Stability analysis of nonlinear Roesser-type two-dimensional systems via a homogenous polynomial technique*

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This paper is concerned with the problem of stability analysis of nonlinear Roesser-type two-dimensional (2D) systems. Firstly, the fuzzy modeling method for the usual one-dimensional (1D) systems is extended to the 2D case so that the underlying nonlinear 2D system can be represented by the 2D Takagi–Sugeno (TS) fuzzy model, which is convenient for implementing the stability analysis. Secondly, a new kind of fuzzy Lyapunov function, which is a homogeneous polynomially parameter dependent on fuzzy membership functions, is developed to conceive less conservative stability conditions for the TS Roesser-type 2D system. In the process of stability analysis, the obtained stability conditions approach exactness in the sense of convergence by applying some novel relaxed techniques. Moreover, the obtained result is formulated in the form of linear matrix inequalities, which can be easily solved via standard numerical software. Finally, a numerical example is also given to demonstrate the effectiveness of the proposed approach.

Keywords: stability analysis, Roesser model, two-dimensional nonlinear systems, parameter-dependent Lyapunov function

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1. Introduction

In recent decades, two kinds of two-dimensional (2D) systems[1,2] have attracted considerable attention due to their extensive applications of practical systems,[3−5] such as those in pattern recognition, image data processing and transmission, modeling of thermal process, signal filtering of complex systems, etc. More recently, the 2D system theory has been frequently developed as an analytical tool for some new problems, e.g., the problem of H_∞ control for 2D discrete state delay systems described by the second Fornasini–Marchesini (FM) state-space model has also been investigated in Ref. [6], and the H_∞ filtering design for 2D Markovian jump systems has also been investigated in Ref. [7]. Furthermore, stability analysis of 2D discrete systems described by the FM second model with state saturation was studied in Ref. [8]. However, all the above results are only for linear 2D systems. As is well known, most of the practical 2D systems in the real world belong to nonlinear systems and the above results fail to work in the nonlinear case.

On the other hand, the Takagi–Sugeno (TS) fuzzy model has become a kind of analysis tool for dealing with the problem of stability analysis and control design of complex nonlinear systems;[9−15] essentially because the fuzzy models could effectively approximate a wide class of nonlinear systems by "blending" several local linear sub-systems. The earlier results on stability analysis and control synthesis are based on a common quadratic Lyapunov function.[16,17] It applies a single Lyapunov matrix for all sub-systems and hence the results are rather conservative.[18] With the purpose of further reducing the conservatism, the so-called fuzzy Lyapunov function was proposed in Ref. [19], and its extension versions have been developed in Refs. [20]–[24]. Nowadays, it seems that the challenge is to improve the quality of the relaxations. For instance, in terms of efficiency, i.e., the ability to achieve the same results with lower computational complexity or the ease of extending a given

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stability condition to deal with similar problems. It is worth noting that the underlying 2D nonlinear system's information is propagated along two independent directions and this fact makes the problem of stability analysis more complicated than the usual 1D nonlinear systems.

Motivated by the above concerns, in this paper, the problem of stability analysis of nonlinear Roesser-type 2D systems will be investigated by applying the TS fuzzy modeling approach. Firstly, LMI-based stability conditions are proposed by applying a novel homogeneous polynomially parameter-dependent Lyapunov function (HPPD-LF). Due to extra degrees of freedom being introduced, the conservatism of the obtained stability conditions could be gradually reduced as the degrees of the homogeneous polynomial matrices increase. Secondly, for given homogeneous polynomial matrices, convergent stability conditions are proposed by further exploiting the algebraic property of membership functions, i.e., attaining asymptotically necessary and sufficient stability conditions in a convergent sense. In brief, the main contributions of this paper are summarized as the following. 1) The so-called homogeneous polynomially parameter-dependent Lyapunov function (HPPD-LF) for Roesser-type discrete-time 2D TS fuzzy systems is proposed to implement the problem of stability analysis. Some homogenous polynomial matrices are introduced and hence more design flexibility could be provided than the usual. As the degree of the related homogeneous polynomially parameter-dependent matrix increases, the conservatism of the obtained sufficient conditions may be gradually reduced. 2) It is worth noting that the underlying Roesser-type discrete-time 2D TS system's information is propagated along two independent directions and this fact makes the problem of stability analysis more complicated than for a conventional 1D TS system, i.e., both the horizonal direction and the vertical direction one-step ahead membership functions are produced in the proof of our main result, which makes the stability analysis more complicated. In this paper, the generalization of Polya's theorem for the case of positive polynomials with matrix-valued coefficients is developed for relaxing the relation among the above homogeneous polynomially parameter-dependent matrices, and a systematic procedure for constructing a family of LMI-based stability conditions is given where the exactness is satisfied.

The rest of this paper is organized as follows: following the introduction, a system description and some preliminaries are given in Section 2. Stability analysis of the underlying nonlinear Roesser-type 2D systems is developed in Section 3. In Section 4, an example is given to demonstrate the effectiveness of the proposed approach. Finally, some conclusions are drawn in Section 5.

For simplicity, the notations used are fairly standard. For example, $\boldsymbol{X} > 0$ (or $\boldsymbol{X} \geq 0$) means the matrix $\boldsymbol{X}$ is symmetric and positive-definite (or symmetric and positive-semidefinite). For a square matrix $\boldsymbol{E}$, $\mathrm{He}(\boldsymbol{E})$ is defined as $\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}$. $\boldsymbol{X}^{\mathrm{T}}$ denotes the transpose of $\boldsymbol{X}$. The symbol $\boldsymbol{I}$ represents the identity matrix with an appropriate dimension. For a matrix $\boldsymbol{M}$, $\min(\boldsymbol{M})$ (respectively, $\max(\boldsymbol{M})$) means the smallest (respectively, largest) eigenvalue of $\boldsymbol{M}$. $\mathbb{Z}^+$ denotes the set of negative integers $\{0, 1, 2, ...\}$ and $n!$ represents factorial, i.e., $n! = n(n-1)(n-2)\cdots(2)(1)$ for $n \in \mathbb{Z}^+$ with $0! = 1$. A star $*$ in a symmetric matrix denotes the transposed element in the symmetric position.

2. System description and preliminaries

2.1. Representation of nonlinear Roesser-type discrete-time 2D systems by the TS fuzzy modeling approach

Consider a class of nonlinear Roesser-type discrete-time 2D systems described as follows:

$$\boldsymbol{x}^{+}(s,l) = \mathscr{F}(\boldsymbol{x}(s,l)) \tag{1}$$

with

$$\boldsymbol{x}(s,l) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{h}}(s,l) \\ \boldsymbol{x}^{\mathrm{v}}(s,l) \end{bmatrix},\ \boldsymbol{x}^{+}(s,l) = \begin{bmatrix} \boldsymbol{x}^{\mathrm{h}}(s+1,l) \\ \boldsymbol{x}^{\mathrm{v}}(s,l+1) \end{bmatrix},$$

where $\boldsymbol{x}^{\mathrm{h}}(\cdot,\cdot)$ is the horizonal state in $\mathbf{R}^{n_1}$, $\boldsymbol{x}^{\mathrm{v}}(\cdot,\cdot)$ is the vertical state in $\mathbf{R}^{n_2}$, where n_1 and n_2 are dimensions of the horizonal state vector and the vertical state vector, respectively. $\mathscr{F}(\cdot)$ is general nonlinear function satisfying $\mathscr{F} \in C^1$. s, l are two integers in $\mathbb{Z}^+$. Moreover, the boundary conditions along two independent directions is defined as $\boldsymbol{x}^{\mathrm{h}}(0,l) = f(l), \boldsymbol{x}^{\mathrm{v}}(s,0) = g(s)$, where $f(l)$ and $g(s)$ are corresponding boundary conditions along two independent directions.

Extending the usual 1D TS fuzzy modeling method to the 2D case, a Roesser-type discrete-time 2D TS fuzzy model described by the following rules

could be developed to represent the nonlinear Roesser-type discrete-time 2D system (1)

$$\begin{aligned}&\text{IF } z_1(s,l) \text{ is } M_{i1},\ z_2(s,l) \text{ is } M_{i2},\\&\text{and ..., and } z_L(s,l) \text{ is } M_{iL},\ \text{Then,}\\&\boldsymbol{x}^+(s,l) = A_i\boldsymbol{x}(s,l),\ i=1,\ldots,r,\\&\boldsymbol{x}^{\rm h}(0,l) = f(l),\ \boldsymbol{x}^{\rm v}(s,0) = g(s),\end{aligned} \tag{2}$$

where $z_p(s,l)$, for $p=1,\ldots,L$ are the premise variables, M_{ip} is the fuzzy set, r is the number of IF-THEN rules. Each sub-system matrix A_i belongs to the following structure

$$A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix},$$

with $A_i^{11} \in \mathbf{R}^{n_1\times n_1}$, $A_i^{12} \in \mathbf{R}^{n_1\times n_2}$, $A_i^{21} \in \mathbf{R}^{n_2\times n_1}$, $A_i^{22} \in \mathbf{R}^{n_2\times n_2}$, respectively.

By using the product of inference, the singleton fuzzifier, and the center-average defuzzifer, the overall Roesser-type discrete-time 2D TS fuzzy system can be expressed as follows:

$$\begin{aligned}&\boldsymbol{x}^+(s,l) = \sum_{i=1}^{r} h_i(z(s,l))A_i\boldsymbol{x}(s,l),\\&\boldsymbol{x}^{\rm h}(0,l) = f(l),\ \boldsymbol{x}^{\rm v}(s,0) = g(s),\end{aligned} \tag{3}$$

where

$$\begin{aligned}h_i(z(s,l)) &= \frac{\mu_i(z(s,l))}{\sum_{i=1}^{r}\mu_i(z(s,l))},\\ \mu_j(z(s,l)) &= \Pi_{m=1}^{L} M_{mj}(z(s,l)).\end{aligned}$$

Denoting $X_r = \sup\{\|\boldsymbol{x}(s,l)\| : r = s+l\}$, we then give the definition of asymptotical stability for the Roesser-type discrete-time 2D TS fuzzy system (3).

Definition 1 The Roesser-type discrete-time 2D TS fuzzy system (3) is asymptotically stable if $\lim\limits_{s\to\infty,\ l\to\infty} X_r = 0$ with the initial and boundary conditions $\boldsymbol{x}^{\rm h}(0,l) = f(l)$ and $\boldsymbol{x}^{\rm v}(s,0) = g(s)$.

In the following sections of this paper, for any matrix G_i, the following notations will be adopted for simplicity:

$$\begin{aligned}&h_i = h_i(z(s,l)),\quad \boldsymbol{G}_z = \sum_{i=1}^{r} h_i G_i,\\&\boldsymbol{G}_z^{-1} = \left(\sum_{i=1}^{r} h_i\boldsymbol{G}_i\right)^{-1}.\end{aligned} \tag{4}$$

2.2. Homogeneous matrix polynomials and some useful technical lemmas

The following definitions of homogeneous matrix polynomials are given as follows, which are consistent with those in Ref. [25].

Define $\mathscr{K}(g)$ as the set of r-tuples obtained as all possible combinations of $k_1k_2\ldots k_r$, $k_i \in \mathbb{Z}^+, i=1,\ldots,r$ such that $k_1+k_2+\cdots+k_r = g$. $\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_r^{k_r}, \alpha\in\Delta_r, k_i\in\mathbb{Z}_+, i=1,2,\ldots,r$ are the monomials, $k = k_1k_2\cdots k_r$, and $P_k \in \mathbf{R}^{n\times n}, \forall k \in \mathscr{K}(g)$ are matrix-valued coefficients. Here, by definitions, $\mathscr{K}(g)$ is the set of r-tuples obtained as all possible combinations of nonnegative integers $k_i, i=1,2,\ldots,r$, such that $k_1+k_2+\cdots+k_r = g$. Since the number of vertices is equal to r, the number of elements in $\mathscr{K}(g)$ is given by

$$J(g) = \frac{(r+g-1)!}{g!(r-1)!}. \tag{5}$$

To give an example, for homogeneous polynomials of degree $g=3$ with $r=3$ variables, the possible values of the partial degrees are $\mathscr{K}(3) = \{300,030,003,210,201,120,021,102,012,111\}$, $J(3) = 10$, corresponding to the generic form $P_4(\alpha) = \alpha_1^3P_{300} + \alpha_2^3P_{030} + \alpha_3^3P_{003} + \alpha_1^2\alpha_2P_{210} + \alpha_1^2\alpha_3P_{201} + \alpha_1\alpha_2^2P_{120} + \alpha_2^2\alpha_3P_{021} + \alpha_1\alpha_3^2P_{102} + \alpha_2\alpha_3^2P_{012} + +\alpha_1\alpha_2\alpha_3P_{111}$.

By definition, for r-tuples k and k', one writes $k \geq k'$ if $k_i \geq k_i'$, $(i=1,\ldots,r)$. The usual operations of summation, $k+k'$, and subtraction, $k-k'$ (whenever $k\geq k'$), are defined componentwise. Moreover, let us define some new mathematical notations as follows:

$$\begin{aligned}&\pi(k) \triangleq (k_1!)(k_2!)\cdots(k_r!),\ k\in\mathscr{K}(d),\\&e_i = 0\cdots0\ \underbrace{1}_{i\text{-th}}\ 0\cdots0,\ \text{rec}(e_i) = i,\\&h^k = h_1^{k_1}h_2^{k_2}\cdots h_r^{k_r},\ h^{+k} = h_1^{+k_1}h_2^{+k_2}\cdots h_r^{+k_r},\\&R_g(h) = \sum_{k\in\mathscr{K}(g)} h^kR_k,\quad R_g(h^+) = \sum_{k\in\mathscr{K}(g)} h^{+k}R_k.\end{aligned} \tag{6}$$

For instance, considering homogeneous polynomials of degree $g=2$ with $r=3$ variables, the possible values of the partial degrees are $\mathscr{K}(4) = \{200,020,002,110,101,011\}$, $J(4)=6$, corresponding to the generic forms of $R_g(h)$ and $R_g(h^+)$:

$$\begin{aligned}R_g(h) =\ & h_1^2R_{200} + h_2^2R_{020} + h_3^2R_{002}\\&+ h_1h_2R_{110} + h_1h_3R_{101} + h_2h_3R_{101},\\R_g(h^+) =\ & h_1^{+2}R_{200} + h_2^{+2}R_{020} + h_3^{+2}R_{002}\\&+ h_1^+h_2^+R_{110} + h_1^+h_3^+R_{101} + h_2^+h_3^+R_{011}.\end{aligned}$$

Now, we end this section with two useful lemmas.

Lemma 1[26] For two symmetric matrices $P > 0$, $P_+ > 0$, the inequality $A^{\mathrm{T}}P_+A - P < 0$ holds, if there exist a matrix G such that $\begin{bmatrix} P & (*) \\ GA & G + G^{\mathrm{T}} - P_+ \end{bmatrix} > 0$.

Lemma 2[27] (Polya's Theorem) Let $F(\xi) \doteq F(\alpha_1, \alpha_1, \ldots, \alpha_r)$ be a real homogenous polynomial which is positive $\forall \alpha \in \Delta_N$. Then for a sufficiently large $d \in \mathbb{Z}^+$, the product $(\alpha_1 + \alpha_2 + \cdots + \alpha_r)^d F(\alpha)$ has all its coefficients strictly positive, where Δ_r is defined as follows:

$$\Delta_r = \left\{ \alpha \in \mathbf{R}^r; \sum_{i=1}^{r} \alpha_i = 1; \alpha_i \geq 0 \right\}. \tag{7}$$

3. Main results

In this section, with the purpose of achieving high quality relaxations, novel stability conditions are developed by means of a new homogeneous polynomially parameter-dependent Lyapunov function (HPPD-LF) and a novel slack variable technique.

Theorem 1 The Roesser-type discrete-time 2D TS fuzzy system (3) is asymptotically stable if there exist appropriately dimensional matrices $P_k > 0$, G_k, and $P(kk')$ ($k \in \mathscr{K}(g)$, $k' \in \mathscr{K}(g)$), $g \in \mathbb{Z}^+$, with

$$P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix}, \quad P_k^1 \in \mathbf{R}^{n_1 \times n_1}, \quad P_k^2 \in \mathbf{R}^{n_2 \times n_2},$$

$$G_k = \begin{bmatrix} G_k^1 & 0 \\ 0 & G_{k'}^2 \end{bmatrix}, \quad G_k^1 \in \mathbf{R}^{n_1 \times n_1}, \; G_k^2 \in \mathbf{R}^{n_2 \times n_2},$$

$$P(kk') = \begin{bmatrix} \dfrac{g!}{\pi(k')} P_k^1 & 0 \\ 0 & \dfrac{g!}{\pi(k)} P_{k'}^2 \end{bmatrix},$$

such that the following LMIs hold:

$$\begin{aligned} &\Pi(kk'k'') > 0, \\ &\forall\, k \in \mathscr{K}(g),\; k' \in \mathscr{K}(g),\; k'' \in \mathscr{K}(g+1), \end{aligned} \tag{8}$$

where

$$\begin{aligned} \Pi(kk'k'') = & \sum_{i \in \{1,\ldots,r\}, k''-e_i \geq 0} \frac{g!}{\pi(k)} \frac{g!}{\pi(k')} \\ & \times \begin{bmatrix} P_{k''-e_i} & * \\ G_{k''-e_i} A_i & G_{k''-e_i} + G_{k''-e_i}^{\mathrm{T}} \end{bmatrix} \\ & + \frac{(g+1)!}{\pi(k'')} \begin{bmatrix} 0 & * \\ 0 & -P(kk') \end{bmatrix}. \end{aligned}$$

Proof Consider a new fuzzy Lyapunov function for the Roesser-type discrete-time 2D TS fuzzy system (3) as follows:

$$V(\boldsymbol{x}(s,l)) = \boldsymbol{x}^{\mathrm{T}}(s,l) P_g(h) \boldsymbol{x}(s,l). \tag{9}$$

where the Lyapunov matrix $P_g(h)$ has the form:

$$P_g(h) = \sum_{k \in \mathscr{K}(g)} h^k P_k = \sum_{k \in \mathscr{K}(g)} h^k \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix}.$$

Firstly, let us check the validity of the fuzzy Lyapunov function (9). Noting that if the LMI-based stability conditions of Theorem 1 hold, it follows from inequality (8) that $P_g(h) > 0$ always holds, which ensures that the fact $\min(P_g(h)) > 0$ exists. Therefore, we could write

$$\underline{\lambda} \boldsymbol{x}^{\mathrm{T}}(s,l) \boldsymbol{x}(s,l) \leqslant V(\boldsymbol{x}(s,l)) \leqslant \overline{\lambda} \boldsymbol{x}^{\mathrm{T}}(s,l) \boldsymbol{x}(s,l), \tag{10}$$

where $\underline{\lambda} = \min(P_g(h))$ and $\overline{\lambda} = \max(P_g(h))$. Thus, inequality (10) ensures that $V(\boldsymbol{x}(s,l))$ is a candidate Lyapunov function.

Then, the variation of $V(\boldsymbol{x}(s,l))$ could be written as

$$\begin{aligned} \triangle V(\boldsymbol{x}(s,l)) = & \boldsymbol{x}^{\mathrm{T}}(s,l) \left(A_z^{\mathrm{T}} P_g(h^+) A_z - P_g(h) \right) \\ & \times \boldsymbol{x}(s,l), \end{aligned} \tag{11}$$

where

$$P_g(h^+) = \begin{bmatrix} \sum\limits_{k \in \mathscr{K}(g)} h(z(s+1,l))^k P_k^1 & 0 \\ 0 & \sum\limits_{k' \in \mathscr{K}(g)} h(z(s,l+1))^{k'} P_{k'}^2 \end{bmatrix},$$

$$h(z(s+1,l))^k = h_1(z(s+1,l))^{k_1} \times h_2(z(s+1,l))^{k_2} \times \cdots \times h_r(z(s+1,l))^{k_r},$$

$$h(z(s,l+1))^{k'} = h_1(z(s,l+1))^{k_1'} \times h_2(z(s,l+1))^{k_2'} \times \cdots \times h_r(z(s,l+1))^{k_r'}.$$

From inequality (11), it could be concluded that $\Delta V(\boldsymbol{x}(s,l)) < 0$ holds if one has the following inequality

$$A_z^{\mathrm{T}} P_g(h^+) A_z - P_g(h) < 0. \tag{12}$$

Using Lemma 1 with $A = A_z$ and $G = G_g(h)$, inequality (12) is ensured by the following inequality:

$$\begin{bmatrix} P_g(h) & * \\ G_g(h)A_z & G_g(h) + G_g(h)^{\mathrm{T}} - P_g(h^+) \end{bmatrix} > 0. \tag{13}$$

On the other hand, we have

$$\begin{aligned} &\begin{bmatrix} P_g(h) & * \\ G_g(h)A_z & G_g(h) + G_g(h)^{\mathrm{T}} - P_g(h^+) \end{bmatrix} \\ &= \sum_{k\in\mathscr{K}(g)} \sum_{k'\in\mathscr{K}(g)} h(z(s+1,l))^k h(z(s,l+1))^{k'} \begin{bmatrix} \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}P_g(h) & * \\ \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}G_g(h)A_z & \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}(G_g(h) + G_g(h)^{\mathrm{T}}) - P(kk') \end{bmatrix} \\ &= \sum_{k\in\mathscr{K}(g)} \sum_{k'\in\mathscr{K}(g)} \sum_{k''\in\mathscr{K}(g+1)} h(z(s+1,l))^k h(z(s,l+1))^{k'} h^{k''} \times \mathit{\Pi}(kk'k''), \end{aligned} \tag{14}$$

where we have

$$\mathit{\Pi}(kk'k'') = \sum_{i\in\{1,\ldots,r\},k''-e_i\geq 0} \frac{g!}{\pi(k)}\frac{g!}{\pi(k')} \begin{bmatrix} P_{k''-e_i} & * \\ G_{k''-e_i}A_i & G_{k''-e_i} + G_{k''-e_i}^{\mathrm{T}} \end{bmatrix} + \frac{(g+1)!}{\pi(k'')} \begin{bmatrix} 0 & * \\ 0 & -P(kk') \end{bmatrix}.$$

Thus, if $\mathit{\Pi}(kk'k'') > 0$ for all $k \in \mathscr{K}(g)$, $k' \in \mathscr{K}(g)$, $k'' \in \mathscr{K}(g+1)$ hold, inequality (13) evidently holds. In other words, inequality (12) holds if those LMIs (8) hold, which guarantee the asymptotical stability for the Roesser-type discrete-time 2D TS fuzzy system (3).

This completes the proof.

Remark 1 In the above theorem, LMI-based stability conditions for the Roesser-type discrete-time 2D TS fuzzy system (3) are developed by means of a new HPPD-LF. It is worth noting that the HPPD-LF (9) reduces to the usual fuzzy Lyapunov function when $g = 1$, i.e., the conventional fuzzy Lyapunov function is a special case of ours. On the other hand, more slack matrix variables will be introduced while the value of g increases, and hence less conservative stability conditions could be obtained by using Theorem 1 with a larger g. As the value of g increases, less conservative results will be obtained while the resulting computational burden will also increase a lot as a trade-off. In the following, for a given value of g, we will exploit the algebraic property of membership functions for developing relaxed stability conditions for the Roesser-type discrete-time 2D TS fuzzy system in a convergent sense.

Theorem 2 The Roesser-type discrete-time 2D TS fuzzy system (3) is asymptotically stable if there exist appropriately dimensional matrices $P_k > 0$, G_k, and $P(kk')$ ($k \in \mathscr{K}(g)$, $k' \in \mathscr{K}(g)$), $g \in \mathbb{Z}^+$, with

$$P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix}, \quad P_k^1 \in \mathbf{R}^{n_1\times n_1}, \quad P_k^2 \in \mathbf{R}^{n_2\times n_2},$$

$$G_k = \begin{bmatrix} G_k^1 & 0 \\ 0 & G_k^2 \end{bmatrix}, \quad G_k^1 \in \mathbf{R}^{n_1\times n_1}, \quad G_k^2 \in \mathbf{R}^{n_2\times n_2},$$

$$P(kk') = \begin{bmatrix} \dfrac{g!}{\pi(k')}P_k^1 & 0 \\ 0 & \dfrac{g!}{\pi(k)}P_{k'}^2 \end{bmatrix},$$

such that the following LMIs hold,

$$\mathit{\Xi}(kk'k'') > 0, \quad \forall\, k \in \mathscr{K}(g), \quad k' \in \mathscr{K}(g), \quad k'' \in \mathscr{K}(g+d+1), \tag{15}$$

where

$$\begin{aligned} \mathit{\Xi}(kk'k'') = &\sum_{kg\in\mathscr{K}(g),i\in\{1,\ldots,r\},k''-kg-e_i\geq 0} \frac{d!}{\pi(k''-kg-e_i)}\frac{g!}{\pi(k)} \times \frac{g!}{\pi(k')} \begin{bmatrix} P_{kg} & * \\ G_{kg}A_i & G_{kg} + G_{kg}^{\mathrm{T}} \end{bmatrix} \\ &+ \frac{(g+d+1)!}{\pi(k'')} \begin{bmatrix} 0 & * \\ 0 & -P(kk') \end{bmatrix}. \end{aligned}$$

Proof From expressions (13) and (14) in the proof of Theorem 1, it follows that discrete-time 2D TS fuzzy system (3) is asymptotically stable if the following matrix inequality holds:

$$\Gamma = \begin{bmatrix} \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}P_g(h) & * \\ \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}G_g(h)A_z & \dfrac{g!}{\pi(k)}\dfrac{g!}{\pi(k')}(G_g(h)+G_g(h)^{\mathrm{T}})-P(kk') \end{bmatrix} > 0, \quad \forall k \in \mathscr{K}(g),\ k' \in \mathscr{K}(g). \tag{16}$$

Recalling the fact that $h_1 + h_2 + \cdots + h_r = 1$, it is easily seen that the inequality (16) is equivalent to the following inequality:

$$\begin{aligned} \Gamma &= (h_1 + h_2 + \cdots + h_r)^d \Gamma \\ &= \sum_{k'' \in \mathscr{K}(g+d+1)} h^{k''} \times \Xi(kk'k''), \\ &> 0, \end{aligned} \tag{17}$$

where $\Xi(kk'k'')$ has been defined by inequality (15), $k \in \mathscr{K}(g)$, $k' \in \mathscr{K}(g)$, $k'' \in \mathscr{K}(g+d+1)$, and $d \in \mathbb{Z}^+$.

From expression (17), if $\Xi(kk'k'') > 0$ for any $k \in \mathscr{K}(g)$, $k' \in \mathscr{K}(g)$, and $k'' \in \mathscr{K}(g+d+1)$, then the inequality (16) evidently holds. In other words, the discrete-time 2D TS fuzzy system (3) is asymptotically stable if the LMI-based condition (15) holds.

By using Lemma 2 (named Polya's Theorem), we have

$$\Gamma > 0 \Longleftrightarrow \Xi(kk'k'') > 0, \exists d \in \mathbb{Z}^+, \tag{18}$$

which means that expression (18) becomes a convergent condition for ensuring expression (16) while the value of d increases, i.e., LMI-based stability condition (15) is a kind of asymptotically sufficient and necessary stability condition for Roesser-type discrete-time 2D TS fuzzy system (3).

This completes the proof.

Remark 2 Since the generalization of Polya's theorem for the case of positive polynomials with matrix-valued coefficients is elaborately used for relaxing the relation among the above homogeneous polynomially parameter-dependent matrices, a systematic procedure for constructing a family of LMI-based stability conditions is developed in Theorem 2 where the exactness is also satisfied. In the past decades, 2D systems have attracted considerable attention due to their extensive applications of practical systems, such as those in image data processing and transmission, thermal process, signal filtering, etc. Nowadays, stability analysis of 2D nonlinear systems deserves to be investigated by researchers in this field, which motivates us to carry out out study. Actually, the results proposed in this paper could be extended to other research fields for Roesser-type 2D nonlinear systems, such as robust control synthesis, filtering design, repetitive process control, etc.

4. Numerical examples

Consider a nonlinear differential equation which is described as follows:

$$\frac{\partial^2 q(x,t)}{\partial x \partial t} = a_1 \frac{\partial q(x,t)}{\partial t} + a_2 \frac{\partial q(x,t)}{\partial x} + a_0 \sin^2(q(x,t))q(x,t),$$

where the initial and boundary conditions $q(x,0) = q_1(x)$ and $q(0,t) = q_2(t)$, $q(x,t)$ is the variable function, a_0, a_1, a_2 are real coefficients.

First, we establish the state space model for the above nonlinear differential equation. Let us define

$$\begin{aligned} x_c^{\mathrm{h}}(x,t) &= \frac{\partial q(x,t)}{\partial t} - a_2 q(x,t), \\ x_c^{\mathrm{v}}(x,t) &= q(x,t), \end{aligned}$$

then the following 2D state space model can be easily obtained

$$\begin{bmatrix} \dfrac{\partial x_c^{\mathrm{h}}(x,t)}{\partial x} \\ \dfrac{\partial x_c^{\mathrm{v}}(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \sin^2(x_c^{\mathrm{v}}(x,t)) \\ 1 & a_2 \end{bmatrix} \times \begin{bmatrix} x_c^{\mathrm{h}}(x,t) \\ x_c^{\mathrm{v}}(x,t) \end{bmatrix},$$

with boundary conditions: $x_c^{\mathrm{h}}(0,t) = \dot{q}_2(t) - a_2 q_2(t)$, $x_c^{\mathrm{v}}(x,t) = q_1(x)$.

To obtain a 2D TS fuzzy representation for this 2D nonlinear system, consider the two following rules obtained for $\sin^2(x_c^{\mathrm{v}}(x,t))$:
IF $\sin^2(x_c^{\mathrm{v}}(x,t))$ is about 0, THEN

$$\begin{bmatrix} \dfrac{\partial x_c^{\rm h}(x,t)}{\partial x} \\ \dfrac{\partial x_c^{\rm v}(x,t)}{\partial t} \end{bmatrix} = A_1^c \times \begin{bmatrix} x_c^{\rm h}(x,t) \\ x_v^{\rm h}(x,t) \end{bmatrix};$$

IF $\sin^2(x_c^{\rm v}(x,t))$ is about ∓ 1, THEN

$$\begin{bmatrix} \dfrac{\partial x_c^{\rm h}(x,t)}{\partial x} \\ \dfrac{\partial x_c^{\rm v}(x,t)}{\partial t} \end{bmatrix} = A_2^c \times \begin{bmatrix} x_c^{\rm h}(x,t) \\ x_v^{\rm h}(x,t) \end{bmatrix}.$$

Here, $A_1^c = \begin{bmatrix} a_1 & a_1a_2 \\ 1 & a_2 \end{bmatrix}$, $A_2^c = \begin{bmatrix} a_1 & a_1a_2 + a_0 \\ 1 & a_2 \end{bmatrix}$.

Without loss of generality, we choose the membership functions as:

$$h_1(x,t) = 1 - \sin^2(x_c^{\rm v}(x,t)),\ h_2(x,t) = \sin^2(x_c^{\rm v}(x,t)).$$

The above 2D TS fuzzy systems are discretized with sampling times T_1 and T_2 corresponding to variables x and t, respectively. The obtained discrete-time 2D fuzzy systems are given by:
IF $\sin^2(x^{\rm v}(k,l))$ is about 0, THEN

$$\begin{bmatrix} x^{\rm h}(k+1,l) \\ x^{\rm v}(k,l+1) \end{bmatrix} = A_1 \times \begin{bmatrix} x^{\rm h}(k,l) \\ x^{\rm v}(k,l) \end{bmatrix};$$

IF $\sin^2(x^{\rm v}(k,l))$ is about ∓ 1, THEN

$$\begin{bmatrix} x^{\rm h}(k+1,l) \\ x^{\rm v}(k,l+1) \end{bmatrix} = A_2 \times \begin{bmatrix} x^{\rm h}(k,l) \\ x^{\rm v}(k,l) \end{bmatrix}.$$

Here,

$$A_1 = \begin{bmatrix} 1 + a_1T_1 & a_1a_2T_1 \\ T_2 & 1 + a_2T_2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 + a_1T_1 & (a_1a_2 + a_0)T_1 \\ T_2 & 1 + a_2T_2 \end{bmatrix}.$$

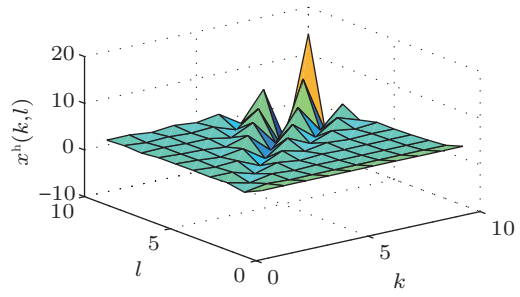
Then, the membership functions of the above discrete-time Roesser-type 2D TS fuzzy system become $h_1(k,l) = 1 - \sin^2(x^{\rm v}(k,l)), h_2(k,l) = \sin^2(x^{\rm v}(k,l))$. Consider the following parameter values about a_0, a_1, T_1, and T_2: $a_0 = -2, a_1 = -3, T_1 = 0.4, T_2 = 0.7$, then we can calculate the feasible parameter intervals by evaluating the feasibility of the associated LMI problems with Theorems 1 and Theorem 2 for varying values of a_2.

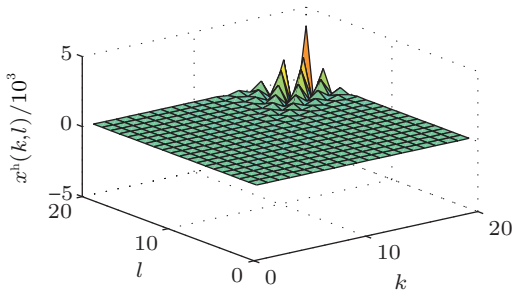
Table 1 shows the parameter feasible intervals of a_2 in which the discrete-time 2-D T–S fuzzy system is asymptotically stable could be ensured by applying the approaches provided by Theorem 1 and Theorem 2 of this paper, respectively. In details: Method 1 −− Theorem 1 with $g = 1$; Method 2−− Theorem 1 with $g = 2$; Method 3−− Theorem 2 with $g = 2$ and $d = 4$;. From Table 1, it can be seen that Method 3 provides the most relaxed results.

Furthermore, Figs. 1–4 show the state trajectory of the system state variables $x^h(k,l)$ and $x^v(k,l)$ with $a_2 = -1.8$ and $a_2 = -1.117$, respectively. From Table 1, one could find that both $a_2 = -1.8$ is out of the calculated stability region by using the approach proposed in this paper, and their state trajectory of the system state variables are not asymptotically stable from Figs. 1 and 2. On the other hand, from Table 1, one could find that both $a_2 = -1.117$ is within the calculated stability region by using the approach proposed in this paper, and their state trajectory of the system state variables are asymptotically stable from Figs. 3 and 4. Therefore, the effectiveness of the proposed approach is illustrated in the above numerical example.

Table 1. Feasible stability parameter intervals of a_2 via several methods proposed in this paper.

Methods	Method 1	Method 2	Method 3
Intervals	[–1.116, –0.461]	[–1.116, –0.429]	[–1.117, –0.426]





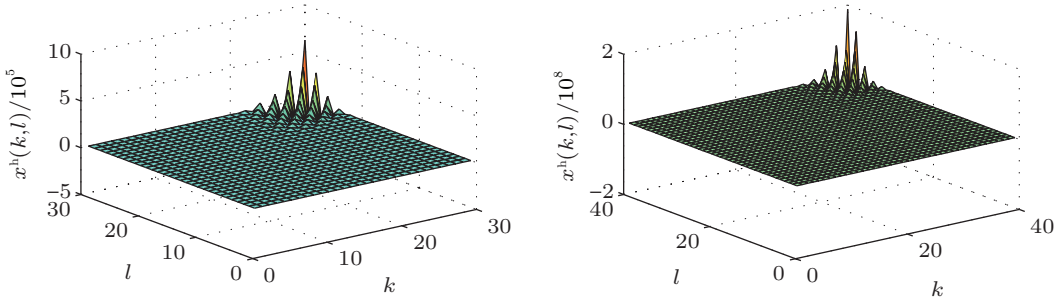


Fig. 1. (color online) State trajectory of system state variable $x^{\rm h}(k,l)$ with $a_2 = -1.8$.

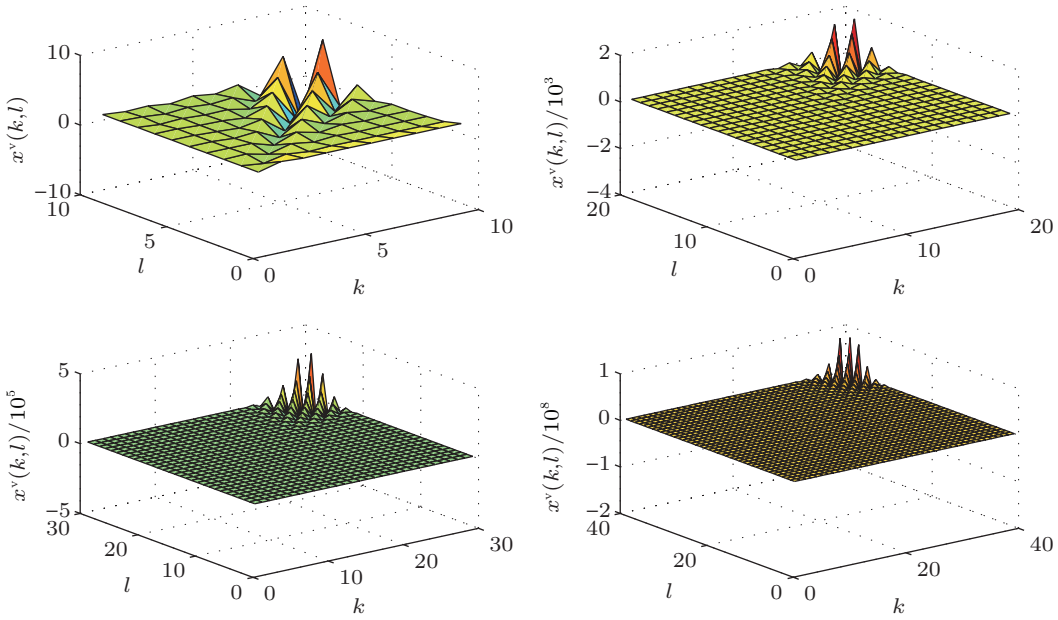


Fig. 2. (color online) State trajectory of the system state variable $x^{\rm v}(k,l)$ with $a_2 = -1.8$.

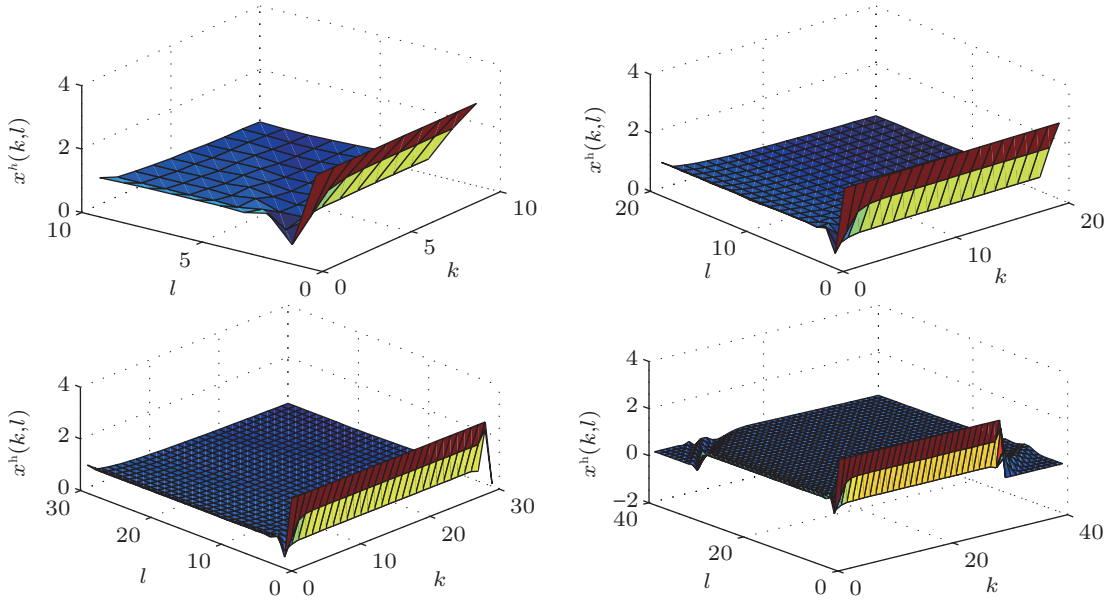


Fig. 3. (color online) State trajectory of the system state variable $x^{\rm h}(k,l)$ with $a_2 = -1.117$.

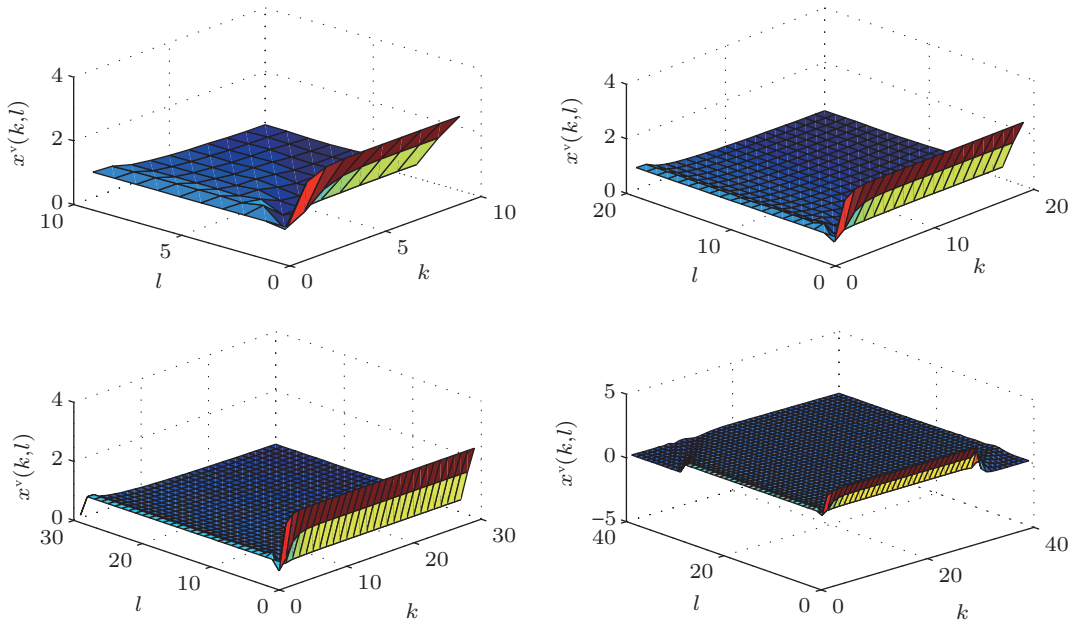


Fig. 4. (color online) State trajectory of the system state variable $x^{\mathrm{v}}(k,l)$ with $a_2 = -1.117$.

5. Conclusions

The problem of stability analysis of nonlinear Roesser-type 2D systems has been investigated by using the TS fuzzy modeling method. A new homogeneous polynomially parameter-dependent Lyapunov function and some novel relaxed techniques have been proposed to develop the main results. More importantly, the LMI-based stability conditions approach to exactness in the sense of convergence by exploiting the algebraic property of membership functions. Numerical results have illustrated the effectiveness of the proposed approach.

References

[1] Roesser R P 1975 *IEEE Trans. Autom. Control* **20** 1

[2] Fornasini E and Marchesini G 1976 *IEEE Trans. Autom. Control* **21** 484

[3] Owens D H, Amann N, Rogers E and French M 2000 *Multidimensional Systems and Signal Processing* **11** 125

[4] Sulikowski B, Galkowski K, Rogers E and Owens D H 2004 *Automatica* **40** 2167

[5] Sulikowski B, Galkowski K, Rogers E and Owens D H 2006 *Automatica* **42** 877

[6] Xu J M and Yu L 2008 *Automatica* **44** 809

[7] Wu L G, Shi P, Gao H J and Wang C H 2008 *Automatica* **44** 1849

[8] Singh V 2008 *IEEE Trans. Circ. Sys.* II*: Express Briefs* **55** 793

[9] Li D, Wang H, Yang D, Zhang X H and Wang S L 2008 *Chin. Phys.* B **17** 4091

[10] Zhao Y, Zhang H G and Zheng C D 2008 *Chin. Phys.* B **17** 529

[11] Zhang H G, Zhao Y, Yu W and Yang D S 2008 *Chin. Phys.* B **17** 4056

[12] Ma D Z, Zhang H G, Wang Z S and Feng J 2010 *Chin. Phys.* B **19** 050506

[13] Sun Q Y, Zhang H G and Zhao Y 2010 *Chin. Phys.* B **19** 070512

[14] Feng Y F and Zhang Q L 2011 *Chin. Phys.* B **20** 010101

[15] Gong D W, Zhang H G and Wang Z S 2012 *Chin. Phys.* B **21** 030204

[16] Kim E and Lee H 2000 *IEEE Trans. Fuzzy Sys.* **8** 523

[17] Sala A and Arino C 2007 *Fuzzy Sets and Systems* **24** 2671

[18] Montagner V F, Oliveira R and Peres P 2009 *IEEE Trans. Fuzzy Sys.* **17** 863

[19] Guerra T M and Vermeiren L 2004 *Automatica* **40** 823

[20] Kruszewski A, Wang R and Guerra T M 2008 *IEEE Trans. Autom. Control* **53** 606

[21] Zhang H G, Xie X P and Tong S C 2011 *IEEE Trans. Sys. Man, Cyb.* Part B: *Cybernetics* **41** 1313

[22] Zhang H G and Xie X P 2011 *IEEE Trans. Fuzzy Sys.* **19** 478

[23] Jiang B, Zhang K and Shi P 2011 *IEEE Transactions on Fuzzy Systems* **19** 291

[24] Zhang K, Jiang B and Shi P 2011 *IEEE Transactions on Fuzzy Systems* **20** 192

[25] Oliveira R and Peres P 2007 *IEEE Transactions on Automatic Control* **52** 1334

[26] De Oliveira J V, Bernussou J and Geromel J C 1999 *Systems & Control Letters* **37** 261

[27] Hardy G H, Littlewood J E and Polya G 1952 *Inequalities* 2nd edn. (Cambridge: Cambridge University Press)