Time evolution of information entropy for a stochastic system with double singularities driven by quasimonochromatic noise*

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This paper deals with the time evolution of information entropy for a stochastic system with double singularities driven by quasimonochromatic noise. The dimension of the Fokker–Planck equation is reduced by the linear transformation. The exact expression of the time dependence of information entropy is obtained based on the definition of Shannon's information entropy. The relationships between the properties of dissipative parameters, system singularity strength parameter, quasimonochromatic noise, and their effects on information entropy are discussed.

Keywords: information entropy, quasimonochromatic noise, Fokker–Planck equation, stochastic system with double singularities

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1. Introduction

Study of the noise driven dynamical system is always an intriguing issue in physics.[1−19] The noise may essentially influence the dynamical system in many aspects, such as stationary probability,[1,2] escape rate,[2,3] noise-induced phase transitions,[4,5] stochastic resonance,[6−9] the time evolution of information entropy,[10−19] etc. Entropy is an appropriate tool in studying the nature of dynamical system driven by noise. The concept of entropy is introduced in accordance with the quantitative description for the natural evolution of the quasi-static isolated system and it plays an important role in describing the evolution, instability, disorder or confusion and information transmission of stochastic systems.[20] In recent years, information entropy as well as its relevant quantities has become an important research area for the understanding of noise-driven dynamical systems in detail.[10−19]

The purpose of the present paper is to show the time evolution of information entropy for a stochastic system with double singularities driven by quasimonochromatic noise (QMN). QMN is "truly colored" noise; it is generated by a variety of systems capable of singling out a frequency, and can be viewed as the result of filtering broadband noise through a highly selective system.[2,3,10,11] Dykman *et al.*[2,3] studied the probability distributions and escape rate for systems driven by QMN. Li *et al.*[9] introduced the stochastic resonance in bistable systems subjected to signal and QMN. Xu *et al.*[10] investigated the entropy flux and entropy production of a dissipative dynamical systems driven by QMN. The upper bound of the time derivative of entropy for a dynamical system driven by QMN have been studied in Ref. [11]. The previous studies indicate that QMN is essentially different from the white and exponentially correlated noise.

The rest of the present paper is organized as follows. In Section 2, a stochastic system with double singularities driven by QMN is introduced and the dimension of the Fokker–Planck equation is reduced by the linear transformation. In Section 3, the exact expression of the time dependence of information entropy is obtained based on the definition of Shannon's information entropy. In Section 4, the relationships between the properties of dissipative parameters, system singularity strength parameter, QMN, and their effects on information entropy are discussed. Some conclusions are drawn in Section 5.

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2. Fokker–Planck equation of a stochastic system with double singularities driven by QMN

The relevant Langevin equation of the stochastic system with double singularities driven by QMN is given as follows:[20]

$$\begin{aligned} &\dot{x} = -\gamma x + |x|^{\beta}\eta, \quad (\gamma > 0), \\ &\ddot{\eta} = -2\Gamma\dot{\eta} - \omega_0^2 + \xi, \end{aligned} \tag{1}$$

where γ $(\gamma > 0)$ is a dissipative parameter, β is the system singularity strength parameter, η is the QMN, Γ has the meaning of the friction coefficient of the oscillator, ω_0 is the eigenfrequency, and ξ is a Gaussian white noise term.

The noise term ξ is characterized by the mean and variance

$$\begin{aligned} &\langle \xi(t) \rangle = 0, \\ &\langle \xi(t)\, \xi(t') \rangle = 2D\delta(t - t'). \end{aligned} \tag{2}$$

Here QMN is a narrow band noise and the power spectrum of QMN contains a relatively narrow peak at a finite frequency ω_0, with the half-width Γ satisfying $\Gamma \ll \omega_0$.[2,3,9–12]

System (1) can be rewritten in an equivalent form as

$$\begin{aligned} &\dot{x} = -\gamma x + |x|^{\beta}\eta, \\ &\dot{\eta} = \zeta, \\ &\dot{\zeta} = -2\Gamma\zeta - \omega_0^2\eta + \xi. \end{aligned} \tag{3}$$

Let $X_1 = x$, $X_2 = \eta$, and $X_3 = \zeta$. The Fokker–Planck equation of Eq. (3) can be written as

$$\begin{aligned} &\dot{X}_1 = -\gamma X_1 + |X_1|^{\beta} X_2, \\ &\dot{X}_2 = X_3, \\ &\dot{X}_3 = -2\Gamma X_3 - \omega_0^2 X_2 + \xi. \end{aligned} \tag{4}$$

In this paper we mainly study the nontrivial case with double singularities, i.e., $\beta < 1$, by using the following transformation:[20]

$$\begin{aligned} Y_1 &= \begin{cases} |X_1|^{1-\beta}, & (X_1 > 0), \\ -|X_1|^{1-\beta}, & (X_1 < 0), \end{cases} \\ Y_2 &= X_2, \\ Y_3 &= X_3. \end{aligned} \tag{5}$$

Equation (5) can be transformed into

$$\begin{aligned} &\dot{Y}_1 = -\gamma(1-\beta) Y_1 + (1-\beta) Y_2, \\ &\dot{Y}_2 = Y_3, \\ &\dot{Y}_3 = -2\Gamma Y_3 - \omega_0^2 Y_2 + \xi. \end{aligned} \tag{6}$$

The Fokker–Planck equation of Eq. (6) can be written as

$$\begin{aligned} &\frac{\partial \rho(Y_1, Y_2, Y_3, t)}{\partial t} \\ &= \gamma(1-\beta)\frac{\partial Y_1 \rho}{\partial Y_1} - (1-\beta)\frac{\partial Y_2 \rho}{\partial Y_1} - \frac{\partial Y_3 \rho}{\partial Y_2} \\ &\quad + \omega_0^2 \frac{\partial Y_2 \rho}{\partial Y_3} + 2\Gamma\frac{\partial Y_3 \rho}{\partial Y_3} + D\frac{\partial^2 \rho}{\partial Y_3^2}, \end{aligned} \tag{7}$$

where $\rho(Y_1, Y_2, Y_3, t)$ is the extended phase-space probability distribution function.

Now making use of the linear transformation $U = Y_1 + aY_2 + bY_3$,[21] the Fokker–Planck equation (7) becomes

$$\frac{\partial \rho(U,t)}{\partial t} = -\frac{\partial A\rho}{\partial U} + B\frac{\partial^2 \rho}{\partial U^2}. \tag{8}$$

Here

$$A = -\lambda U, B = b^2 D, \tag{9}$$

$$\begin{aligned} \lambda U = {} & \gamma(1-\beta) Y_1 + \left[\omega_0^2 b - (1-\beta)\right] Y_2 \\ & + (2\Gamma b - a) Y_3, \end{aligned} \tag{10}$$

with a, b, and λ being constants to be determined. Using $U = Y_1 + aY_2 + bY_3$ in Eqs. (9) and (10), one has

$$\begin{aligned} &\lambda = \gamma(1-\beta), \\ &\lambda a = \omega_0^2 b - (1-\beta), \\ &\lambda b = 2\Gamma b - a. \end{aligned} \tag{11}$$

From Eq. (11), we find

$$\begin{aligned} a &= \frac{2\Gamma(1-\beta) - \gamma(1-\beta)^2}{\omega_0^2 + \gamma^2(1-\beta)^2 - 2\Gamma\gamma(1-\beta)}, \\ b &= \frac{1-\beta}{\omega_0^2 + \gamma^2(1-\beta)^2 - 2\Gamma\gamma(1-\beta)}. \end{aligned} \tag{12}$$

3. Time evolution of information entropy for system (2)

The Fokker–Planck equation (8) can be rewritten as a continuity equation in the general form

$$\frac{\partial \rho(U,t)}{\partial t} = -\nabla_U \cdot j, \tag{13}$$

where ∇_U denotes the phase space divergence and

$$j = A\rho - B\frac{\partial \rho}{\partial U}. \tag{14}$$

Based on the definition of Shannon's information entropy

$$S = -\int \rho(U,t)\ln\rho(U,t)\,\mathrm{d}U, \tag{15}$$

the time evolution equation for information entropy S can be written as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \nabla_U \cdot j \ln\rho \mathrm{d}U. \tag{16}$$

Performing partial integration on Eq. (16) and then imposing the natural boundary conditions:[22] $j|_{\text{boundary}} = 0$, and $j\ln\rho|_{\text{boundary}} = 0$, one obtains

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int \frac{1}{\rho} j\cdot\nabla_U \rho \mathrm{d}U. \tag{17}$$

Applying the method of Refs. [10]–[18] we then search for the Green's function or conditional probability solution for Eq. (8). We now look for a solution of Eq. (8) which is in the following form:

$$\rho(U,t|\,U',0) = \exp\left[G(t)\right], \tag{18}$$

where

$$G(t) = \frac{-(U-\alpha(t))^2}{\sigma(t)} + \ln v(t). \tag{19}$$

Equation (8) can be solved for the following initial conditions:

$$\rho(U,0|\,U',0) = \lim_{\varepsilon\to\infty}\sqrt{\frac{\varepsilon}{\pi}}\exp\left[-\varepsilon(U-U')^2\right]. \tag{20}$$

Comparing Eq. (20) with Eq. (18) and $G(0)$ we have

$$\sigma(0) = \frac{1}{\varepsilon}, \quad \alpha(0) = U', \quad v(0) = \sqrt{\frac{\varepsilon}{\pi}}. \tag{21}$$

Substituting Eq. (18) into Eq. (8) we can obtain the following set of equations:

$$\begin{aligned} &\dot\sigma(t) = -2\gamma(1-\beta)\sigma(t) + 4b^2 D, \\ &\dot\alpha(t) = -\gamma(1-\beta)\alpha(t), \\ &\frac{1}{v(t)}\dot v(t) = -\frac{1}{2\sigma(t)}\dot\sigma(t). \end{aligned} \tag{22}$$

The relevant solutions of $\sigma(t)$ and $\alpha(t)$ which satisfy the above initial conditions are given by

$$\begin{aligned} \alpha(t) &= \alpha(0)\exp\left[-\gamma(1-\beta)t\right], \\ \sigma(t) &= \frac{2b^2 D\left\{1-\exp\left[-2\gamma(1-\beta)t\right]\right\}}{\gamma(1-\beta)} \\ &\quad + \sigma(0)\exp\left[-2\gamma(1-\beta)t\right]. \end{aligned} \tag{23}$$

Substituting Eqs. (18), (19), and (23) into Eq. (17), the explicit time evolution equation for information entropy can be written as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\gamma(1-\beta) + \frac{2b^2 D}{\sigma(t)}. \tag{24}$$

Integrating Eq. (24) we have

$$S(t) = S(0) + \frac{1}{2}\ln\left(\frac{\sigma(t)}{\sigma(0)}\right). \tag{25}$$

4. Influences of dissipative parameters, system singularity strength parameter, and QMN on information entropy $S(t)$

Now we discuss the influences of dissipative parameters, system singularity strength parameter, and QMN on information entropy $S(t)$. From Eq. (25) we can see that $S(t)$ is dependent on γ, β, Γ, D, and ω_0. In this paper, $S(0) = 0.0$, $\sigma(0) = 0.5$, and $\alpha(0) = 1.0$ are chosen. The three-dimensional plot of $S(t)$ as a function of the time t is given in Figs. 1–5 when parameters γ, β, Γ, D, and ω_0 are varied. From Figs. 1–5, we can see clearly that the curves of $S(t)$ increase monotonically with the increase of time t for different values of γ, β, Γ, D, and ω_0. As $t\to\infty$, $S(t)$ ultimately reaches a limiting value, which shows that the system turns from a non-equilibrium state relaxation to equilibrium state in the long run.

Figure 1 shows the variations of $S(t)$ with parameters γ and t. From Fig. 1, it is seen that the values of $S(t)$ decrease monotonically with the increase of γ for different values of t, showing that the increase of γ can enhance the system conversion from a non-equilibrium state relaxation into equilibrium state. Figure 2 exhibits the variations of $S(t)$ with parameters β and t. From Fig. 2, it is seen that the values of $S(t)$ first increase up to their corresponding maxima, then decrease monotonically with the increase of β for different values of t. The maxima increase and the positions of the extreme points move rightward with the increase of β. It means that the parameter β influences the variation of $S(t)$ remarkably in the process of the system conversion from a non-equilibrium state relaxation to equilibrium state.

Figure 3 shows the variations of $S(t)$ with parameters D and t. From Fig. 3, it is seen that values of $S(t)$ increase monotonically with the increase of D for different values of t. Figure 4 displays the variations of $S(t)$ with parameters Γ and t. From Fig. 4, it is seen that the values of $S(t)$ increase monotonically with the increase of Γ for different values of t. Figures 3 and 4 show that the increases of D and Γ can slow

down the system conversion from a non-equilibrium state relaxation to equilibrium state. Figure 5 indicates the variations of $S(t)$ with parameters ω_0 and t. From Fig. 5, it is seen that the values of $S(t)$ first increase up to their corresponding maxima, then decrease monotonically with the increase of ω_0 for different values of t, which also shows that parameter ω_0 influences the variation of $S(t)$ remarkably in the process of the system conversion from a non-equilibrium state relaxation to equilibrium state.

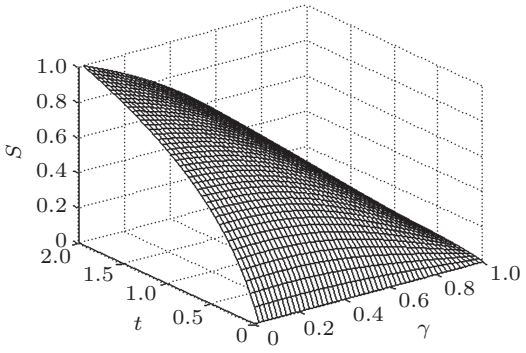


Fig. 1. Variation of $S(t)$ with γ and t at $D = 0.5$, $\beta = 0.1$, $\Gamma = 0.3$, $\omega_0 = 1.0$ for Eq. (25) (units are arbitrary).

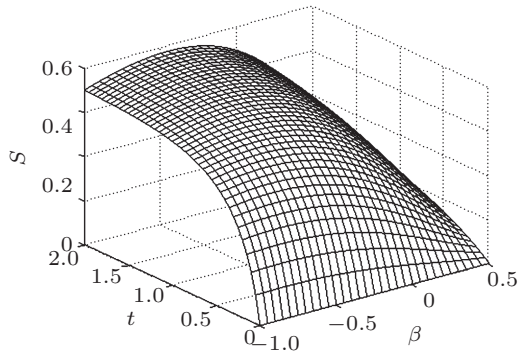


Fig. 2. Variation of $S(t)$ with β and t at $D = 0.5$, $\gamma = 0.5$, $\Gamma = 0.3$, $\omega_0 = 1.0$ for Eq. (25) (units are arbitrary).

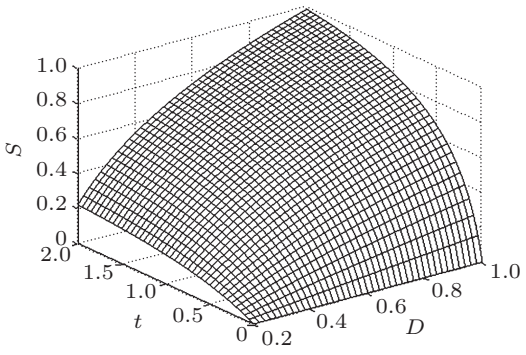


Fig. 3. Variations of $S(t)$ with D and t at $\gamma = 0.5$, $\beta = 0.1$, $\Gamma = 0.3$, $\omega_0 = 1.0$ for Eq. (25) (units are arbitrary).

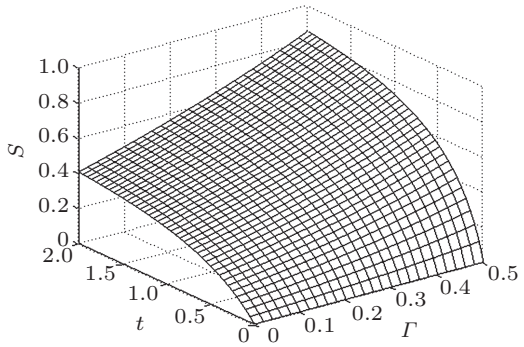


Fig. 4. Variations of $S(t)$ with Γ and t at $D = 0.5$, $\gamma = 0.5$, $\beta = 0.1$, $\omega_0 = 1.0$ for Eq. (25) (units are arbitrary).

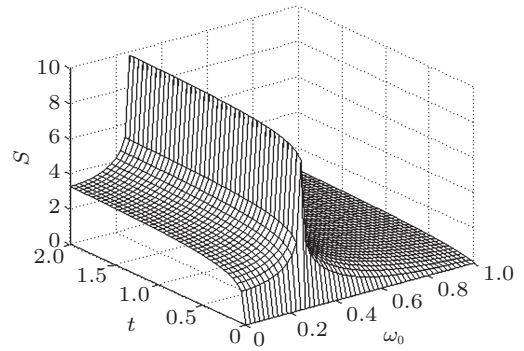


Fig. 5. Variations of $S(t)$ with ω_0 and t at $D = 0.5$, $\gamma = 0.5$, $\beta = 0.1$, $\Gamma = 0.3$ for Eq. (25) (units are arbitrary).

5. Conclusion

In this paper we studied the time evolution of information entropy for a stochastic system with double singularities driven by QMN. The dimension of the Fokker–Planck equation is reduced by the linear transformation. The exact expression of the time dependence of information entropy is obtained based on the definition of Shannon's information entropy. The relationships between the properties of dissipative parameters, system singularity strength parameter, QMN, and their effects on information entropy are also discussed. It is found that parameters γ, β, Γ, D, and ω_0 influence the variation of information entropy for a stochastic system with double singularities remarkably.

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