Approximate any l-state solutions of the Dirac equation for modified deformed Hylleraas potential by using the Nikiforov–Uvarov method

H. Hassanabadi[a)†], E. Maghsoodi[a)], S. Zarrinkamar[b)], and H. Rahimov[c)]

[a)] *Physics Department, Shahrood University of Technology, Shahrood, Iran*
[b)] *Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[c)] *Computer Engineering Department, Shahrood University of Technology, Shahrood, Iran*

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We investigate the spin and pseudospin symmetries of the Dirac equation under modified deformed Hylleraas potential via a Pekeris approximation and the Nikiforov–Uvarov technique. A tensor interaction of Coulomb form is considered and its degeneracy-removing role is discussed in detail. The solutions are reported for an arbitrary quantum number in a compact form and useful numerical data are included.

Keywords: Dirac equation, spin symmetry, pseudospin symmetry, modified Hylleraas potential

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1. Introduction

It is well known that the spin and pseudospin symmetries of the Dirac equation have successfully accounted for the particle and nuclear spectrums.[1,2] The Dirac equation, after separation of variables, leads to a Schrödinger-like equation. Therefore, it seems logical that the methods originally used to solve the nonrelativistic Schrödinger equation can be applied to the problem. This has been done by many authors under various potentials and various techniques have been used. This lengthy list includes Coulomb, harmonic oscillator, Hartmann potential, Woods–Saxon, Morse, Eckart, Pöschl–Teller, pseudoharmonic, Yukawa, etc.[3−29] Within our study, we consider a modified deformed Hylleraas interaction.[30−32] In addition, we take into account a tensor interaction which removes the degeneracy between the doublets and therefore plays a significant role in theoretical spectroscopy. To do so, we first review the Dirac equation under these symmetries. Next, we briefly introduce the Nikiforov–Uvarov (NU) technique, which stands as one of the most powerful tools of mathematical physics.[33] After a transformation, we then bring the obtained differential equation into a form solvable via the NU method. In our final section, we give our conclusions and depict the behaviors of the solutions versus the various parameters engaged.

2. Dirac equation

The Dirac equation with the scalar potential $S(r)$, vector potential $V(r)$, and a tensor potential $U(r)$ is

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta(M+S(r))-\mathrm{i}\beta\boldsymbol{\alpha}\cdot\hat{r}U(r)]\,\psi(\boldsymbol{r}) = [E-V(r)]\,\psi(\boldsymbol{r}), \quad (1)$$

where E, $\boldsymbol{p}=-\mathrm{i}\boldsymbol{\nabla}$, and M respectively denote the relativistic energy of the system, the three-dimensional momentum operator, and the mass of the particle. α and β matrices are

$$\alpha=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \quad (2)$$

where I is 2×2 unitary matrix and the spin matrices are

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \quad (3)$$

The total angular momentum operator $\boldsymbol{J}$ and the spin–orbit coupling operator $K=(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$, where $\boldsymbol{L}$ denotes orbital angular momentum, of the spherical nucleons commuting with the Hamiltonian. The eigenvalues of spin–orbit coupling operator are $\kappa=(j+1/2)>0$ and $\kappa=-(j+1/2)<0$ for unaligned

†Corresponding author. E-mail: h.hasanabadi@shahroodut.ac.ir

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spin $j = \ell - 1/2$ and the aligned spin $j = \ell + 1/2$, respectively. The set (H^2, K, J^2, J_z) is taken as the complete set of the conservative quantities. Thus, the spinors can be written as

$$\psi_{nk}(\boldsymbol{r}) = \begin{pmatrix} f_{nk}(\boldsymbol{r}) \\ g_{nk}(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \dfrac{F_{nk}(\boldsymbol{r})}{r} \mathrm{Y}^{\ell}_{jm}(\theta,\varphi) \\ \mathrm{i}\dfrac{G_{nk}(\boldsymbol{r})}{r} \mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) \end{pmatrix}, \quad (4)$$

where $f_{nk}(\boldsymbol{r})$ is the upper (large) component and $g_{nk}(\boldsymbol{r})$ is the lower (small) component of the Dirac spinors. $\mathrm{Y}^{\ell}_{jm}(\theta,\varphi)$ and $\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi)$ respectively stand for spin and pseudospin spherical harmonics and m is the projection of the angular momentum along the z axis. Using the simple but useful relations

$$(\boldsymbol{\sigma}\cdot\boldsymbol{A})(\boldsymbol{\sigma}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{A}\times\boldsymbol{B}), \quad (5a)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p}) = \boldsymbol{\sigma}\cdot\hat{r}(\hat{r}\cdot\boldsymbol{p} + \mathrm{i}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{L}}{r}), \quad (5b)$$

as well as

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) = (\kappa-1)\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi), \quad (6a)$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{L})\mathrm{Y}^{\ell}_{jm}(\theta,\varphi) = -(\kappa-1)\mathrm{Y}^{\ell}_{jm}(\theta,\varphi), \quad (6b)$$

$$(\boldsymbol{\sigma}\cdot\hat{r})\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi) = -\mathrm{Y}^{\ell}_{jm}(\theta,\varphi), \quad (6c)$$

$$(\boldsymbol{\sigma}\cdot\hat{r})\mathrm{Y}^{\ell}_{jm}(\theta,\varphi) = -\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\varphi), \quad (6d)$$

we arrive at the following coupled differential equations for upper and lower radial wavefunctions

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right)F_{nk}(r) = (M + E_{nk} - \Delta(r))G_{nk}(r), \quad (7a)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right)G_{nk}(r) = (M - E_{nk} + \Sigma(r))F_{nk}(r), \quad (7b)$$

where

$$\Delta(r) = V(r) - S(r), \quad (8a)$$

$$\Sigma(r) = V(r) + S(r). \quad (8b)$$

From Eq. (7), we find

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r) + \frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{M + E_{n\kappa} - \Delta(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\kappa}{r} - U(r)\right)\right\}F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))F_{n\kappa}(r), \quad (9)$$

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{\mathrm{d}U(r)}{\mathrm{d}r} - U^2(r) + \frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M - E_{n\kappa} + \Sigma(r)}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{r} + U(r)\right)\right\}G_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))G_{n\kappa}(r), \quad (10)$$

where $\kappa(\kappa-1) = \tilde{\ell}(\tilde{\ell}+1)$ and $\kappa(\kappa+1) = \ell(\ell+1)$.

2.1. Pseudospin symmetry (PSS) limit

The pseudospin symmetry occurs when $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or equivalently $\Sigma(r) = C_{\mathrm{ps}} = \text{const}$. Here, our desired interaction is the modified deformed Hylleraas potential[30–32]

$$\Delta(r) = \frac{V_0}{b}\left(\frac{a - \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}\right) - V_1\frac{\mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}} + V_2\frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2}, \quad (11)$$

$$U(r) = -\frac{H}{r}. \quad (12)$$

Now, using the following alternative for the centrifugal term

$$\frac{1}{r^2} \approx \frac{\alpha^2\,\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2} \quad (13)$$

yields

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{(1 - \mathrm{e}^{-2\alpha r})}\left[\frac{MV_0 a}{b} - \frac{MV_0\,\mathrm{e}^{-2\alpha r}}{b} - \frac{V_0 a E^{\mathrm{ps}}_{n\kappa}}{b} + \frac{V_0 E^{\mathrm{ps}}_{n\kappa}\,\mathrm{e}^{-2\alpha r}}{b} + \frac{V_0 a C_{\mathrm{ps}}}{b} - \frac{V_0 C_{\mathrm{ps}}\,\mathrm{e}^{-2\alpha r}}{b} - MV_1\,\mathrm{e}^{-2\alpha r} + E^{\mathrm{ps}}_{n\kappa}V_1\,\mathrm{e}^{-2\alpha r} - V_1 C_{\mathrm{ps}}\,\mathrm{e}^{-2\alpha r}\right] + \frac{1}{(1 - \mathrm{e}^{-2\alpha r})^2}[-\delta\alpha^2\,\mathrm{e}^{-2\alpha r} + MV_2\,\mathrm{e}^{-2\alpha r} - E^{\mathrm{ps}}_{n\kappa}V_2\,\mathrm{e}^{-2\alpha r} + V_2 C_{\mathrm{ps}}\,\mathrm{e}^{-2\alpha r}] - M^2 - MC_{\mathrm{ps}} + (E^{\mathrm{ps}}_{n\kappa})^2 - C_{\mathrm{ps}}E^{\mathrm{ps}}_{n\kappa}\right\}G^{\mathrm{ps}}_{n\kappa}(r) = 0, \quad (14)$$

where

$$\delta = k(k-1) + 2kH - H + H^2 = (k+H)(k+H-1) = \eta_k(\eta_k - 1) \to \eta_k = k + H, \quad (15)$$

and $\kappa = -\tilde{\ell}$ and $\kappa = \tilde{\ell} + 1$ for $\kappa < 0$ and $\kappa > 0$, respectively.

2.2. Spin symmetry limit

In the spin symmetry limit $\mathrm{d}\Delta(r)/\mathrm{d}r = 0$ or $\Delta(r) = C_{\mathrm{s}} = \text{const}$.[1,2] As the previous section, we consider

$$\Sigma(r) = \frac{V_0}{b}\left(\frac{a - \mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}}\right) - V_1\frac{\mathrm{e}^{-2\alpha r}}{1 - \mathrm{e}^{-2\alpha r}} + V_2\frac{\mathrm{e}^{-2\alpha r}}{(1 - \mathrm{e}^{-2\alpha r})^2}. \quad (16)$$

Substitution of latter into Eq. (9) gives

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{(1-\mathrm{e}^{-2\alpha r})}\Big[-\frac{MV_0a}{b}+\frac{MV_0\,\mathrm{e}^{-2\alpha r}}{b}+MV_1\,\mathrm{e}^{-2\alpha r}-\frac{V_0aE^{\mathrm{s}}_{n\kappa}}{b}+\frac{V_0E^{\mathrm{s}}_{n\kappa}\,\mathrm{e}^{-2\alpha r}}{b}+E^{\mathrm{s}}_{n\kappa}V_1\,\mathrm{e}^{-2\alpha r}+\frac{V_0aC_{\mathrm{s}}}{b}-\frac{V_0C_{\mathrm{s}}\,\mathrm{e}^{-2\alpha r}}{b}-V_1C_{\mathrm{s}}\,\mathrm{e}^{-2\alpha r}\Big]+\frac{1}{(1-\mathrm{e}^{-2\alpha r})^2}[-\gamma\alpha^2\,\mathrm{e}^{-2\alpha r}-MV_2\,\mathrm{e}^{-2\alpha r}-E^{\mathrm{s}}_{n\kappa}V_2\,\mathrm{e}^{-2\alpha r}+V_2C_{\mathrm{s}}\,\mathrm{e}^{-2\alpha r}]-M^2+(E^{\mathrm{s}}_{n\kappa})^2+MC_{\mathrm{s}}-C_{\mathrm{s}}E^{\mathrm{s}}_{n\kappa}\right\}F^{\mathrm{s}}_{n\kappa}(r)=0, \tag{17}$$

where

$$\begin{aligned}\gamma &= (k+H+1)(k+H)\\ &= \eta_k(\eta_k-1)\to\eta_k=k+H+1,\end{aligned} \tag{18}$$

and $\kappa=\ell$ and $\kappa=-\ell-1$ for $\kappa<0$ and $\kappa>0$, respectively. We shall now introduce the NU method to proceed.

3. The NU method

As frequently noted, this mathematical technique solves second-order differential equations. Let us consider the following differential equation

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}s^2}+\frac{\alpha_1-\alpha_2s}{s(1-\alpha_3s)}\frac{\mathrm{d}}{\mathrm{d}s}+\frac{1}{[s(1-\alpha_3s)]^2}\times[-\xi_1s^2+\xi_2s-\xi_3]\right\}\psi=0. \tag{19}$$

According to the NU method, the eigenfunctions and eigenenergies respectively are

$$\begin{aligned}\psi(s) &= s^{\alpha_{12}}(1-\alpha_3s)^{-\alpha_{12}-\alpha_{13}/\alpha_3}\\ &\quad\times P_n^{(\alpha_{10}-1,\alpha_{11}/\alpha_3-\alpha_{10}-1)}(1-2\alpha_3s)\end{aligned} \tag{20}$$

and

$$\alpha_2n-(2n+1)\alpha_5+(2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8})+n(n-1)\alpha_3+\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0, \tag{21}$$

where

$$\begin{aligned}&\alpha_4=\frac{1}{2}(1-\alpha_1),\quad \alpha_5=\frac{1}{2}(\alpha_2-2\alpha_3),\\ &\alpha_6=\alpha_5^2+\xi_1,\quad \alpha_7=2\alpha_4\alpha_5-\xi_2,\quad \alpha_8=\alpha_4^2+\xi_3,\\ &\alpha_9=\alpha_3\alpha_7+\alpha_3^2\alpha_8+\alpha_6,\quad \alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8},\\ &\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}),\\ &\alpha_{12}=\alpha_4+\sqrt{\alpha_8},\end{aligned}$$

$$\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}). \tag{22}$$

In the rather more special case of $\alpha_3=0$,

$$\begin{aligned}&\lim_{\alpha_3\to0}P_n^{(\alpha_{10}-1,\alpha_{11}/\alpha_3-\alpha_{10}-1)}(1-2\alpha_3s)\\ &=L_n^{\alpha_{10}-1}(\alpha_{11}s),\end{aligned} \tag{23a}$$

$$\lim_{\alpha_3\to0}(1-\alpha_3s)^{-\alpha_{12}-\alpha_{13}/\alpha_3}=\mathrm{e}^{\alpha_{13}s}, \tag{23b}$$

and, from Eq. (20), we find for the wavefunction

$$\psi=s^{\alpha_{12}}\,\mathrm{e}^{\alpha_{13}s}L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{24}$$

3.1. Approximate bound-state solutions for pseudospin symmetry limit

To obtain the solution of Eq. (14), bearing in mind Eqs. (19), we apply a transformation $z=\mathrm{e}^{-2\alpha r}$ which brings us to

$$\frac{\mathrm{d}^2G^{\mathrm{ps}}_{n\kappa}(z)}{\mathrm{d}z^2}+\frac{(1-z)}{z(1-z)}\frac{\mathrm{d}G^{\mathrm{ps}}_{n\kappa}(z)}{\mathrm{d}z}+\frac{1}{(z(1-z))^2}(k^{\mathrm{ps}}z^2+L^{\mathrm{ps}}z+f^{\mathrm{ps}})G^{\mathrm{ps}}_{n\kappa}(z)=0, \tag{25}$$

where

$$\begin{aligned}k^{\mathrm{ps}} &= \frac{MV_0}{4\alpha^2b}-\frac{V_0E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2b}+\frac{V_0C_{\mathrm{ps}}}{4\alpha^2b}+\frac{MV_1}{4\alpha^2}-\frac{V_1E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2}+\frac{V_1C_{\mathrm{ps}}}{4\alpha^2}-\frac{M^2}{4\alpha^2}\\ &\quad-\frac{MC_{\mathrm{ps}}}{4\alpha^2}+\frac{(E^{\mathrm{ps}}_{n\kappa})^2}{4\alpha^2}-\frac{C_{\mathrm{ps}}E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2},\end{aligned} \tag{26a}$$

$$\begin{aligned}L^{\mathrm{ps}} &= -\frac{\delta}{4}+\frac{MV_2}{4\alpha^2}-\frac{V_2E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2}+\frac{V_2C_{\mathrm{ps}}}{4\alpha^2}-\frac{MV_0a}{4\alpha^2b}-\frac{MV_0}{4\alpha^2b}+\frac{V_0E^{\mathrm{ps}}_{n\kappa}a}{4\alpha^2b}\\ &\quad+\frac{V_0E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2b}-\frac{V_0aC_{\mathrm{ps}}}{4\alpha^2b}-\frac{V_0C_{\mathrm{ps}}}{4\alpha^2b}-\frac{MV_1}{4\alpha^2}+\frac{V_1E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2}-\frac{V_1C_{\mathrm{ps}}}{4\alpha^2}+\frac{2M^2}{4\alpha^2}\\ &\quad+\frac{2MC_{\mathrm{ps}}}{4\alpha^2}-\frac{2(E^{\mathrm{ps}}_{n\kappa})^2}{4\alpha^2}+\frac{2C_{\mathrm{ps}}E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2},\end{aligned} \tag{26b}$$

$$\begin{aligned}f^{\mathrm{ps}} &= \frac{MV_0a}{4\alpha^2b}-\frac{V_0E^{\mathrm{ps}}_{n\kappa}a}{4\alpha^2b}+\frac{V_0aC_{\mathrm{ps}}}{4\alpha^2b}-\frac{M^2}{4\alpha^2}\\ &\quad-\frac{MC_{\mathrm{ps}}}{4\alpha^2}+\frac{(E^{\mathrm{ps}}_{n\kappa})^2}{4\alpha^2}-\frac{C_{\mathrm{ps}}E^{\mathrm{ps}}_{n\kappa}}{4\alpha^2}.\end{aligned} \tag{26c}$$

By comparing Eqs. (25)–(26c), with Eq. (22), we find

$$\begin{aligned}&\alpha_1=1,\quad \alpha_2=1,\quad \alpha_3=1,\\ &\xi_1=-k^{\mathrm{ps}},\quad \xi_2=L^{\mathrm{ps}},\quad \xi_3=-f^{\mathrm{ps}},\\ &\alpha_4=0,\quad \alpha_5=-\frac{1}{2},\quad \alpha_6=\frac{1}{4}+\xi_1,\quad \alpha_7=-\xi_2,\\ &\alpha_8=\xi_3,\quad \alpha_9=\xi_1-\xi_2+\xi_3+\frac{1}{4},\\ &\alpha_{10}=1+2\sqrt{\xi_3},\\ &\alpha_{11}=2+2\left(\sqrt{\xi_1-\xi_2+\xi_3+\frac{1}{4}}+\sqrt{\xi_3}\right),\end{aligned}$$

$$\alpha_{12} = \sqrt{\xi_3},$$
$$\alpha_{13} = -\frac{1}{2} - \left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right). \quad (27)$$

Thus, by putting Eq. (27) into Eq. (21), we find

$$n + \frac{(2n+1)}{2} + (2n+1)\left(\sqrt{-k^{\rm ps} - L^{\rm ps} - f^{\rm ps} + \frac{1}{4}} + \sqrt{-f^{\rm ps}}\right) + n(n-1) - L^{\rm ps} - 2f^{\rm ps} + 2\sqrt{k^{\rm ps}f^{\rm ps} + L^{\rm ps}f^{\rm ps} + f^{\rm ps\,2} - \frac{f^{\rm ps}}{4}} = 0. \quad (28)$$

For this case the upper and lower components of the wave function are as

$$G^{\rm ps}_{n\kappa}(r) = (\,{\rm e}^{-2\alpha r})^{\sqrt{-f^{\rm ps}}} \times (1 - {\rm e}^{-2\alpha r})^{1/2+\sqrt{-k^{\rm ps}-L^{\rm ps}-f^{\rm ps}+1/4}} P_n^{(2\sqrt{-f^{\rm ps}},2\sqrt{-k^{\rm ps}-L^{\rm ps}-f^{\rm ps}+1/4})}(1 - 2\,{\rm e}^{-2\alpha r}), \quad (29a)$$

and the other component can be simply found via

$$F_{{\rm ps},n\kappa}(r) = \frac{1}{M - E^{\rm ps}_{n\kappa} + C_{\rm ps}} \times \left(\frac{\rm d}{{\rm d}r} - \frac{k}{r} + U(r)\right) G^{\rm ps}_{n\kappa}(r). \quad (29b)$$

3.2. Exact bound-state solutions for spin symmetry limit

In this limit, we have to deal with

$$\frac{{\rm d}^2F^{\rm s}_{n\kappa}(z)}{{\rm d}z^2} + \frac{(1-z)}{z(1-z)}\frac{{\rm d}F^{\rm s}_{n\kappa}(z)}{{\rm d}z} + \frac{1}{(z(1-z))^2} \times (k^{\rm s}z^2 + L^{\rm s}z + f^{\rm s})F^{\rm s}_{n\kappa}(z) = 0, \quad (30)$$

where

$$k^{\rm s} = -\frac{MV_0}{4\alpha^2 b} - \frac{MV_1}{4\alpha^2} - \frac{V_0E^{\rm s}_{n\kappa}}{4\alpha^2 b} - \frac{V_1E^{\rm s}_{n\kappa}}{4\alpha^2} + \frac{V_0C_{\rm s}}{4\alpha^2 b} + \frac{V_1C_{\rm s}}{4\alpha^2} - \frac{M^2}{4\alpha^2} + \frac{(E^{\rm s}_{n\kappa})^2}{4\alpha^2} + \frac{MC_{\rm s}}{4\alpha^2} - \frac{C_{\rm s}E^{\rm s}_{n\kappa}}{4\alpha^2}, \quad (31a)$$

$$L^{\rm s} = -\frac{\gamma}{4} - \frac{MV_2}{4\alpha^2} - \frac{V_2E^{\rm s}_{n\kappa}}{4\alpha^2} + \frac{V_2C_{\rm s}}{4\alpha^2} + \frac{MV_0a}{4\alpha^2 b} + \frac{MV_0}{4\alpha^2 b} + \frac{MV_1}{4\alpha^2} + \frac{V_0E^{\rm s}_{n\kappa}a}{4\alpha^2 b} + \frac{V_0E^{\rm s}_{n\kappa}}{4\alpha^2 b} + \frac{V_1E^{\rm s}_{n\kappa}}{4\alpha^2} - \frac{V_0aC_{\rm s}}{4\alpha^2 b} - \frac{V_0C_{\rm s}}{4\alpha^2 b} - \frac{V_1C_{\rm s}}{4\alpha^2} + \frac{2M^2}{4\alpha^2} - \frac{2(E^{\rm s}_{n\kappa})^2}{4\alpha^2} - \frac{2MC_{\rm s}}{4\alpha^2} + \frac{2C_{\rm s}E^{\rm s}_{n\kappa}}{4\alpha^2}, \quad (31b)$$

$$f^{\rm s} = -\frac{MV_0a}{4\alpha^2 b} - \frac{V_0E^{\rm s}_{n\kappa}a}{4\alpha^2 b} + \frac{V_0aC_{\rm s}}{4\alpha^2 b} - \frac{M^2}{4\alpha^2} + \frac{(E^{\rm s}_{n\kappa})^2}{4\alpha^2} + \frac{MC_{\rm s}}{4\alpha^2} - \frac{C_{\rm s}E^{\rm s}_{n\kappa}}{4\alpha^2}. \quad (31c)$$

By comparing Eqs. (30)–(31c) with Eq. (22), we find the correspondence

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1,$$
$$\xi_1 = -k^{\rm s}, \quad \xi_2 = L^{\rm s}, \quad \xi_3 = -f^{\rm s},$$
$$\alpha_4 = 0, \quad \alpha_5 = -\frac{1}{2}, \quad \alpha_6 = \frac{1}{4} + \xi_1, \quad \alpha_7 = -\xi_2,$$
$$\alpha_8 = \xi_3, \quad \alpha_9 = \xi_1 - \xi_2 + \xi_3 + \frac{1}{4},$$
$$\alpha_{10} = 1 + 2\sqrt{\xi_3},$$
$$\alpha_{11} = 2 + 2\left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right),$$
$$\alpha_{12} = \sqrt{\xi_3},$$
$$\alpha_{13} = -\frac{1}{2} - \left(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3}\right). \quad (32)$$

Substituting Eq. (32) into Eq. (21) immediately gives

$$n + \frac{(2n+1)}{2} + (2n+1) \times \left(\sqrt{-k^{\rm s} - L^{\rm s} - f^{\rm s} + \frac{1}{4}} + \sqrt{-f^{\rm s}}\right) + n(n-1) - L^{\rm s} - 2f^{\rm s} + 2\sqrt{k^{\rm s}f^{\rm s} + L^{\rm s}f^{\rm s} + f^{\rm s\,2} - \frac{f^{\rm s}}{4}} = 0, \quad (33)$$

and the upper and lower components of the wave function are

$$F^{\rm s}_{n\kappa}(r) = (\,{\rm e}^{-2\alpha r})^{\sqrt{-f^{\rm s}}} \times (1 - {\rm e}^{-2\alpha r})^{1/2+\sqrt{-k^{\rm s}-L^{\rm s}-f^{\rm s}+1/4}} \times P_n^{(2\sqrt{-f^{\rm s}},2\sqrt{-k^{\rm s}-L^{\rm s}-f^{\rm s}+1/4})}(1 - 2\,{\rm e}^{-2\alpha r}), \quad (34a)$$

and

$$G^{\rm s}_{n\kappa}(r) = \frac{1}{M + E^{\rm s}_{n\kappa} - C_{\rm s}} \times \left(\frac{\rm d}{{\rm d}r} + \frac{k}{r} - U(r)\right) F^{\rm s}_{n\kappa}(r). \quad (34b)$$

Figures 1 and 2 show energy versus H and α for pseudospin and spin symmetry limits. Figures 3–5 represent energy versus V_0, V_1, and V_2 for pseudospin and spin symmetry limits, respectively. Figures 6 and 7 show energy versus a and b for both the symmetry limits, respectively. In Tables 1 and 2 we have portrayed the energy values for various H. Tables 3 and 4 show the energy for different values of $C_{\rm ps}$ and $C_{\rm s}$ respectively. In Tables 5 and 6 we have reported energy for different various M. The wavefunctions for pseudospin symmetry limit and spin symmetry limit are plotted from Eqs. (29a) and (29b) and Eqs. (34a) and (34b) in Fig. 8 for the case without tensor interaction and for that with tensor interaction.

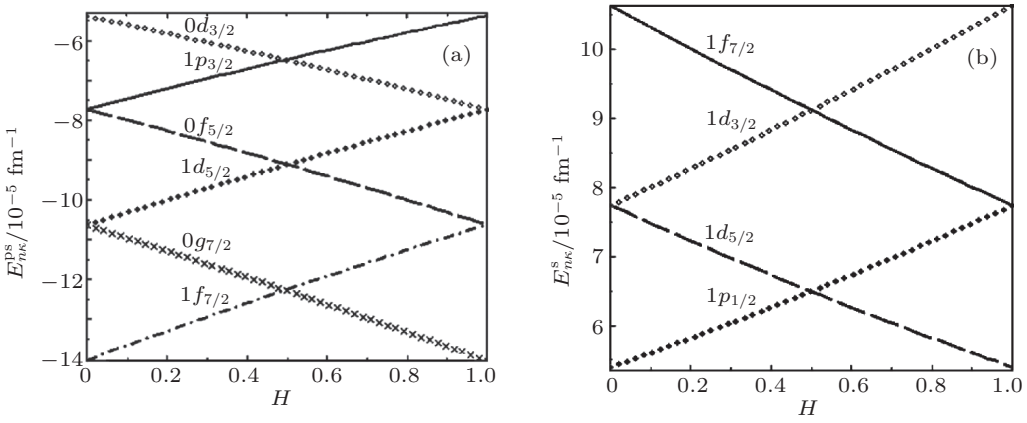


Fig. 1. (a) Energy versus H for pseudospin symmetry limit for $\alpha = 0.01$, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$. (b) Energy versus H for spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, $V_0 = 0.8$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$, 1 fm=10^{-15} m.

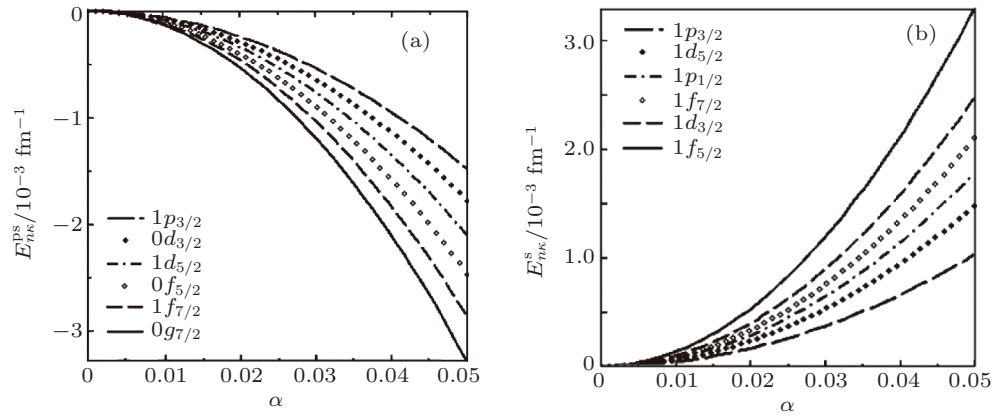


Fig. 2. (a) Energy versus α for pseudospin symmetry limit for $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$. (b) Energy versus α for spin symmetry limit for $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, V_0=0.8, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

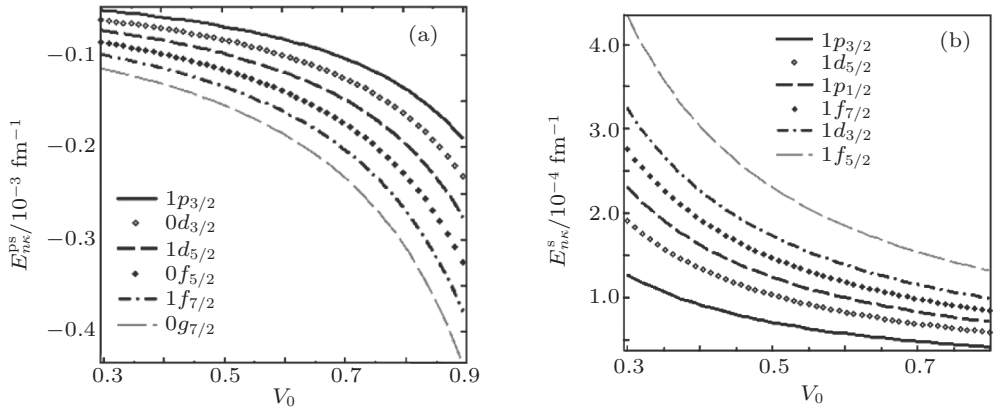


Fig. 3. (a) Energy versus V_0 for pseudospin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $H = 0.75$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$. (b) Energy versus V_0 for spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, $H = 0.75$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

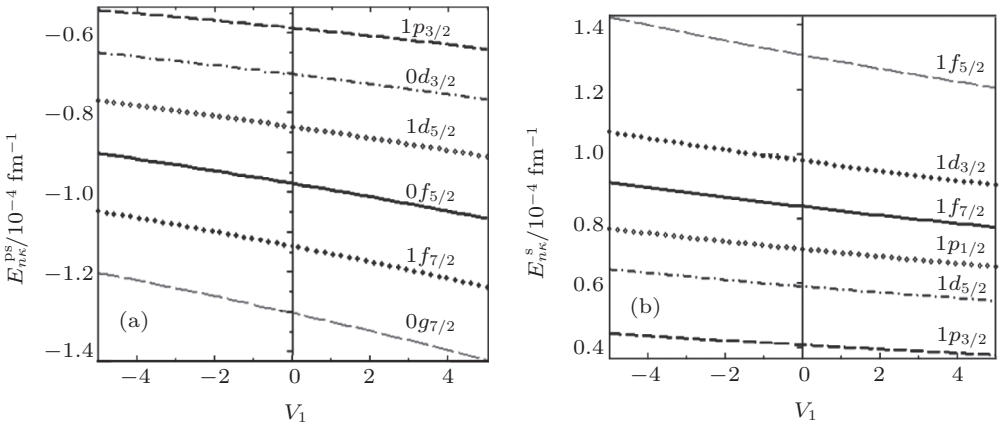


Fig. 4. (a) Energy versus V_1 for pseudospin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $V_0 = -0.8$, $H = 0.75$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$. (b) Energy versus V_1 for spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, V_0=0.8, $H = 0.75$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

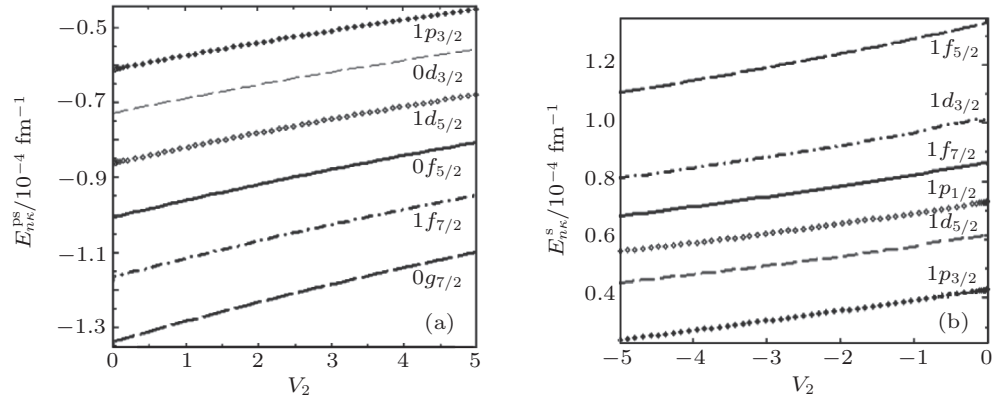


Fig. 5. (a) Energy versus V_2 for pseudospin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, V_0= $-$ 0.8, $V_1 = 0.4$, $H = 0.75$, $a = 0.04$, $b = 0.01$. (b) Energy versus V_2 for spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, V_0=0.8, $V_1 = -0.4$, $H = 0.75$, $a = 0.04$, $b = 0.01$.

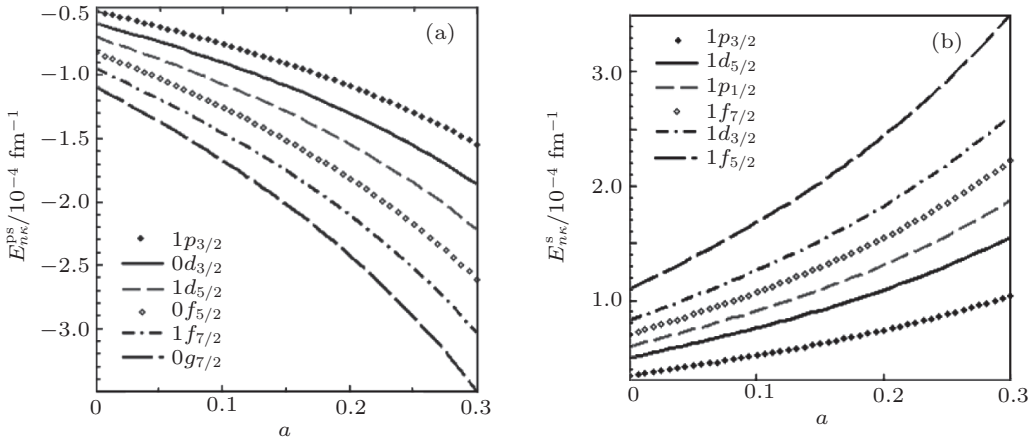


Fig. 6. (a) Energy versus a for pseudospin symmetry limit for α= 0.01, $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $b = 0.01$. (b) Energy versus a for spin symmetry limit for α= 0.01, $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, V_0= 0.8, $V_1 = -0.4$, $V_2 = -0.5$, $b = 0.01$.

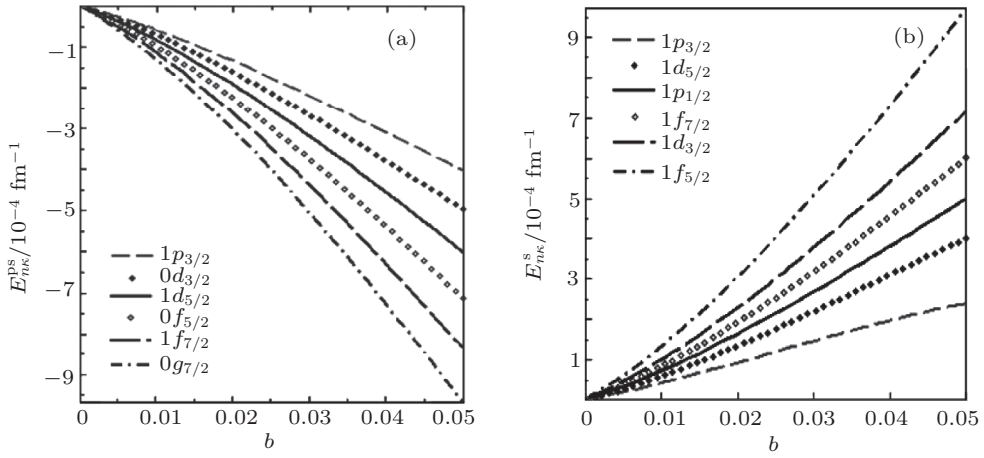


Fig. 7. (a) Energy versus b for pseudospin symmetry limit for α= 0.01, $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$. (b) Energy versus b for spin symmetry limit for α= 0.01, $H = 0.75$, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, $V_0 = 0.8$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$.

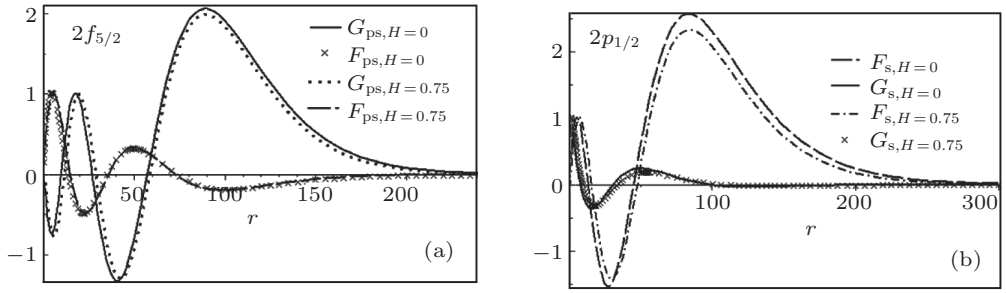


Fig. 8. (a) Wavefunction for pseudospin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, V_0= − 0.8, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$. (b) Wavefunction for spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, V_0=0.8, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

Table 1. Energies in the pseudospin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm ps} = -5$, V_0= − 0.8, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$.

$\tilde{\ell}$	(ℓ, j)	$n, \kappa < 0$	$E^{\rm ps}_{n\kappa}$/fm^{-1}($H = 0$)	$E^{\rm ps}_{n\kappa}$/fm^{-1}($H = 0.75$)	$n-1, \kappa > 0$	$(\ell+2, j+1)$	$E^{\rm ps}_{n\kappa}$/fm^{-1}($H = 0$)	$E^{\rm ps}_{n\kappa}$/fm^{-1}($H = 0.75$)
1	$1s_{1/2}$	1, –1	–0.000053945	–0.000040932	0, 2	$0d_{3/2}$	–0.000053945	–0.000070968
2	$1p_{3/2}$	1, –2	–0.000077369	–0.000059236	0, 3	$0f_{5/2}$	–0.000077369	–0.00098585
3	$1d_{5/2}$	1, –3	–0.000106310	–0.000084110	0, 4	$0g_{7/2}$	–0.000106310	–0.000131398
4	$1f_{7/2}$	1, –4	–0.000140393	–0.000114355	0, 5	$0h_{9/2}$	–0.00140393	–0.000169257
1	$2s_{1/2}$	2, –1	–0.000107554	–0.000087781	1, 2	$1d_{3/2}$	–0.000107554	–0.000131734
2	$2p_{3/2}$	2, –2	–0.000140547	–0.000115212	1, 3	$1f_{5/2}$	–0.000140547	–0.000169031
3	$2d_{5/2}$	2, –3	–0.000179186	–0.000149706	1, 4	$1g_{7/2}$	–0.000179186	–0.000211576
4	$2f_{7/2}$	2, –4	–0.000223007	–0.000189663	1, 5	$1h_{9/2}$	–0.000223007	–0.000259186
1	$3s_{1/2}$	3, –1	–0.000178063	–0.000149730	2, 2	$2d_{3/2}$	–0.000178063	–0.000210133
2	$3p_{3/2}$	3, –2	–0.000221509	–0.000188395	2, 3	$2f_{5/2}$	–0.000221509	–0.000257583
3	$3d_{5/2}$	3, –3	–0.000270245	–0.000233213	2, 4	$2g_{7/2}$	–0.000270245	–0.000310115
4	$3f_{7/2}$	3, –4	–0.000324029	–0.000283222	2, 5	$2h_{9/2}$	–0.000324029	–0.000367638

Table 2. Energies in the spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $C_{\rm s} = 5$, $V_0 = 0.8$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

ℓ	$n, \kappa < 0$	(ℓ, j)	$E^{\rm s}_{n\kappa}$/fm^{-1}($H = 0$)	$E^{\rm s}_{n\kappa}$/fm^{-1}($H = 0.75$)	$n, \kappa > 0$	(ℓ, j)	$E^{\rm s}_{n\kappa}$/fm^{-1}($H = 0$)	$E^{\rm s}_{n\kappa}$/fm^{-1}($H = 0.75$)
1	0, –2	$0p_{3/2}$	0.000018243	0.000011211	0, 1	$0p_{1/2}$	0.000018243	0.000028493
2	0, –3	$0d_{5/2}$	0.000032567	0.000021324	0, 2	$0d_{3/2}$	0.000032567	0.000046697
3	0, –4	$0f_{7/2}$	0.000052036	0.000036960	0, 3	$0f_{5/2}$	0.000052036	0.000069926
4	0, –5	$0g_{9/2}$	0.000076512	0.000057688	0, 4	$0g_{7/2}$	0.000076512	0.000098126
1	1, –2	$1p_{3/2}$	0.000053945	0.000040932	1, 1	$1p_{1/2}$	0.000053945	0.000070968
2	1, –3	$1d_{5/2}$	0.000077369	0.000059236	1, 2	$1d_{3/2}$	0.000077369	0.000098585
3	1, –4	$1f_{7/2}$	0.000106310	0.000084110	1, 3	$1f_{5/2}$	0.000106310	0.000131398
4	1, –5	$1g_{9/2}$	0.000140393	0.000114355	1, 4	$1g_{7/2}$	0.000140393	0.000169257
1	2, –2	$2p_{3/2}$	0.000107554	0.000087781	2, 1	$2p_{1/2}$	0.000107554	0.000131734
2	2, –3	$2d_{5/2}$	0.000140547	0.000115212	2, 2	$2d_{3/2}$	0.000140547	0.000169031
3	2, –4	$2f_{7/2}$	0.000179186	0.000149706	2, 3	$2f_{5/2}$	0.000179186	0.000211576
4	2, –5	$2g_{9/2}$	0.000223007	0.000189663	2, 4	$2g_{7/2}$	0.000223007	0.000259186

Table 3. Energies in the pseudospin symmetry limit for α= 0.01, $M = 5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$.

$C_{\rm ps}$	$E^{\rm ps}_{n\kappa}$/fm^{-1}			
	$1p_{3/2}$ ($H = 0$)	$1p_{3/2}$ ($H = 0.75$)	$2f_{7/2}$ ($H = 0$)	$2f_{7/2}$ ($H = 0.75$)
–12	–7.000052622	–7.000040354	–7.000151648	–7.000129082
–10	–5.000061352	–5.000047021	–5.000176819	–5.000150462
–8	–3.000068289	–3.000052316	–3.000196821	–3.000167445
–6	–1.000074461	–1.000057021	–1.000214622	–1.000182549
–4	0.999919812	0.999938618	0.999768864	0.999803443
–2	2.999914386	2.999934488	2.999753212	2.999790173
0	4.999909177	4.999930530	4.999738188	4.999777442
2	6.999904135	6.999926699	6.999723645	6.999765121

Table 4. Energies in the spin symmetry limit for α= 0.01, $M = 5$ fm^{-1}, $V_0 = 0.8$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

$C_{\rm s}$	$E^{\rm s}_{n\kappa}$/fm^{-1}			
	$1p_{3/2}$ ($H = 0$)	$1p_{3/2}$ ($H = 0.75$)	$2f_{7/2}$ ($H = 0$)	$2f_{7/2}$ ($H = 0.75$)
–2	–6.999933285	–6.999949534	–6.999778153	–6.999814861
0	–4.999936761	–4.999952124	–4.999789775	–4.999824506
2	–2.999940356	–2.999954804	–2.999801788	–2.999834479
4	–0.999944105	–0.999957609	–0.999814308	–0.999844885
6	1.000051933	1.000039422	1.000172473	1.000144122
8	3.000047657	3.000036212	3.000158216	3.000132256
10	5.000042846	5.000032588	5.000142186	5.000118902
12	7.000036781	7.000028012	7.000121997	7.000102071

Table 5. Energies in the pseudospin symmetry limit for α= 0.01, $C_{\rm ps} = -5$, $V_0 = -0.8$, $V_1 = 0.4$, $V_2 = 0.5$, $a = 0.04$, $b = 0.01$.

M/fm^{-1}	$E^{\rm ps}_{n\kappa}$/fm^{-1}			
	$1s_{1/2}$ ($H = 0$)	$1s_{1/2}$ ($H = 0.75$)	$0h_{9/2}$ ($H = 0$)	$0h_{9/2}$ ($H = 0.75$)
1	–4.00003183	–4.000024269	–4.000082369	–4.000099256
2	–3.000040071	–3.000030499	–3.000103914	–3.00012524
3	–2.000045342	–2.00003447	–2.000117741	–2.000141921
4	–1.000049844	–1.000037854	–1.000129582	–1.000156209
5	–0.000053945	–0.000040932	–0.000140393	–0.000169257
6	0.999942208	0.999956189	0.999849445	0.999818478
7	1.999938541	1.999953449	1.999839738	1.999806759
8	2.999935009	2.999950814	2.99983037	2.999795447
9	3.999931584	3.999948268	3.999821267	3.999784455
10	4.999928245	4.999945787	4.999812376	4.999773717

Table 6. Energies in the spin symmetry limit for α= 0.01, $C_s = 5$, $V_0 = 0.8$, $V_1 = -0.4$, $V_2 = -0.5$, $a = 0.04$, $b = 0.01$.

M/fm^{-1}	$E^s_{n\kappa}/\mathrm{fm}^{-1}$			
	$0p_{3/2}$ $(H = 0)$	$0p_{3/2}$ $(H = 0.75)$	$1g_{7/2}$ $(H = 0)$	$1g_{7/2}$ $(H = 0.75)$
1	4.000010699	4.000006585	4.000082369	4.000099256
2	3.0000135	3.000008304	3.000103914	3.00012524
3	2.000015298	2.000009406	2.000117741	2.000141921
4	1.000016837	1.00001035	1.000129582	1.000156209
5	0.000018243	0.000011211	0.000140393	0.000169257
6	–0.999980435	–0.99998798	–0.999849445	–0.999818478
7	–1.999979172	–1.999987208	–1.999839738	–1.999806759
8	–2.999977953	–2.999986462	–2.99983037	–2.999795447
9	–3.999976768	–3.999985741	–3.999821267	–3.999784455
10	–4.999975611	–4.999985034	–4.999812376	–4.999773717

4. Conclusion

Due to the interesting feature of the modified deformed Hylleraas potential (an Eckart potential plus deformed Hylleraas potential), which is the more general case of the exponential-type ones frequently present in various physical sciences, we investigated the relativistic symmetries of the Dirac equation under this equation. After a transformation, we converted the problem into a form that can be solved by the powerful NU technique and the corresponding eigenfunctions and eigenvalues were reported. Our numerical data describe the energy splitting in detail. For $H = 0$, one can clearly see that the degeneracy between two states in the pseudospin doublets, i.e., $ns_{1/2}, (n-1)d_{3/2}$ for $\tilde{\ell} = 1$ $(\ell = 0)$, $np_{3/2}, (n-1)f_{5/2}$ for $\tilde{\ell} = 2$ $(\ell = 1)$, $nd_{5/2}$, $(n-1)g_{7/2}$ for $\tilde{\ell} = 3$ $(\ell = 2)$, and $nf_{7/2}$, $(n-1)h_{9/2}$ for $\tilde{\ell} = 4$ $(\ell = 3)$, etc. for different various H, exist other states from degeneracy. From the spin symmetry limit in Table 4, for $H = 0$, one can clearly see the degeneracy between two states in the spin doublets, i.e., $(np_{1/2}, np_{3/2})$ for $\ell = 1$, $(nd_{3/2}, nd_{5/2})$ for $\ell = 2$, $(nf_{5/2}, nf_{7/2})$ for $\ell = 3$, and $(ng_{7/2}, ng_{9/2})$ for $\ell = 4$, etc.

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