# Stochastic resonance of a damped oscillator with frequency fluctuation driven by a periodic external force*

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Considering a damped linear oscillator model subjected to a white noise with an inherent angular frequency and a periodic external driving force, we derive the analytic expression of the first moment of output response, and study the stochastic resonance phenomenon in a system. The results show that the output response of this system behaves as a simple harmonic vibration, of which the frequency is the same as the external driving frequency, and the variations of amplitude with the driving frequency and the inherent frequency present a bona fide stochastic resonance.

**Keywords:** frequency noise, the first moment, stochastic resonance

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## 1. Introduction

It is well known that noise has two opposite natures, of which the disadvantages are often observed, and the advantage has a good manifestation in the phenomenon of stochastic resonance (SR). SR had been generated only in nonlinear systems before it was found in a linear system in the early 1990s.[1−7] On the basis of the conventional SR described by the relation between signal-to-noise ratio and the noise intensity, Berdichevsky and Gitterman[8] in 1999 first found SR in a broad sense, which exists as non-monotonic behaviour of the other quantity (such as moment, correlation function, power spectrum, etc.) of system response to input parameters except noise intensity, such as driving frequency, driving amplitude, or noise-correlation-time. In previous studies, the one-order differential Langevin equation is considered, and it is often limited in the case of over damping. In recent years, the study of the two-order stochastic differential equation has aroused the wide interest of investigators. For example, Wang *et al.*[9] in 1996 performed experimental research on the SR in a Duffing system by a simulation method using an electronic circuit, and found that the output signal-to-noise ratio can be increased by changing the damping coefficient in a certain condition. The linear harmonic oscillator driven by noise is a stochastic dynamic system, and some researchers have done a lot of work on it. For example, Gitterman[10] studied the output signal of an undamped linear oscillator with a random frequency subjected to a periodic external force, and showed non-monotonic dependence on the strength and the coloured noise rate. Jin and Hu[11] in 2009 investigated a linearly damped oscillator subjected to random coloured noise and periodic modulation noise, and found that the amplitude of the output signal presents conventional SR, bona fide SR, and SR in a broad sense.[11]

For a stochastic dynamic system, we found that the noise probably exists in the phase of frequency-modulated signal using the SR method,[12] and indicated that the frequency drift of water acoustic wave signal detected in a low frequency signal spectrum is an example of phase fluctuation.[13] We studied the SR of a single-mode laser gain model with frequency fluc-

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tuation and driven by a periodic signal,[14] and found SR existing in variations of the signal-to-noise ratio with the intensity of frequency noise, pump noise, and quantum noise.

The spring system with frequency fluctuation is a widespread model,[15,16] and has become an important research object in physics, biology, chemistry, and engineering. In this paper, we investigate the analytic expression of the first moment of linear oscillator, which is subjected to a white noise with inherent frequency and a periodic external driving force, and then study the SR in the system. The results show that the output response behaves as a simple harmonic vibration, and bona fide SR appears in the variations of amplitude.

## 2. First moment of damped oscillator subjected to periodic external driving and frequency noise

The dynamics of a linear harmonic oscillator can be described by the following differential equation:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2n\frac{\mathrm{d}x}{\mathrm{d}t}+\omega^2x=0. \tag{1}$$

By solving the above equation, we can obtain

$$x(t)=x_0\,\mathrm{e}^{-nt}\cos(\omega_1 t)+\frac{u_0+nx_0}{\omega_1}\,\mathrm{e}^{-nt}\sin(\omega_1 t), \tag{2}$$

where $\omega_1=\sqrt{\omega^2-n^2}$, with $\omega$ being the inherent frequency, $n$ the damping coefficient ($\omega>n$), $x_0$ and $u_0$ the initial displacement and velocity of the oscillator, respectively. If the system is driven by a noise, subjected to an inherent frequency and a periodic external force, from Eq. (1), we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2n\frac{\mathrm{d}x}{\mathrm{d}t}+(\omega+\xi(t))^2x=\cos\Omega t, \tag{3}$$

where $\xi(t)$ is white noise, whose statistical properties are

$$\langle\xi(t)\rangle=0,\ \langle\xi(t)\xi(s)\rangle=D\delta(t-s). \tag{4}$$

We set $\tilde{\omega}=\sqrt{(\omega+\xi(t))^2-n^2}$, which is the instantaneous angular frequency. We expand $\tilde{\omega}$ using Taylor series, and neglect their products in second and higher orders, then we can obtain

$$\tilde{\omega}\approx\omega_1+\frac{\omega}{\omega_1}\xi(t).$$

The relation between the instantaneous angular frequency and the phase $\phi(t)$ is

$$\tilde{\omega}=\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}=\omega_1+\frac{\omega}{\omega_1}\xi(t),$$

or,

$$\phi(t)=\int_0^t\tilde{\omega}\,\mathrm{d}t=\omega_1 t+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'. \tag{5}$$

The formal solution of Eq. (3) can be given as

$$\begin{aligned}x(t)=\;&x_0\,\mathrm{e}^{-nt}\cos\left(\omega_1 t+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right)\\&+\frac{u_0+nx_0}{\tilde{\omega}}\,\mathrm{e}^{-nt}\sin\left(\omega_1 t+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right)\\&+\int_0^t\frac{\mathrm{e}^{-n(t-s)}}{\tilde{\omega}}\sin\left[\omega_1(t-s)\right.\\&\left.+\frac{\omega}{\omega_1}\int_0^{t-s}\xi(s')\,\mathrm{d}s'\right]\cos(\Omega s)\,\mathrm{d}s.\end{aligned} \tag{6}$$

The statistical mean expression of Eq. (6) is

$$\begin{aligned}\langle x(t)\rangle=\;&x_0\,\mathrm{e}^{-nt}\left\langle\cos\left[\omega_1 t+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right]\right\rangle\\&+\mathrm{e}^{-nt}(u_0+nx_0)\left\langle\frac{1}{\tilde{\omega}}\sin\left[\omega_1 t\right.\right.\\&\left.\left.+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right]\right\rangle+\int_0^t\mathrm{e}^{-n(t-s)}\\&\times\left\langle\frac{1}{\tilde{\omega}}\sin\left[\omega_1(t-s)\right.\right.\\&\left.\left.+\frac{\omega}{\omega_1}\int_0^{t-s}\xi(s')\,\mathrm{d}s'\right]\right\rangle\cos(\Omega s)\,\mathrm{d}s.\end{aligned} \tag{7}$$

Expanding Eq. (7) using series, and taking the first two terms, we can obtain

$$\begin{aligned}\frac{1}{\tilde{\omega}}&=\frac{1}{\omega_1(1+\omega\xi(t)/\omega_1^2)}\\&\approx\frac{1}{\omega_1}\left(1-\frac{\omega}{\omega_1^2}\xi(t)\right),\quad(\omega D<\omega_1^2),\end{aligned} \tag{8}$$

$$\begin{aligned}&\left\langle\frac{1}{\tilde{\omega}}\sin\left[\omega_1 t+\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right]\right\rangle\\&=\frac{1}{\omega_1}\left\langle\left(1-\frac{\omega}{\omega_1^2}\xi(t)\right)\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\omega_1 t}\right.\right.\\&\quad\times\exp\left(\mathrm{i}\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right)\\&\quad\left.\left.-\mathrm{e}^{-\mathrm{i}\omega_1 t}\exp\left(-\mathrm{i}\frac{\omega}{\omega_1}\int_0^t\xi(s')\,\mathrm{d}s'\right)\right)\right\rangle\\&=\frac{1}{\omega_1}\left(\exp\left(-\frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2Dt\right)\sin\omega_1 t\right.\\&\quad\left.-\frac{D\omega^2}{2\omega_1^3}\exp\left(-\frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2Dt\right)\cos\omega_1 t\right).\end{aligned} \tag{9}$$

In order to obtain Eq. (9), we adopt the following formulae:[17]

$$\begin{aligned}
&\left\langle \exp\left(\mathrm{i}\frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right)\right\rangle \\
&= \left\langle \exp\left(-\mathrm{i}\frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right)\right\rangle \\
&= \exp\left(-\frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2 Dt\right), \\
&\left\langle \xi(t)\exp\left(\mathrm{i}\frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right)\right\rangle \\
&= \frac{\mathrm{i}D\omega}{2\omega_1}\exp\left(-\frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2 Dt\right)
\end{aligned}$$

$$\begin{aligned}
&\left\langle \xi(t)\exp\left(-\mathrm{i}\frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right)\right\rangle \\
&= \frac{-\mathrm{i}D\omega}{2\omega_1}\exp\left(-\frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2 Dt\right). \qquad (10)
\end{aligned}$$

When $t\to\infty$, from Eq. (7), we have

$$\left\langle \cos\left[\omega_1 t + \frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right]\right\rangle = 0, \qquad (11)$$

$$\left\langle \frac{1}{\tilde{\omega}}\sin\left[\omega_1 t + \frac{\omega}{\omega_1}\int_0^t \xi(s')\mathrm{d}s'\right]\right\rangle = 0. \qquad (12)$$

The first moment of the system can be obtained by solving Eq. (7) as

$$\begin{aligned}
\langle x(t)\rangle &= \int_0^t \mathrm{e}^{-n(t-s)}\left\langle \frac{1}{\tilde{\omega}}\sin\left[\omega_1(t-s)+\frac{\omega}{\omega_1}\int_0^{t-s}\xi(s')\mathrm{d}s'\right]\right\rangle\cos(\Omega s)\mathrm{d}s \\
&= \frac{1}{\omega_1}\int_0^t\left[\mathrm{e}^{-D_0(t-s)}\sin\omega_1(t-s) - \frac{D\omega^2}{2\omega_1^3}\mathrm{e}^{-D_0(t-s)}\cos\omega_1(t-s)\right]\cos(\Omega s)\mathrm{d}s \\
&= \left[\frac{4\omega_1^4(D_0^2-\Omega^2+\omega_1^2)-2\omega^2 DD_0(D_0^2+\omega_1^2+\Omega^2)}{\omega_1^4 k_1 k_2}\right]\cos\Omega t + \frac{8\Omega D_0}{k_1 k_2}\sin\Omega t \\
&= A\cos(\Omega t+\varphi), \qquad (13)
\end{aligned}$$

where

$$D_0 = n + \frac{1}{2}\left(\frac{\omega}{\omega_1}\right)^2 D,$$

$k_1 = 2[D_0^2+(\Omega-\omega_1)^2]$, and $k_2 = 2[D_0^2+(\Omega+\omega_1)^2]$. $D$ is regarded as the damping coefficient of the oscillator with frequency fluctuation, and it is related to $n$ and $D$.

Inspecting Eq. (13), we can find that the output response of system behaves also as a simple harmonic vibration, of which the frequency is the same as the external driving frequency, and the amplitude is constant and expressed as

$$A = \left\{\left[\frac{4\omega_1^4(D_0^2-\Omega^2+\omega_1^2)-2\omega^2 DD_0(D_0^2+\omega_1^2+\Omega^2)}{\omega_1^4 k_1 k_2}\right]^2 + \left[\frac{8\Omega D_0}{k_1 k_2}\right]^2\right\}^{1/2}. \qquad (14)$$

The phase difference between the system frequency and the external driving frequency is

$$\varphi = \arctan\left[\frac{-8\Omega D_0\omega_1^4}{4\omega_1^4(D_0^2-\Omega^2+\omega_1^2)-2\omega^2 DD_0(D_0^2+\omega_1^2+\Omega^2)}\right]. \qquad (15)$$

If $D=0$, Eqs. (14) and (15) come back to the expressions in the determinism

$$A = \frac{1}{(\Omega^2-\omega^2)^2+4\Omega^2 n^2}, \qquad (16)$$

$$\varphi = \arctan\left[\frac{-2\Omega n}{(\omega^2-\Omega^2)}\right]. \qquad (17)$$

Both the displacement and velocity oscillator are related to the phase. Therefore, the noise intensity of inherent angular frequency should affect the amplitude, the damping coefficient, and the phase difference of output response.

# 3. SR of amplitude of output response

## 3.1. Variation of response amplitude with angular frequency of external driving

Using Eq. (14), the curves of response amplitude $A$ versus external driving frequency are plotted in Fig. 1. From them, we can see that the ampli-

tude $A$ presents one resonance peak as the value of $\Omega$ varies except the case of $n = 4$. This is bona fide SR. Taking the frequency noise intensity as a parameter, the position of the peak turns to the right and goes down with the increase of $D$ as shown in Fig. 1(a). When $D = 0.1$, the position of the peak corresponds to $\Omega < 5$, namely $\Omega < \omega$, and it is at $\Omega > \omega$ only when noise intensity is large enough ($D = 3$).

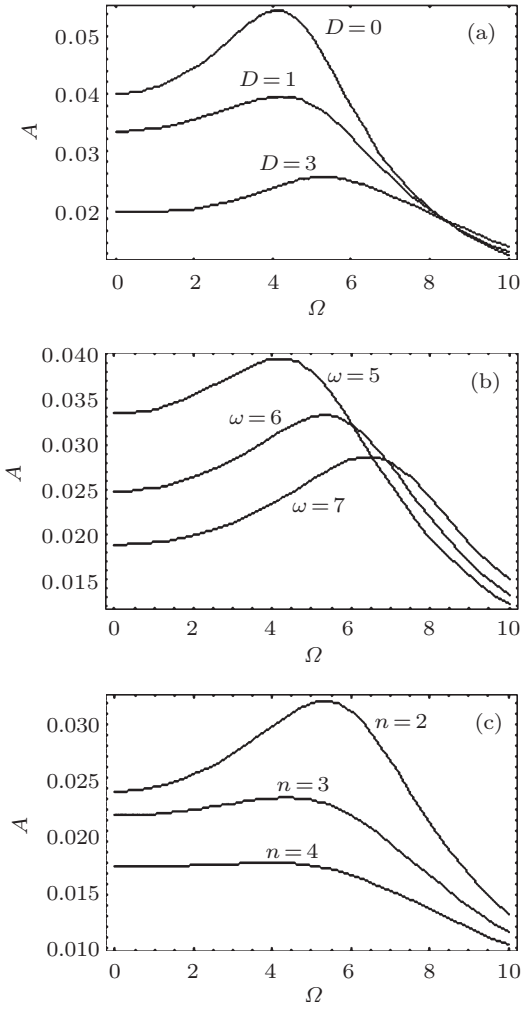



**Fig. 1.** Curves of response amplitude $A$ versus external driving frequency $\Omega$, with parameters being (a) $n = 2$, $\omega = 5$; (b) $n = 2$, $D = 1$; (c) $\omega = 6$, $D = 2$.

Figure 1(b) depicts the curve of $A$ versus $\Omega$ with taking the inherent frequency $\omega$ as a parameter. It shows that the larger the value of $\omega$, the lower the peak is, and the position shifts to the right. In Fig. 1(c), we take damping coefficient $n$ as a parameter, and can see that as $n$ increases, the position of peak shifts to the left, and as $n$ increases to a certain value ($n = 4$), the system presents an overdamping vibration, and the resonance peak disappears.

## 3.2. Variation of response amplitude with inherent angular frequency

In order to discuss the influence of inherent angular frequency $\omega$ on output amplitude $A$, we adopt the conditions of $\omega > n$ and $\omega_1^2 > \omega D$ according to the results in Section 2, and plot the curves of $A$ versus $\omega$ as shown in Fig. 2. We can see that the curves also present bona fide SR phenomena when the parameters of $D$, $\Omega$, and $n$ take different values.

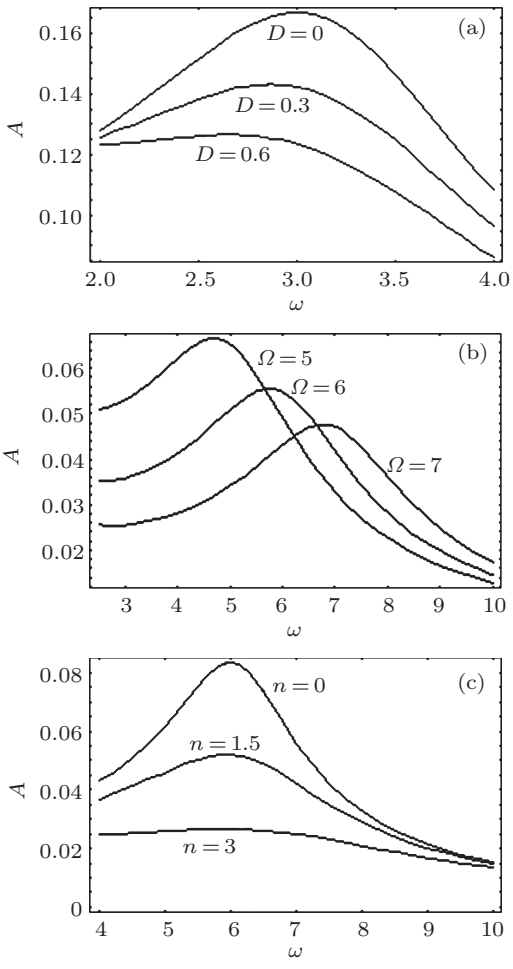



**Fig. 2.** Curves of response amplitude $A$ versus inherent angular frequency $\omega$, with parameters being (a) $n = 1$, $\Omega = 3$; (b) $n = 1$, $D = 1$; (c) $\Omega = 6$, $D = 0.2$.

As $D$ increases, the position of the peak turns to the left and goes down. However, when $D$ increases to a certain value ($D = 0.6$), the system experiences an overdamping vibration, and the resonance peak also disappears, as shown in Fig. 2(a). Taking the external driving frequency $\Omega$ to be different values, all the curves present SRs, and with the increase of $\Omega$, the position of the peak turns rightward and goes down (shown in Fig. 2(b)). From Fig. 2(c), it is noted that the damping coefficient $n$ only affects the value of the

peak, but does not change its position. The weaker the damping, the more obvious the SR phenomenon is. The resonance peak also disappears when $n$ increases to a certain value.

## 4. Conclusion

In this paper, for a damped linear oscillator model subjected to a white noise existing in the inherent angular frequency and a periodic external driving force, we successfully derived the analytic expression of the first moment of output response of system, and investigated the phenomenon of stochastic resonance of the curves of $A$ versus $\Omega$, and $A$ versus $\omega$. The main results show that the output response of system behaves as a simple harmonic vibration, of which the frequency is the same as the external driving frequency, and there exist bona fide SR phenomena in these curves. The parameter $D$ makes the position of resonance peak shift to the right, and the parameter $n$ makes it shift to the left, so the angular frequency corresponding to the resonance peak may be at $\Omega > \omega$ or $\Omega < \omega$. The noise intensity of inherent frequency $D$ plays a role in damping the system, and when D increases to a certain value, the SR phenomenon disappears.

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