Synchronization between a novel class of fractional-order and integer-order chaotic systems via a sliding mode controller*

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In order to figure out the dynamical behaviour of a fractional-order chaotic system and its relation to an integer-order chaotic system, in this paper we investigate the synchronization between a class of fractional-order chaotic systems and integer-order chaotic systems via sliding mode control method. Stability analysis is performed for the proposed method based on stability theorems in the fractional calculus. Moreover, three typical examples are carried out to show that the synchronization between fractional-order chaotic systems and integer-orders chaotic systems can be achieved. Our theoretical findings are supported by numerical simulation results. Finally, results from numerical computations and theoretical analysis are demonstrated to be a perfect bridge between fractional-order chaotic systems and integer-order chaotic systems.

Keywords: synchronization, fractional-order system, integer-order chaotic system, sliding mode

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1. Introduction

Since the pioneering work of Pecora and Carroll[1] who proposed a successful method to synchronize two identical chaotic systems, chaos synchronization has attracted lots of attention in a variety of research fields over the last two decades. This is because chaos synchronization can be used in many areas of physics and engineering, and in particular in secure communication.[2−5] A variety of approaches have been proposed. Li *et al.*[6] investigated a master-slave synchronization scheme by using a feedback control mechanism with time-varying delay. Zhang and Yang[7] dealt with the lag synchronization of fractional-order chaotic systems with uncertain parameters. Chen *et al.*[8] investigated the chaos control of a class of fractional-order chaotic systems via a sliding mode. A kind of modified scaling function projective synchronization of uncertain chaotic systems using an adaptive controller was studied by Xu.[9] The properties of the leader-laggard chaos synchronization (LLCS) in two mutually coupled external-cavity semiconductor lasers were studied systematically by Jiang *et al.*[10]

Unfortunately, most of the above-mentioned work on chaos synchronization has focused on fractional-order chaos, integer-order systems, and synchronization in practical applications. To the best of our knowledge, there is little information about the synchronization between a fractional-order chaotic system and a chaotic system of integer orders, available in the literature.[11]

There are three advantages which make our approach attractive, as compared with prior work. First, based on sliding mode control (SMC) and stability theorems in the fractional calculus, a new synchronization method is presented for a wide class of nonlinear systems. Second, it guarantees that the synchronization between fractional-order chaotic systems and integer-order chaotic systems can be achieved even though the systems may be identical or different. Third, it is also applicable to the synchronization between different fractional-order chaotic systems.

The rest of this paper is organized as follows. In Section 2 the novel class of fractional-order chaotic systems is briefly presented. Furthermore, a controller is proposed to synchronize chaotic systems introduced based on the active sliding mode control theory. The stability analysis for fractional-order systems

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is discussed in Section 3. In Section 4, three typical examples are given and numerical simulation is carried out to illustrate the effectiveness of the proposed controller. Finally, conclusions are drawn from the present study in Section 5.

2. Problem formulation and control design

Consider the following class of chaotic system as a drive system:

$$D^{q_d}x = A_1x + g_1(x), \tag{1}$$

where $x(t) \in R^3$ denotes a three-dimensional state vector of the system; q_{d_i} are the chaotic orders, and for $q_{d_i} = 1$ $(i = 1, 2, 3)$ system (1) is named as a integer-order system, otherwise it is called a fractional-order system; $A_1 \in R^{3\times3}$ represents the linear part of the system dynamics; $g_1 : R^3 \to R^3$ is the nonlinear part of the system.

A class of chaotic fractional-order response system is described as

$$D^{q_r}y = A_2y + g_2(y), \tag{2}$$

where $y(t) \in R^3$ is a three-dimensional state vector of the response system, $q_r = [q_{r_1},\ q_{r_2},\ q_{r_3}]^{\mathrm{T}}$ are fractional orders satisfying $0 < q_{r_i} \leq 1$, $A_2 \in R^{3\times3}$ and $g_2 : R^3 \to R^3$ imply the same roles as $\boldsymbol{A}_1$ and $\boldsymbol{g}_1$ for the drive system, respectively.

From the viewpoint of control, the synchronization task is to design a nonlinear controller which obtains signals from the drive system (master system) to tune the behavior of the response system (slave system).

The controller $u(t) \in R^3$ is added into the response system (2), so it is given by

$$D^{q_r}y = A_2y + g_2(y) + u(t). \tag{3}$$

To achieve the control law, $u(t)$, the synchronization error is defined as

$$e_i = y_i - x_i. \tag{4}$$

To design a sliding mode controller, we have two steps: the first step is to construct a sliding surface that represents the desired system dynamics, and the next step is to develop a switching control law such that a sliding mode exists at each point of the sliding surface, and any state outside the surface is driven to the surface in a finite time.[12] As a choice for the sliding surface, we have

$$s(t) = Ce, \tag{5}$$

where $\boldsymbol{C} = [c_1,\ c_2,\ c_3]^{\mathrm{T}}$ is a constant vector. In the sliding mode, the sliding surface and its derivative must satisfy

$$s(t) = 0, \quad \dot{s}(t) = 0. \tag{6}$$

The corresponding equivalent control force can be obtained by $\dot{s}(t) = 0$. The following equation fractionalizes the classic derivative into a fractional type as given below.

$$\dot{s}(t) = D^{1-q_r}(D^{q_r}s(t)) = 0 \Rightarrow D^{q_r}s(t) = 0. \tag{7}$$

Substituting Eq. (5) into Eq. (7), then one will obtain

$$D^{q_r}(Ce) = 0 \Rightarrow CD^{q_r}e = 0. \tag{8}$$

Now, let the controller $u(t)$ be

$$u(t) = u_1(t) + u_2(t), \tag{9}$$

where $u_2(t) \in R^3$ is a vector control function, and will be designed. The $u_1(t) \in R^3$ is a compensation controller, and $u_1(t) = D^{q_r}x - A_2x - g_2(x)$. According to Eq. (9), the response system (3) can be rewritten as

$$D^{q_r}e = A_2e + g_2(y) - g_2(x) + u_2(t). \tag{10}$$

In accordance with the active control design procedure, the nonlinear part of the error dynamics is eliminated by the following choice of the input vector:

$$u_2(t) = g_2(x) - g_2(y) + Kw(t). \tag{11}$$

Error system (10) is then rewritten as

$$D^{q_r}e = A_2e + Kw(t), \tag{12}$$

where $K = [k_1,\ k_2,\ k_3]^{\mathrm{T}}$ is a constant gain vector and $w(t) \in R$ is the control input that satisfies

$$w(t) = \begin{cases} w^+(t), & s(e) \geq 0, \\ w^-(t), & s(e) < 0. \end{cases} \tag{13}$$

Subtracting Eq. (12) from Eq. (8) yields

$$C(A_2e + Kw(t)) = 0. \tag{14}$$

To satisfy the sliding condition, the discontinuous reaching law is chosen as follows:

$$Ds(t) = -p\,\mathrm{sign}(s) - rs, \tag{15}$$

where

$$\mathrm{sign}(s) = \begin{cases} +1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0, \end{cases} \tag{16}$$

and $p > 0$, $r > 0$ are the gains of the controller.

In the sliding phase, it implies $Ds(t) = \dot{s}(t) = 0$. Considering Eqs. (14) and (15), one has

$$w(t) = -(CK)^{-1}\left[C(rI + A)e + p\,\mathrm{sign}(s)\right]. \tag{17}$$

Now, the total control law can be defined as follows:

$$\begin{aligned} u(t) = {} & D^{q_r}x - Ax - g(y) - K(CK)^{-1} \\ & \times \left[C(rI + A)e + p\,\mathrm{sign}(s)\right]. \end{aligned} \tag{18}$$

The error dynamics can be obtained by substituting Eq. (17) into Eq. (12) as

$$\begin{aligned} D^{q_r}e = {} & [A - K(CK)^{-1}C(rI + A)]e \\ & - K(CK)^{-1}p\,\mathrm{sign}(s). \end{aligned} \tag{19}$$

Remark 1 If chaotic orders in drive system (1) meet $q_{d_i} = 1$ $(i = 1, 2, 3)$, then the synchronization between fractional-order chaotic systems and chaotic systems of integer orders can be achieved.

Remark 2 If chaotic orders in drive system (1) satisfy $0 < q_{d_i} \leq 1$ $(i = 1, 2, 3)$ except $q_{d_1} = q_{d_2} = q_{d_3} = 1$, then the synchronization between different fractional-order chaotic systems can be achieved.

3. Stability analysis

Theorem 1[13] Consider the following fractional-order system:

$$D^q x = Ax, \quad x(0) = x_0, \tag{20}$$

where $0 < q \leq 1$, $x \in R^n$, and $A \in R^{n\times n}$. System (20) is asymptotically stable when $|\arg(\lambda_i)| > q\pi/2$ where λ_i are the eigenvalues of matrix A. Also, this system is stable when $|\arg(\lambda_i)| \geq q\pi/2$, and those critical eigenvalues that satisfy $|\arg(\lambda_i)| = q\pi/2$ all have geometric multiplicity one.

Theorem 2[13] Consider a system given in the following linear state space form with inner dimension n:

$$\begin{cases} D^q x = Ax + Bu, \\ y = Cx, \end{cases} \quad x(0) = x_0, \tag{21}$$

where $0 < q \leq 1$, $x \in R^n$, $y \in R^p$, and $A \in R^{n\times n}$. Assuming that the triplet (A, B, C) is minimal, system (21) is bounded-input bounded-output stable when $|\arg(\mathrm{eig}\,(A))| > q\pi/2$.

According to Theorem 1 and Theorem 2, the error dynamics on the sliding surface defined by Eq. (18) is asymptotically stable, as long as all eigenvalues of $[A - K(CK)^{-1}C(rI + A)]$ satisfy the condition $|\arg(\lambda_i)| > q\pi/2$. In the sliding phase, as a linear fractional-order system with bounded input ($-K(CK)^{-1}p$ for $s > 0$ and $K(CK)^{-1}p$ for $s < 0$) error system (19) is stable if $\left|\arg(\mathrm{eig}\,([A - K(CK)^{-1}C(rI + A)]))\right| > q\pi/2$. It can be shown that choosing appropriate values of K, C and r can make the error dynamics stable. Hence, the synchronization is realized. And the detailed description of the system dynamics which is affected by the fractional parameters $q_-\{1, 2, 3\}$ can be seen in Ref. [14].

As we can see, for a certain and same parameter matrix $\boldsymbol{A}$, eigenvalues of matrix $\boldsymbol{A}$ are all certain. And thus, the stability of synchronization has relations with varying fractional orders $q_-\{1, 2, 3\}$. When the fractional orders are small, it is easy to satisfy the inequality $|\arg(\mathrm{eig}\,(A))| > q\pi/2$; when the orders are bigger, the possibility of the inequality is smaller, for the matrix $\boldsymbol{A}$ is certain.

However, considering the chaos synchronization, the fractional orders are certain whereas the matrix $[A - K(CK)^{-1}C(rI + A)]$ is uncertain in which the parameter $\boldsymbol{K}$ is to be designed for the error dynamics system. From the dissection, we can easily find that when fractional orders are certain and different, the chosen range of parameter $\boldsymbol{K}$ is different. When the fractional orders are smaller, the chosen range is larger.

4. Numerical simulation results

In this section of the paper we present three illustrative examples to verify and demonstrate the effectiveness of the proposed control scheme. The numerical simulations are carried out using the Caputo version and a predictor–corrector algorithm for fractional-order differential equations, which is the generalization of the Adams–Bashforth–Moulton one in time steps of 0.01.

Case 1 Synchronization between fractional-order and integer-order Chen systems with different parameters.

Consider the Chen system which is written as[15]

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x}{\mathrm{d}t^{q_1}} = a(y-x), \\ \dfrac{\mathrm{d}^{q_2}y}{\mathrm{d}t^{q_2}} = (c-a)x - xz + cy, \\ \dfrac{\mathrm{d}^{q_3}z}{\mathrm{d}t^{q_3}} = xy - bz. \end{cases} \tag{22}$$

Take the integer-order system as a drive system, and the fractional-order Chen chaotic system with chaotic orders $q_{r1} = q_{r2} = q_{r3} = 0.95$ as a response system with the following initial conditions: $[x_d,\ y_d,\ z_d]^{\mathrm{T}} = [0.1,\ 0,\ -0.9]^{\mathrm{T}}$ and $[x_r,\ y_r,\ z_r]^{\mathrm{T}} = [1,\ 1,\ 1]^{\mathrm{T}}$ and system parameters: $(a,\ b,\ c)$=(35, 3, 28) and $(a,\ b,\ c)$=(40, 3, 28). The chaotic attractors are shown in Figs. 1(a) and 1(b), respectively.

We choose the design parameters in the simulations to be $K = [-2,\ -8,\ -2]^{\mathrm{T}}$, $C = [5,\ -5,\ 5]$, $r = 5$ and $p = 0.2$. This selection of parameters results in eigenvalues $(\lambda_1,\ \lambda_2,\ \lambda_3) = (-3.25 + 18.26\mathrm{i},\ -3.25 - 18.26\mathrm{i},\ -5)$ which are located in the stable region. According to Eq. (18), we can yield the controller easily, that is,

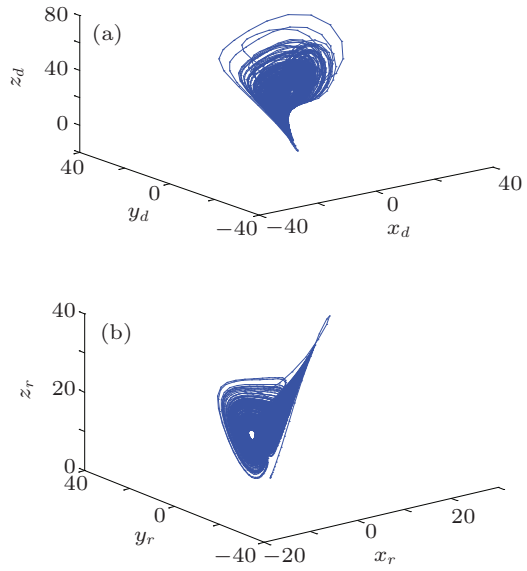


Fig. 1. (color online) Chaotic attractors of an integer-order system (a) and a fractional-order system (b) of Chen systems.

$$\begin{cases} u_1 = \dfrac{\mathrm{d}^{q_{r1}}x_d}{\mathrm{d}t} - \dfrac{23}{2}(x_r - x_d) + \dfrac{7}{2}(y_r - y_d) + (z_r - z_d) + 40x_d - 40y_d + \dfrac{1}{50}\,\mathrm{sign}\,(s_1), \\ u_2 = \dfrac{\mathrm{d}^{q_{r2}}y_d}{\mathrm{d}t} - 46(x_r - x_d) + 14(y_r - y_d) + 4(z_r - z_d) + 12x_d - 28y_d + x_r z_r + \dfrac{2}{25}\,\mathrm{sign}\,(s_2), \\ u_3 = \dfrac{\mathrm{d}^{q_{r3}}z_d}{\mathrm{d}t} - \dfrac{23}{2}(x_r - x_d) + \dfrac{7}{2}(y_r - y_d) + (z_r - z_d) + 3z_d - x_r y_r + \dfrac{1}{50}\,\mathrm{sign}\,(s_3). \end{cases} \tag{23}$$

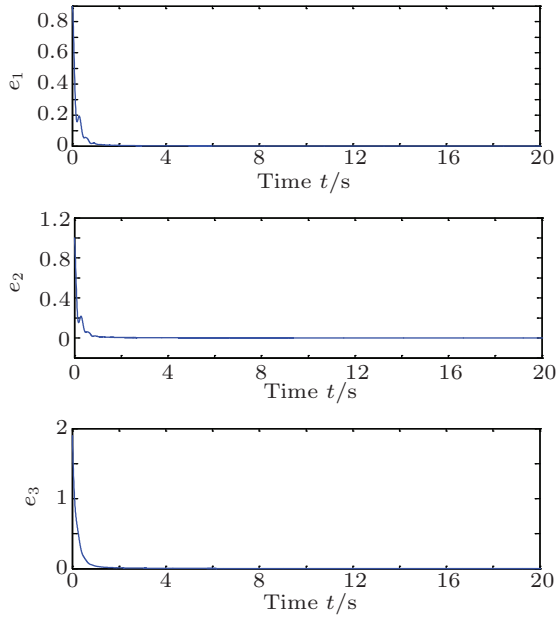


Fig. 2. (color online) Synchronization errors between Chen systems $e_1 = x_r - x_d, e_2 = y_r - y_d, e_3 = z_r - z_d$.

The numerical simulation results are given in Fig. 2. As one can see, the errors converge to zero immediately, which implies that the chaos synchronization between the two systems is realized.

Case 2 Synchronization between a fractional-order Lü chaotic system and an integer-order Liu chaotic system.

Consider the synchronization of a Liu chaotic system with a response fractional-order Lü chaotic system. The Liu chaotic system is described by[16]

$$\begin{cases} \dfrac{\mathrm{d}x_d}{\mathrm{d}t} = a(y_d - x_d), \\ \dfrac{\mathrm{d}y_d}{\mathrm{d}t} = bx_d - kx_d z_d, \\ \dfrac{\mathrm{d}z_d}{\mathrm{d}t} = -cz_d + hx_d^2. \end{cases} \tag{24}$$

The Liu system exhibits a chaotic behaviour at the parameters $(a,\ b,\ c,\ k,\ h) = (10,\ 40,\ 2.5,\ 1,\ 4)$ with

the initial conditions: $[x_d,\ y_d,\ z_d]^{\mathrm{T}}=[10,\ 0,\ 30]^{\mathrm{T}}$, and the chaotic attractor is shown in Fig. 3(a).

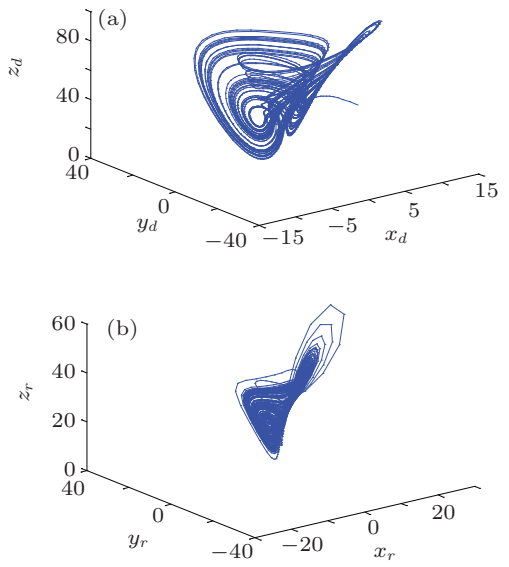


Fig. 3. (color online) Chaotic attractors of an integer-order Liu chaotic system (a) and a fractional-order Lü chaotic system (b).

The fractional-order Lü chaotic system[17,18] is

$$\begin{cases} \dfrac{\mathrm{d}^{q_{r1}}x_r}{\mathrm{d}t^{q_{r1}}} = a(y_r - x_r), \\ \dfrac{\mathrm{d}^{q_{r2}}y_r}{\mathrm{d}t^{q_{r2}}} = -x_r z_r + c y_r, \\ \dfrac{\mathrm{d}^{q_{r3}}z_r}{\mathrm{d}t^{q_{r3}}} = x_r y_r - b z_r. \end{cases} \tag{25}$$

Fractional-order system (25) exhibits a chaotic behaviour. For $q_{r1} = q_{r2} = q_{r3} = 0.90$ and $(a,\ b,\ c)=(35,\ 3,\ 28)$, the chaotic attractor is shown in Fig. 3(b) with the initial values $[x_r,\ y_r,\ z_r]^{\mathrm{T}}=[0,\ 3,\ 9]^{\mathrm{T}}$.

The controller parameters $K = [-2,\ -8,\ -2]^{\mathrm{T}}$, $C = [5,\ -5,\ 5]$, $r = 5$ and $p = 0.2$ are adopted here and the eigenvalues $(\lambda_1,\ \lambda_2,\ \lambda_3) = (-7.5 + 21.812\mathrm{i},\ -7.5 - 21.812\mathrm{i},\ -5)$ are located in the stable region. According to the above-mentioned analyses, we can obtain the controller $u(t)$ for response system (25) as follows:

$$\begin{cases} u_1 = \dfrac{\mathrm{d}^{q_{r1}}x_d}{\mathrm{d}t} - 15(x_r - x_d) + (y_r - y_d) + (z_r - z_d) + 35x_d - 35y_d + \dfrac{1}{50}\,\mathrm{sign}\,(s_1), \\ u_2 = \dfrac{\mathrm{d}^{q_{r2}}y_d}{\mathrm{d}t} - 60(x_r - x_d) + 4(y_r - y_d) + 4(z_r - z_d) - 28y_d + x_r z_r + \dfrac{2}{25}\,\mathrm{sign}\,(s_2), \\ u_3 = \dfrac{\mathrm{d}^{q_{r3}}z_d}{\mathrm{d}t} - 15(x_r - x_d) + (y_r - y_d) + (z_r - z_d) + 3z_d - x_r y_r + \dfrac{1}{50}\,\mathrm{sign}\,(s_3). \end{cases} \tag{26}$$

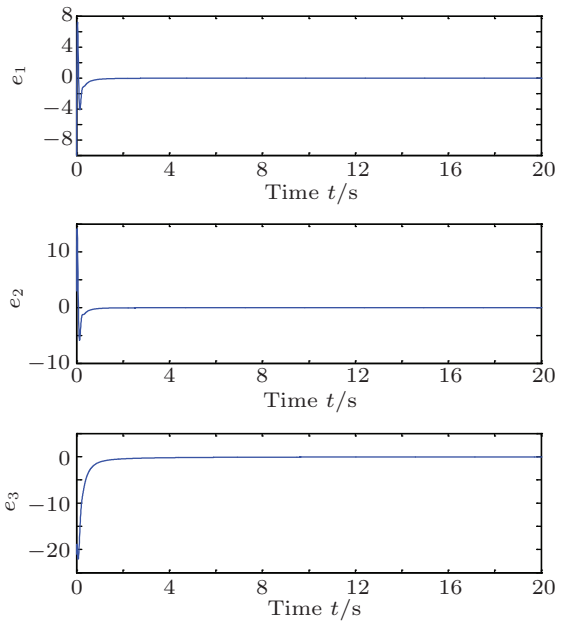


Fig. 4. (color online) Synchronization errors between the integer-order Liu chaotic system and the fractional-order Lü chaotic system $e_1 = x_r - x_d, e_2 = y_r - y_d, e_3 = z_r - z_d$.

The synchronization errors are shown in Fig. 4, showing that the proposed method is successful in synchronizing the two systems.

Case 3 Synchronization of a fractional-order Lü chaotic system with a fractional-order Chen chaotic system.

Take the fractional-order Lü chaotic system mentioned in Case 2 as a drive system, and the fractional-order Chen system presented in Case 1 as a response system.

The controller parameters are chosen to be

$$K = [-2,\ -6,\ -2]^{\mathrm{T}},\quad C = [5,\ -5,\ 5],\quad r = 5,\quad p = 0.2$$

and the eigenvalues

$$(\lambda_1,\ \lambda_2,\ \lambda_3) = (-10 + 26.42\mathrm{i},\ -10 - 26.42\mathrm{i},\ -5)$$

are located in the stable region. Subsequently, according to Eq. (18), the control inputs are taken as

$$\begin{cases} u_1 = \dfrac{\mathrm{d}^{q_{r1}} x_d}{\mathrm{d}t} - 23(x_r - x_d) + 2(y_r - y_d) + 2(z_r - z_d) + 35x_d - 35y_d + \dfrac{1}{25}\operatorname{sign}(s_1), \\ u_2 = \dfrac{\mathrm{d}^{q_{r2}} y_d}{\mathrm{d}t} - 69(x_r - x_d) + 6(y_r - y_d) + 6(z_r - z_d) + 7x_d - 28y_d + x_r z_r + \dfrac{3}{25}\operatorname{sign}(s_2), \\ u_3 = \dfrac{\mathrm{d}^{q_{r3}} z_d}{\mathrm{d}t} - 23(x_r - x_d) + 2(y_r - y_d) + 2(z_r - z_d) + 3z_d - x_r y_r + \dfrac{1}{25}\operatorname{sign}(s_3). \end{cases} \tag{27}$$

Figure 5 illustrates the synchronization errors of the two systems. As one can see, the designed controller is effectively able to synchronize two different fractional-order chaotic systems.

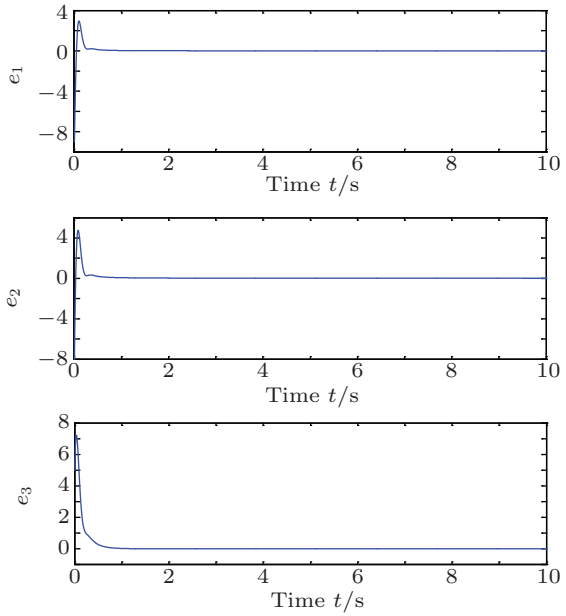


Fig. 5. (color online) Synchronization errors between the fractional-order Lü chaotic system with the fractional-order Chen chaotic system $e_1 = x_r - x_d$, $e_2 = y_r - y_d$, $e_3 = z_r - z_d$.

5. Conclusion

In this paper, we investigated the chaos synchronization of a class of fractional-order systems by utilizing three variant synchronization examples: synchronization between fractional-order system and its integer-order chaotic system with different parameters, fractional-order system synchronized with different forms of integer-order chaotic system, synchronization between different fractional-order chaotic systems. The method can be regarded as a bridge between fractional-order systems and integer-order systems, and also a bridge between different fractional-order systems, which makes systems of different chaotic orders and different types integrate together. The conditions suitable for achieving synchronization based on the fractional stability theory are proposed, which show that the proposed synchronization approach is simple and theoretically rigorous. Numerical simulation results illustrated the effectiveness of the proposed scheme.

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