + a\sqrt{\lambda}y - c\sqrt{\lambda}^3 z - b\sqrt{\lambda}^3 t)}, \end{aligned} \tag{34}$$

$$\begin{aligned} u_{2-\text{negaton}} &= 2\partial_x \ln Wr(\cosh(\eta x + a\eta y + c\eta^3 z + b\eta^3 t), \partial_\lambda \cosh(\eta x + a\eta y + c\eta^3 z + b\eta^3 t)) \\ &= \frac{4\sqrt{-\lambda}\cosh^2(\sqrt{-\lambda}x + a\sqrt{-\lambda}y + c\sqrt{-\lambda}^3 z + b\sqrt{-\lambda}^3 t)}{\cosh\theta_1 \sinh\theta_1 + \sqrt{-\lambda}x + a\sqrt{-\lambda}y + 3c\sqrt{-\lambda}^3 z + 3b\sqrt{-\lambda}^3 t}, \end{aligned} \tag{35}$$

$$\begin{aligned} u_{2-\text{positon}} &= 2\partial_x \ln Wr(\cos(\eta x + a\eta y - c\eta^3 z - b\eta^3 t), \partial_\lambda \cos(\eta x + a\eta y - c\eta^3 z - b\eta^3 t)) \\ &= \frac{4\sqrt{\lambda}\cos^2(\sqrt{\lambda}x + a\sqrt{\lambda}y - c\sqrt{\lambda}^3 z - b\sqrt{\lambda}^3 t)}{\cos\theta_2 \sin\theta_2 + \sqrt{\lambda}x + a\sqrt{\lambda}y - 3c\sqrt{\lambda}^3 z - 3b\sqrt{\lambda}^3 t}, \end{aligned} \tag{36}$$

where $\theta_1=\sqrt{\lambda}x + a\sqrt{\lambda}y + c\sqrt{\lambda}^3 z + b\sqrt{\lambda}^3 t$, and $\theta_2=\sqrt{\lambda}x + a\sqrt{\lambda}y - c\sqrt{\lambda}^3 z - b\sqrt{\lambda}^3 t$. Parts of these solutions are given in Figs. 5–8.

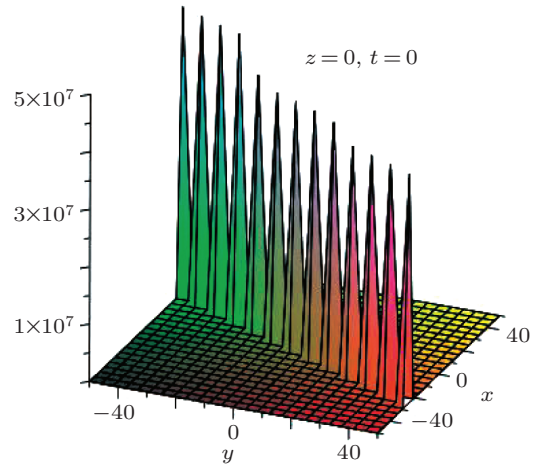


Fig. 5. (color online) 2-negaton solution for $a = 0.5$, $b = 0.5$.

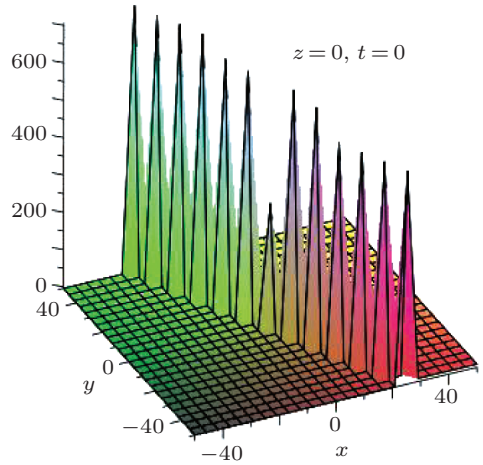


Fig. 6. (color online) 3-negaton solution for $a = 0.5$, $b = 0.5$.

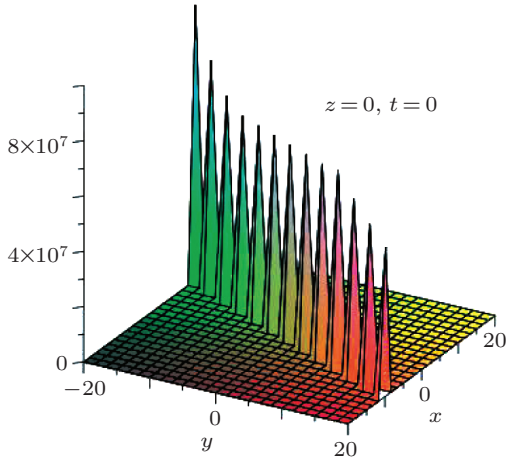


Fig. 7. (color online) 2-positon solution for $a = 0.5$, $b = 0.5$.

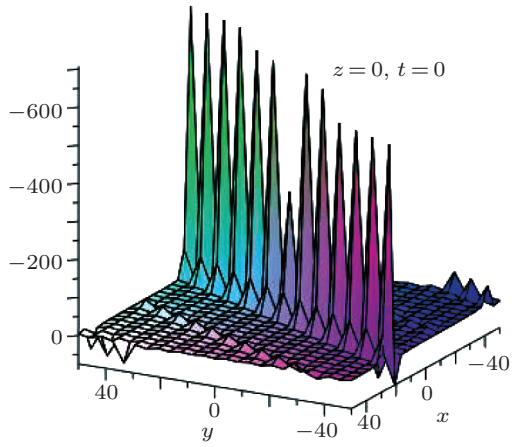


Fig. 8. (color online) 3-positon solution for $a = 0.5$, $b = 0.5$.

3. Grammian solutions

Besides the Wronskian determinant solutions, we can also prove that equation (3) possesses Grammian solutions.[27]

Theorem 2 Equation (3) possesses a solution in terms of the Grammian determinant defined by

$$f_N = \det(a_{i,j})_{1\le i,j\le N},$$
$$a_{ij} = c_{ij} + \int^x f_i g_j \,\mathrm{d}x, \quad c_{ij} = \text{const}, \tag{37}$$

the function $f_i = f_i(x, y, z, t)$ and $g_i = g_i(x, y, z, t)$ satisfy the following conditions:

$$f_{i,y} = a f_{i,x}, \quad f_{i,t} = b f_{i,xxx}, \quad f_{i,z} = c f_{i,xxx},$$
$$g_{i,y} = a g_{i,x}, \quad g_{i,t} = b g_{i,xxx},$$
$$g_{i,z} = c g_{i,xxx}, \quad (i, j = 1, 2, \ldots, N), \tag{38}$$

where the three parameters a, b, c satisfy $2ab - 3c + 4a = 0$.[28]

Proof For convenience, we rewrite f_N as a Pfaffian

$$f_N = (1, 2, \ldots, N, N^*, \ldots, 2^*, 1^*), \tag{39}$$
$$(i, j^*) = a_{ij} = c_{ij} + \int^x f_i g_j \,\mathrm{d}x,$$
$$(i, j) = (i^*, j^*) = 0, \tag{40}$$

where Pfaffians entries $(m, n = 1, 2, \ldots, N)$ are defined by

$$(d_n, j^*) = \frac{\partial^n}{\partial x^n} g_j, \quad (d_m, d_n^*) = 0,$$
$$(d_n^*, i) = \frac{\partial^n}{\partial x^n} f_i, \quad (d_n, i) = (d_m^*, i^*) = 0. \tag{41}$$

Based on the above Pfaffian, the derivatives of the elements a_{ij} of f_N are given by

$$\frac{\partial}{\partial x} a_{ij} = f_i g_j = (d_0, d_0^*, i, j^*), \tag{42}$$

$$\begin{aligned}\frac{\partial}{\partial y}a_{ij} &= a\int^x (f_{i,x}g_j + f_i g_{j,x})\,\mathrm{d}x\\ &= af_ig_j = a(d_0, d_0^*, i, j^*), \qquad (43)\\ \frac{\partial}{\partial t}a_{ij} &= b\int^x (f_{i,xxx}g_j + f_i g_{j,xxx})\,\mathrm{d}x\\ &= b(f_{i,xx}g_j - f_{i,x}g_{j,x} + f_i g_{j,xx})\\ &= b[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*)\\ &\quad + (d_0, d_2^*, i, j^*)], \qquad (44)\\ \frac{\partial}{\partial z}a_{ij} &= c\int^x (f_{i,xxx}g_j + f_i g_{j,xxx})\,\mathrm{d}x\\ &= c(f_{i,xx}g_j - f_{i,x}g_{j,x} + f_i g_{j,xx})\\ &= c[(d_2, d_0^*, i, j^*) - (d_1, d_1^*, i, j^*)\\ &\quad + (d_0, d_2^*, i, j^*)]. \qquad (45)\end{aligned}$$

For convenience, we adopt the simple notation $f_N = (\bullet)$. By means of the above results (42)–(45), we get the derivatives of f_N as

$$\begin{aligned}&f_{N,x} = (d_0, d_0^*, \bullet), \quad f_{N,y} = a(d_0, d_0^*, \bullet),\\ &f_{N,xx} = (d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet),\\ &f_{N,xy} = a[(d_1, d_0^*, \bullet) + (d_0, d_1^*, \bullet)],\\ &f_{N,t} = b[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\ &f_{N,z} = c[(d_2, d_0^*, \bullet) - (d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\ &f_{N,xz} = c[(d_3, d_0^*, \bullet) - (d_1, d_1^*, d_0, d_0^*, \bullet) + (d_0, d_3^*, \bullet)],\\ &f_{N,yt} = ab[(d_3, d_0^*, \bullet) - (d_1, d_1^*, d_0, d_0^*, \bullet) + (d_0, d_3^*, \bullet)],\\ &f_{N,xxx} = [(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\ &f_{N,xxy} = b[(d_2, d_0^*, \bullet) + 2(d_1, d_1^*, \bullet) + (d_0, d_2^*, \bullet)],\\ &f_{N,xxxy} = a[(d_3, d_0^*, \bullet) + 2(d_1, d_1^*, d_0, d_0^*, \bullet)\\ &\qquad + 3(d_1, d_2^*, \bullet) + 3(d_2, d_1^*, \bullet) + (d_0, d_3^*, \bullet)].\end{aligned}$$

By virtue of the identities of determinant, the following identity is easily obtained:

$$\begin{aligned}&[(d_3, d_0^*, \bullet) + 2(d_1, d_1^*, d_0, d_0^*, \bullet)\\ &- (d_1, d_2^*, \bullet) - (d_2, d_1^*, \bullet) + (d_0, d_3^*, \bullet)](\bullet)\\ &= [(d_0, d_1^*, \bullet) - (d_1, d_0^*, \bullet)]^2. \qquad (46)\end{aligned}$$

Substituting the derivatives of f_N and Eq. (46) into Eq. (3), through a tedious calculation, we obtain the following formula:

$$\begin{aligned}&(3f_{xz} - 2f_{yt} - f_{xxxy})f - 3(f_z - f_{xxy})f_x\\ &+ (2f_t + f_{xxx})f_y - 3f_{xy}f_{xx}\\ &= -12a(d_1, d_0^*, \bullet)(d_0, d_1^*, \bullet)\\ &+ 12a(d_1, d_1^*, \bullet)(d_0, d_0^*, \bullet)\\ &- 12a(d_1, d_1^*, d_0, d_0^*, \bullet)(\bullet). \qquad (47)\end{aligned}$$

Obviously, equation (47) satisfies the Jacobi identity, thus it equals zero. Therefore, $f_N = (\bullet)$ is the solution of Eq. (1), and then $u = u_0 + 2\partial_x \ln(1, 2, \ldots, N, N^*, \ldots, 2^*, 1^*)$ is the solution of the Jimbo–Miwa equation.

In order to better understand the Grammian solution, we would like to give several special exact Grammian solutions of lower order. System (38) has solutions in the following form:

$$f_i = \sum_{i=1}^{l} k_i e^{\eta_i},\ \eta_i = p_i x + a p_i y + c {p_i}^3 z + b {p_i}^3 t, \qquad (48)$$

$$g_i = \sum_{i=1}^{l} k_i e^{\eta_i},\ \eta_i = -q_i x - a q_i y - c {q_i}^3 z - b {q_i}^3 t, \qquad (49)$$

where k_i $(i \geq 1)$ are free parameters, l is an arbitrary natural number, p_i and q_i satisfy the dispersion relation

$$3P_i\Omega_i - 2Q_i\Phi_i - Q_i {P_i}^3 = 0, \qquad (50)$$

with $P_i = p_i - q_i$, $Q_i = a(p_i - q_i)$, $\Phi_i = b({p_i}^3 - {q_i}^3)$, and $\Omega_i = c({p_i}^3 - {q_i}^3)$. By means of the dispersion relation (50), we get $q_i = -p_i$, $i \in N^*$. When $l = 1$, equation (3) has the following special solution:[29,30]

$$f = \det(a_{ij}),\ \ a_{ij} = c_{ij} + \int^x f_i^2\,\mathrm{d}x. \qquad (51)$$

And then, we get several special exact Grammian solutions of Eq. (1) as follows:

$$u_1 = u_0 + 2\partial_x \ln(1, 1^*) = u_0 + \frac{4p_1\,\mathrm{e}^{\eta_1}}{2p_1 + \mathrm{e}^{\eta_1}}, \qquad (52)$$

$$\begin{aligned}u_2 &= u_0 + 2\partial_x \ln[(1, 1^*)(2, 2^*) - (1, 2^*)(2, 1^*)]\\ &= u_0 + \frac{2\,\mathrm{e}^{\eta_1} + 2\,\mathrm{e}^{\eta_2} + ({p_1}^{-1} + {p_2}^{-1})\,\mathrm{e}^{\eta_1+\eta_2} - 4(p_1 + p_2)^{-1}\,\mathrm{e}^{2\eta_1+2\eta_2}}{4 + 2{p_2}^{-1}\,\mathrm{e}^{\eta_2} + 2{p_1}^{-1}\,\mathrm{e}^{\eta_1} + {p_1}^{-1}{p_2}^{-1}\,\mathrm{e}^{\eta_1+\eta_2} - 8(p_1 + p_2)^{-1}\,\mathrm{e}^{2\eta_1+2\eta_2}}, \qquad (53)\end{aligned}$$

where $\eta_i = 2p_i x + 2ap_i y + 2c{p_i}^3 z + 2b{p_i}^3 t$, $1 \leq i \leq N$, and p_i are arbitrary real constants. With the help of Maple, take different values for p_i, we display some parts of these solutions in Figs. 9 and 10.

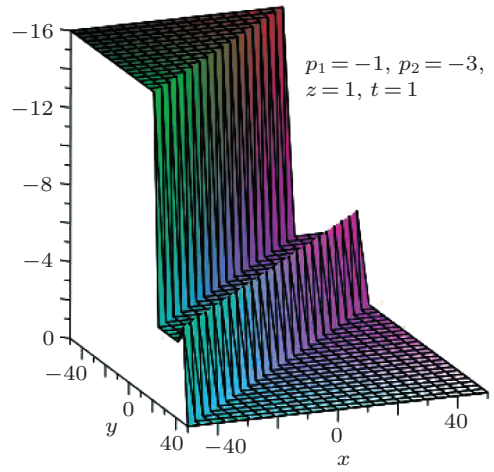


Fig. 9. (color online) 2-Grammian solution for $a = 1$, $b = 1$, $c = 2$.

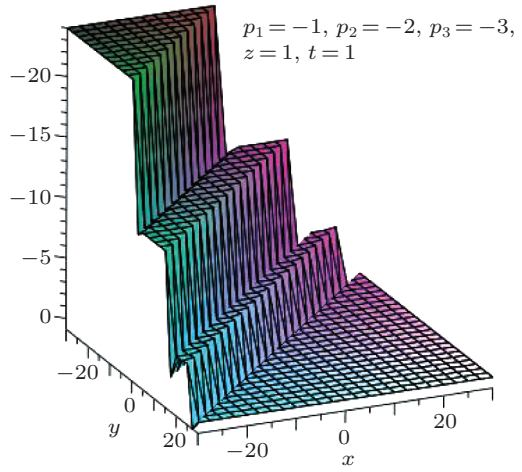


Fig. 10. (color online) 3-Grammian solution for $a = 1$, $b = 1$, $c = 2$.

4. Conclusion

In this paper, a broad set of extended sufficient conditions have been obtained through the Wronskian and Grammian technique, and we have presented the Wronskian and Grammian solutions of the (3+1)-dimensional Jimbo–Miwa equation. By means of the arbitrariness of the coefficients which appear in the conditions, we have obtained some special exact solutions and verified the correctness of these solutions by using Maple. Our results show that equation (1) has different exact solutions.

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