Area spectrum of the three-dimensional Gödel black hole*

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The aim of this paper is to investigate the area spectrum of the three-dimensional Gödel black hole by using two different methods. The result shows that the area spectrum of the black hole is $\Delta A = 8\pi l_P^2$, which confirms the initial proposal of Bekenstein that the area spectrum is independent of black hole parameters and the spacing is $8\pi l_P^2$.

Keywords: area spectrum, Gödel black hole, adiabatic invariant quantity

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1. Introduction

The quantum effect and thermodynamic properties of black holes have attracted many people's attention in recent years.[1−14] In particular, there have been many attempts to derive the area spectrum[15−32] for a black hole horizon since Bekenstein showed that black hole entropy is proportional to its horizon area as well as that the horizon area is quantized in units of l_P^2.[33,34] Bekenstein first proposed that the black hole horizon area is an adiabatic invariant and the spacing of the area spectrum is $\Delta A = 8\pi l_P^2$. Later on, Hod suggested that the quantum of the black hole area can be determined by considering the real part of the asymptotic quasinormal mode of a black hole.[15,16]

Based on Hod's proposal for the relationship between the quasinormal frequencies and Bekenstein's proposal for the adiabaticity of the black hole horizon area, Kunstatter proposed a different method to quantize the horizon area of a black hole by using an adiabatic invariant.[17] He presented that the entropy spectrum of a d-dimensional black hole is quantized and the quantum of the horizon area is proportional to the logarithm. Kunstatter's method allows us to fix the spacing of the area spectrum for the Schwarzschild black hole, and the result of the area spectrum agrees with Hod's result.

In 2007, Maggiore provided a new interpretation of the quantum effect of the black hole horizon area with respect to the quasinormal frequencies.[18] In support of the theory that the quasinormal modes of the perturbed Schwarzschild black holes can be described as damped harmonic oscillators, Maggiore proposed that in the semiclassical limit, the area spectrum of the event horizon is determined by the physical frequency of the harmonic oscillator whose frequencies are complex. It was proposed that we cannot neglect the imaginary part of the quasinormal frequency, while in Hod's work one was interested only in the real parts of the frequencies. Following Maggiore's interpretation for the black hole quasinormal frequencies and the black hole properties of adiabaticity, the area spectra of black holes in a variety of different spacetimes have been studied.[26−32]

Recently, Majhi and Vagenas derived the entropy spectrum and the horizon area quantum effect of the more general class of spherically symmetric static black holes utilizing solely the adiabaticity of black holes.[24] Following that, this method of the adiabatic invariant action variable was extended to the charged Reissner–Nordström black hole[35] and the BTZ black hole.[36] It is noteworthy that there is no use at all of the quasinormal frequencies. The resulting black hole entropy spectrum is equidistant and the corresponding horizon area quantum effect is identical to Bekenstein's result. Very recently, Zeng and Liu *et al.*[37] proposed a new scheme to quantize the horizon

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area of a black hole. It was found that the period of the gravity system with respect to the Euclidean time can determine the area spectrum of a black hole exclusively. In their analysis, quasinormal mode frequency is not used and there is also no confusion on whether the rear part or imaginary part is responsible for the area spectrum. They exclusively utilized the period of motion of an outgoing wave, which is shown to be related to the vibrational frequency of the perturbed black hole, to quantize the horizon areas of a Schwarzschild black hole and a Kerr black hole. The result showed that the area spacing is $8\pi l_P^2$ without any assumptions. In this paper, we would like to extend above two methods to investigate the area spectrum of a Gödel black hole in three-dimensional spacetime.

2. Periodicity and area spectrum of the three-dimensional Gödel black hole

The three-dimensional Gödel black hole[38] is an exact solution derived from a three-dimensional Einstein–Maxwell theory with a negative cosmological constant $\Lambda = -1/l^2$ and a Chern–Simons term

$$I = \frac{1}{16\pi G}\int \mathrm{d}^3x\left[\sqrt{-g}\left(R + \frac{2}{l^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) - \frac{\alpha}{2}\epsilon^{\mu\nu\varphi}A_\mu F_{\nu\varphi}\right], \tag{1}$$

where G is the three-dimensional gravitational constant. The corresponding metric and gauge potential are

$$\mathrm{d}s^2 = (\mathrm{d}t - 2\alpha r\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{\Delta(r)} - \Delta(r)\mathrm{d}\varphi^2, \tag{2}$$

and

$$A_\varphi = -\frac{4GQ}{\alpha} + \frac{2r}{l}\sqrt{1-\alpha^2 l^2} \tag{3}$$

with the metric function

$$\Delta(r) = \frac{2r^2}{l}(1+\alpha^2 l^2) - 8G\nu r + \frac{4GQ}{\alpha}, \tag{4}$$

where α is the Chern–Simons coupling constant, ν and J are two integral constants, which may be related to the mass and angular momentum of black hole. The asymptotic geometry of the three-dimensional Gödel black hole does not behave as either a de Sitter nor an anti-de Sitter due to the presence of a nontrivial gauge field. The horizons are determined by the equation

$$\Delta(r) = \frac{2r^2}{l}(1+\alpha^2 l^2) - 8G\nu r + \frac{4GQ}{\alpha} = 0, \tag{5}$$

and the corresponding outer and inter horizons $r_\pm$ are

$$r_\pm = \frac{l^2}{1+\alpha^2 l^2} \times \left[2G\nu \pm \sqrt{4G^2\nu^2 - \frac{2GJ}{\alpha}\cdot\frac{(1+\alpha^2 l^2)}{l^2}}\right]. \tag{6}$$

The Hawking temperature T_{BH} and angular velocity at the outer event horizon are given by[39,40]

$$T_{\mathrm{BH}} = \frac{1+\alpha^2 l^2}{4\pi\alpha l^2}\cdot\frac{r_+ - r_-}{r_+} \tag{7}$$

and

$$\Omega_+ = \frac{1}{2\alpha r_+}. \tag{8}$$

The area of the event horizon A and the entropy for Einstein gravity are, respectively,

$$A = 4\pi\alpha r_+ \tag{9}$$

and

$$S = \frac{A}{4G} = \frac{A}{4l_P^2}, \tag{10}$$

which is the well-known Bekenstein–Hawking area formula for entropy. Now, let us focus on investigating the periodicity and spectrum of the Gödel black hole. In order to derive the area spectrum, we firstly find the periodicity of the outgoing wave. According to the Klein–Gordon equation

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right) - \frac{\mu^2}{\hbar^2}\Phi = 0, \tag{11}$$

we can obtain wave function. For convenience, we may get the wave function from the Hamilton–Jacobi equation

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + \mu^2 = 0, \tag{12}$$

where the wave function Φ and the action satisfy the following relationship

$$\Phi = \exp\left[\frac{\mathrm{i}}{\hbar}S(t,r,\varphi)\right]. \tag{13}$$

For the three-dimensional Gödel black hole, to avoid the dragging effect, we should perform a dragging coordinate transformation

$$\mathrm{d}\varphi = \frac{1}{2\alpha r}\mathrm{d}t = \Omega\mathrm{d}t. \tag{14}$$

Then, the metric of the Gödel black hole becomes

$$\mathrm{d}s^2 = -\frac{\Delta(r)}{4\alpha^2 r^2}\mathrm{d}t^2 + \frac{1}{\Delta(r)}\mathrm{d}r^2. \tag{15}$$

Near the horizon, in the dragging coordinate frame, the action S can be decomposed as

$$S(t,r) = -(E - m\Omega_+)t + R(r), \tag{16}$$

where E is the energy of the emitted particle, m is the angular quantum about ϕ. Instituting Eqs. (15) and (16) into Eq. (12), near the horizon, we can get

$$R(r) = \mathrm{i}\frac{2\alpha r_+}{\Delta'(r_+)}(E - m\Omega_+), \tag{17}$$

where

$$\Delta'(r_+) = \frac{\partial \Delta}{\partial r}\,|_{r=r_+}\,.$$

Here, we only consider the outgoing wave near the horizon. It is evident that the wave function Φ outside the horizon can be taken the form of

$$\Phi = \exp\left[-\frac{\mathrm{i}}{\hbar}(E - m\Omega_+)t\right]\Psi(r_+), \tag{18}$$

where

$$\Psi(r_+) = \exp\left[-\frac{\pi(E - m\Omega_+)}{\hbar\Delta'(r_+)}\right]. \tag{19}$$

From Eq. (18), we can obviously see that wave function Φ is a periodic function with the period

$$T = \frac{2\pi\hbar}{E - m\Omega_+}. \tag{20}$$

Considering the relationship $E = \hbar\omega$, we can obtain

$$T = \frac{2\pi}{\omega - m\Omega_+/\hbar}. \tag{21}$$

According to Ref. [33], a particle's period is the same as that of the periodic gravity system, namely

$$T = \frac{2\pi}{\kappa_+} = \frac{\hbar}{T_{\mathrm{BH}}}. \tag{22}$$

Combining Eqs. (20) and (21), we have

$$\mathrm{d}\nu = \hbar\omega = m\Omega_+ + 2\pi T_{\mathrm{BH}}, \tag{23}$$

where one identifies the parameter ν as the mass and the parameter J as the angular moment of the black hole. This identification is effective and reasonable, at least, in handling with the quantization of black hole entropy.[35,36] From Eq. (9), we can obtain the change of horizon area of the Gödel black hole

$$\begin{aligned}\Delta A &= \frac{8\pi G\alpha l^2}{1+\alpha^2 l^2}\left[\frac{\sqrt{4G^2\nu^2 - 2GJ(1+\alpha^2 l^2)/\alpha l^2} + 2G\nu}{\sqrt{4G^2\nu^2 - 2GJ(1+\alpha^2 l^2)/\alpha l^2}}\mathrm{d}\nu - \frac{1+\alpha^2 l^2}{2\alpha l^2\sqrt{4G^2\nu^2 - 2GJ(1+\alpha^2 l^2)/\alpha l^2}}\mathrm{d}J\right] \\ &= 8\pi G\left[\frac{\mathrm{d}\nu}{2\pi T_{\mathrm{BH}}} - \frac{\mathrm{d}J}{2\sqrt{4G^2\nu^2 - 2GJ(1+\alpha^2 l^2)/\alpha l^2}}\right].\end{aligned} \tag{24}$$

Substituting Eq. (23) into Eq. (24), we have

$$\Delta A = 8\pi G\left[1 + \frac{m\Omega_+}{2\pi T_{\mathrm{BH}}} - \frac{\mathrm{d}J}{2\sqrt{4G^2\nu^2 - 2GJ(1+\alpha^2 l^2)/\alpha l^2}}\right]. \tag{25}$$

Thus, using Eqs. (7), (8), and (25), we find the spacing of the equal space area spectrum as follows:

$$\Delta A = 8\pi l_P^2. \tag{26}$$

This result indicates that the area spectrum of the three-dimensional Gödel black hole is equally spaced.

3. Three-dimensional Gödel black hole spectroscopy via adiabatic invariance

Now, we would like to employ an adiabatic-invariant quantity $\int p_i \mathrm{d}q_i$ to quantize the horizon area. Firstly, we Euclideanize the metric, given in Eq. (15), by using the transformation $t \to -\mathrm{i}\tau$. This leads to

$$\mathrm{d}s^2 = \frac{\Delta(r)}{4\alpha^2 r^2}\mathrm{d}\tau^2 + \frac{1}{\Delta(r)}\mathrm{d}r^2. \tag{27}$$

Using coordinate transformation

$$\mathrm{d}r^* = \frac{2\alpha r}{\Delta(r)}\mathrm{d}r, \tag{28}$$

we can obtain

$$\begin{aligned}\mathrm{d}s^2 = {} & \frac{\Delta(r)}{4\alpha^2 r^2}\exp\left[-\frac{(1+\alpha^2 l^2)(r_+ - r_-)}{\alpha l^2 r_+}r^*\right] \\ & \times(\mathrm{d}T^2 + \mathrm{d}R^2), \quad (r > r_+),\end{aligned} \tag{29}$$

where

$$T = \frac{2\alpha l^2 r_+ \sin\kappa_+\tau}{(1+\alpha^2 l^2)(r_+ - r_-)} \times \exp\left[\frac{(r_+ - r_-)}{r_+}\frac{(1+\alpha^2 l^2)}{2\alpha l^2} r^*\right], \quad (30)$$

$$R = \frac{2\alpha l^2 r_+ \cos\kappa_+\tau}{(1+\alpha^2 l^2)(r_+ - r_-)} \times \exp\left[\frac{(1+\alpha^2 l^2)(r_+ - r_-)}{2\alpha l^2 r_+} r^*\right], \quad (31)$$

$$\kappa_+ = \frac{r_+ - r_-}{r_+}\frac{1+\alpha^2 l^2}{2\alpha l^2}. \quad (32)$$

From Eqs. (30) and (31), we can see that T and R are periodic functions with respect to the Euclidean time τ with period $2\pi/\kappa_+$. We will use this period to investigate the area spectrum of the three-dimensional Gödel black hole via an adiabatic-invariant action variable.

According to classical mechanics, the action I, action variable I_ν, and the Hamiltonian $\mathscr{H}$ of any single periodic system satisfy the relationship

$$I_\nu = I + \int \mathscr{H}\mathrm{d}\tau. \quad (33)$$

In the dragged coordinate system, the coordinate ϕ does not appear in the line element expressions in Eqs. (26), (27), and (29). That is to say, ϕ is an ignorable coordinate. To eliminate this freedom completely, the action I should be written as[41–44]

$$\begin{aligned} I &= \int p_r \mathrm{d}r - \int p_\phi \mathrm{d}\phi \\ &= \iint_0^{\mathscr{H}(r;\,\phi;\,p_\phi)} \frac{\mathrm{d}\mathscr{H}'(r;\,\phi;\,p_\phi)}{\dot r}\mathrm{d}r \\ &\quad - \iint_0^{\mathscr{H}(\phi;\,r;\,p_r)} \frac{\mathrm{d}\mathscr{H}'(\phi;\,r;\,p_r)}{\dot r}\mathrm{d}r \\ &= \iint_0^{p_r} \mathrm{d}p_r'\mathrm{d}r - \iint_0^{p_\phi} \mathrm{d}p_\phi'\mathrm{d}\phi \\ &= \iint_{(0,0)}^{(\nu,J)} (\mathrm{d}\nu' - \Omega\mathrm{d}J')\frac{\mathrm{d}r}{\dot r}, \end{aligned} \quad (34)$$

where we eliminate the momentum in favor of energy by using the following Hamilton's equation

$$\dot r = \left.\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}p_r'}\right|_{(r;\,\phi;\,p_\phi)} = \frac{\mathrm{d}\nu'}{\mathrm{d}p_r'}, \quad (35)$$

$$\dot\phi = \left.\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}p_\phi'}\right|_{(\phi;\,r;\,p_r)} = \Omega\frac{\mathrm{d}J'}{\mathrm{d}p_\phi'}. \quad (36)$$

So the invariant action variable I_ν can be rewritten as

$$I_\nu = \iint_0^{\mathscr{H}} \mathrm{d}\mathscr{H}'\mathrm{d}\tau + \iint_{(0,0)}^{(\nu,J)} (\mathrm{d}\nu' - \Omega\mathrm{d}J')\frac{\mathrm{d}r}{\dot r}. \quad (37)$$

To obtain $\dot r$, we should consider the radial null geodesics while a particle runs out. Solving Eq. (27), we obtain

$$\dot r \equiv \frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\mathrm{i}\frac{\Delta(r)}{2\alpha r} = \mathscr{W}_\pm(r), \quad (38)$$

where the positive (negative) sign stands for the outgoing (ingoing) radial null paths. So, our subsequent analysis will focus on the outgoing paths. According to Eq. (38), we have

$$\begin{aligned} \iint_0^{\mathscr{H}} \mathrm{d}\mathscr{H}'\mathrm{d}\tau &= \iint_0^{\mathscr{H}} \mathrm{d}\mathscr{H}'\frac{\mathrm{d}r}{\mathscr{W}_+(r)} \\ &= \iint_{(0,0)}^{(\nu,J)} (\mathrm{d}\nu' - \Omega\mathrm{d}J')\frac{\mathrm{d}r}{\dot r}, \end{aligned} \quad (39)$$

where in the last step, we have used the energy conservation condition $\mathrm{d}\mathscr{H}' = \mathrm{d}\nu' - \Omega\mathrm{d}J'$, and thus the adiabatic invariant quantity given by Eq. (37) reads

$$I_\nu = 2\iint_{(0,0)}^{(\nu,J)} (\mathrm{d}\nu' - \Omega\mathrm{d}J')\frac{\mathrm{d}r}{\dot r}. \quad (40)$$

Note that τ has a periodicity of $2\pi/\kappa_+$. Since we are considering only the outgoing paths, the integration limit for τ will be $0 \leqslant \tau \leqslant \pi/\kappa_+$. Hence, the adiabatic invariant quantity reads

$$I_\nu = 2\pi\int_{(0,0)}^{(\nu,J)} \frac{\mathrm{d}\nu' - \Omega\mathrm{d}J'}{\kappa_+}. \quad (41)$$

It is known that the temperature of a black hole is proportional to the surface gravity of the horizon

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi}, \quad (42)$$

and thus the adiabatic-invariant quantity given in Eq. (42) becomes

$$I_\nu = \hbar\int_{(0,0)}^{(\nu,J)} \frac{\mathrm{d}\nu' - \Omega\mathrm{d}J'}{T_{\mathrm{H}}} = \hbar S_{\mathrm{BH}}, \quad (43)$$

where the first law of the three-dimensional Gödel black hole thermodynamics $\mathrm{d}\nu = T\mathrm{d}S + \Omega\mathrm{d}J$ has been used. According to the Bohr–Sommerfield quantization rule

$$I_\nu = \oint p\mathrm{d}q = nh = 2\pi n\hbar,\ n = 1, 2, 3, \ldots \quad (44)$$

we derive the black hole entropy spectrum

$$S_{\mathrm{BH}} = 2\pi n\hbar,\ n = 1, 2, 3, \ldots \quad (45)$$

It is evident that the spacing in the entropy spectrum is given by

$$\Delta S_{\mathrm{BH}} = S_{(n+1)\mathrm{BH}} - S_{(n)\mathrm{BH}} = 2\pi. \quad (46)$$

Since black hole entropy is proportional to the black hole horizon area

$$S_{\rm BH} = \frac{A}{4G} = \frac{A}{4l_P^2}, \tag{47}$$

the quantum of the black hole horizon area is of the form

$$\Delta A = 8\pi l_P^2. \tag{48}$$

This is identical to the area quantum effect derived by the previous method.

4. Conclusion

In this paper, we have succeeded in quantizing the horizon area of the three-dimensional Gödel black hole. It is noteworthy that the derived area spectrum has not been employed in quasinormal modes. On the other hand, the obtained area spectrum is $\Delta A = 8\pi l_P^2$, which is consistent with the claim of Ref. [45], that the quantum area ΔA may be universal for all black holes.

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