Alternative routes to equivalent classical models of a quantum system*

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Coarse-graining of some sort is a fundamental and unavoidable step in any attempt to derive the classical mechanical behavior from the quantum formalism. We utilize the two-mode Bose–Hubbard model to illustrate how different coarse-grained systems can be naturally associated with a fixed quantum system if it is compatible with different dynamical algebras. Alternative coarse-grained systems generate different evolutions of the same physical quantities, and the difference becomes negligible only in the appropriate macro-limit.

Keywords: two-mode Bose–Hubbard model, coarse-grained system, Hamilton operator, system dynamical algebra

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1. Introduction

Reduction of classical mechanics onto a presumably more fundamental quantum formalism is one of the main topics of the theory of quantum to classical relation (QCR). The relevant literature on the topic is vast and we shall single out only a few reviews as illustrative examples. Some of the more formal mathematical aspects of the QCR are treated in Ref. [1]. Putative physical mechanisms and the appropriate ontological considerations underlying the QCR are discussed from different points of view, for example in Refs. [2]–[6]. During the last couple of decades detailed experimental studies of the problems related to QCR have been performed (see for example Refs. [7]–[9]).

In this paper, we analyse possible alternative routes to the classical limit utilizing a particular quantum system, namely the two-mode Bose–Hubbard model (2mBH). The main tool of our analyses is the recently developed formalization of the process that associates classical models with a given quantum system.[10,11] The construction of the classical model proceeds in two clearly distinguished steps: i) a particular, system-dependent coarse-graining, formalized by the appropriate constraint on the quantum system; and ii) the appropriate macro-limit. The system appropriate coarse-graining is determined by the system's dynamical algebra. If, as in the case of the 2mBH model, the quantum system is compatible with different dynamical algebras, then the quantum systems can be coarse-grained into different Hamiltonian dynamical systems. Our main conclusion will be that the alternative coarse-grained systems generate different evolutions of the relevant physical quantities in general. Only in the relevant macro-limit of the coarse-grained systems do the corresponding classical models become canonically equivalent.

2. 2mBH system and its alternative mechanical models

2mBH system is given, up to c-number terms, by the following Hamiltonian (see for example Ref. [12] and the references therein)

$$\hat{H} = \epsilon_1 \hat{a}_1^\dagger \hat{a}_1 + \epsilon_2 \hat{a}_2^\dagger \hat{a}_2 - \delta(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{c}{2}\left((\hat{a}_1^\dagger)^2 \hat{a}_1^2 + (\hat{a}_2^\dagger)^2 \hat{a}_2^2\right), \quad (1)$$

where the operators $\hat{a}_1^\dagger$, $\hat{a}_1$, and $\hat{a}_2^\dagger$, $\hat{a}_2$ are the bosonic creation and annihilation operators for the two degrees of freedom. The Hamiltonian (1) approximates the physical situation created experimentally by confining an atomic Bose–Einstein condensate in a double-well

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trap.[13,14] In this case operators $a_1^\dagger$ and $a_2^\dagger$ are interpreted as creating atoms in the condensate confined in potential wells 1 and 2, respectively. The term proportional to δ describes the tunneling of atoms between the two wells and the term proportional to c describes the non-harmonicity of the potential wells. The operator $\hat{N} = \hat{N}_1 + \hat{N}_2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2$ commutes with the Hamiltonian (1) and represents the conserved total number of atoms in the trapped condensate.

The Hamiltonian (1) can be interpreted as corresponding to alternative but equivalent quantum mechanical models. One is the model of two coupled nonlinear oscillators ($\hat{q}_1$, $\hat{p}_1$, $\hat{q}_2$, $\hat{p}_2$) and the other is the angular momentum $\hat{\boldsymbol{J}}$. Substituting the operators $\hat{q}_1$, $\hat{q}_2$, $\hat{p}_1$, $\hat{p}_2$ related to $\hat{a}_1^\dagger$, $\hat{a}_1$, $\hat{a}_2^\dagger$, $\hat{a}_2$ in the standard way

$$\hat{a}_k = \frac{\hat{q}_k + \mathrm{i}\hat{p}_k}{\sqrt{2}}, \quad \hat{a}_k^\dagger = \frac{\hat{q}_k - \mathrm{i}\hat{p}_k}{\sqrt{2}}, \quad (k = 1, 2), \tag{2}$$

the Hamiltonian (1) becomes, up to c-number terms,

$$\hat{H} = \epsilon_1 \frac{\hat{q}_1^2 + \hat{p}_1^2}{2} + \epsilon_2 \frac{\hat{q}_2^2 + \hat{p}_2^2}{2} - \delta(\hat{q}_1\hat{q}_2 + \hat{p}_1\hat{p}_2) + \frac{c}{2}\left(\frac{(\hat{q}_1^2 + \hat{p}_1^2)^2}{4} + \frac{(\hat{q}_2^2 + \hat{p}_2^2)^2}{4}\right). \tag{3}$$

The operators of reduced coordinate and momentum $\hat{q}_k, \hat{p}_l$ satisfy the commutation relations $[\hat{q}_k, \hat{p}_l] = \mathrm{i}\delta_{kl}$ $(k, l = 1, 2)$. The Hamiltonian (3) corresponds to a system of two interacting nonlinear oscillators with the same type of nonlinearity. Corresponding to the total number of atoms $\hat{N}$ is the quantity $\hat{q}_1^2 + \hat{p}_1^2 + \hat{q}_2^2 + \hat{p}_2^2$, commuting with the Hamiltonian (3). The dynamical algebra of the two models (1) and (3) is of course the same: the direct product of two Heisenberg algebras $h_4 \otimes h_4$.

Two comments concerning the Hamiltonian (3) are in order. The first is concerned with the term interpreted as the interaction between the two oscillators containing products of the momenta $\hat{p}_1\hat{p}_2$. Interaction in the standard models of interacting mechanical oscillators usually depends only on the oscillators coordinates, and therefore the interaction term in Eq. (3) indicates the non-classical origin of the model. The second remark concerns the conserved quantity $\hat{N}$ or $\hat{q}_1^2 + \hat{p}_1^2 + \hat{q}_2^2 + \hat{p}_2^2$. Although it commutes with the Hamiltonian, this quantity is represented by an operator on the systems state space and is therefore considered as all other quantum dynamical observables.

Introducing operators

$$\hat{J}_x = (\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1)/2, \tag{4a}$$

$$\hat{J}_y = \mathrm{i}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1)/2, \tag{4b}$$

$$\hat{J}_z = (\hat{a}_2^\dagger \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_1)/2, \tag{4c}$$

that satisfy the standard $su(2)$ commutation relations the Hamiltonian (1) becomes, up to a constant term,

$$\hat{H} = (\epsilon_2 - \epsilon_1)\hat{J}_z - 2\delta\hat{J}_x + c\hat{J}_z^2. \tag{5}$$

The invariant subspaces of the total Hilbert space are the spaces of irreducible $su(2)$ representations. The total number operator $\hat{N}$ is related to the Casimir operator of the $su(2)$ by the relation $\hat{J}^2 = \hat{N}/2(\hat{N}/2 + 1)$. The system corresponding to Eq. (5) with $su(2)$ dynamical algebra has one degree of freedom, but it is equivalent to the system related to Eq. (1) or Eq. (3) restricted on one of the invariant subspaces. Notice that the conserved quantity $\hat{N}$ is now treated as a number related to the dimension of the system's state space, while it was represented by an operator in the models (1) and (3) and considered as a dynamical observable.

3. Hamiltonian formulation of constrained quantum dynamics and the classical limit

Emergence of classical systems from the quantum background necessary implies an appropriate coarse-graining. A recently introduced formalization of this coarse-graining procedure[10,11] is based on the treatment of nonlinear constraints[15,16] within the Hamiltonian geometric formulation of quantum mechanics (see for example Refs. [15], [17]–[19]).

Consider a quantum system with a dynamical Lie algebra g. The Hilbert space of the system $\mathscr{H}$ can be considered as a real Riemannian and symplectic manifold $\mathscr{M}$. Thus the manifold $\mathscr{M}$ associated with the Hilbert space $\mathscr{H}$ can be viewed as a phase space of a Hamiltonian dynamical system. A vector $|\psi\rangle$ from $\mathscr{H}$, represented in the coordinate representation, if it is appropriate, by $\psi(q) = (\phi(q) + \mathrm{i}\,\pi(q))\sqrt{2}$ or in some denumerable or finite basis by the set of coefficients $\{c_k = (x_k + \mathrm{i}\,y_k)/\sqrt{2},\ k \in \mathbb{N}\}$, is identified with the point of $\mathscr{M}$, denoted by $X_\psi \in \mathscr{M}$, with the coordinates $\{(x_k, y_k),\ k \in \mathbb{N}\}$ or $\{(\psi(q), \pi(q)),\ q \in \mathbb{R}^d\}$. These coordinates are canonical for the symplectic structure on $\mathscr{M}$. The Schrödinger equation on $\mathscr{H}$ is equivalent with Hamiltonian dynamical equations on $\mathscr{M}$ with the Hamilton's function $\langle\psi|\hat{H}|\psi\rangle$.

The coarse-grained description of the quantum system which is necessary for its classical appearance

is achieved by the appropriate equivalence of points $X \in \mathscr{M}$.[10,11] The equivalence relations relevant for the case of oscillators is

$$X_1 \sim X_2 \Leftrightarrow q_i(X_1) = q_i(X_2) \wedge p_i(X_1) = p_i(X_2), \quad (6)$$

where $i = 1, 2$, and for the case of spin is

$$X_1 \sim X_2 \quad \Leftrightarrow \quad \boldsymbol{J}(X_1) = \kappa \boldsymbol{J}(X_2), \quad (7)$$

where $\boldsymbol{J}(X) = (\langle \hat{J}_x \rangle_X, \langle \hat{J}_y \rangle_X, \langle \hat{J}_z \rangle_X)$ and κ is a positive scalar. In both cases, each equivalence class of quantum pure states $[X]$ contains one and only one coherent state from the corresponding manifolds of coherent states.[20]

Schrödinger evolution equation for $\psi(t)$, or its Hamiltonian form for $X_\psi(t)$, with the Hamiltonian (3) or (5) does not preserve the equivalence classes of quantum states (6) or (7) and the corresponding manifolds of coherent states are not invariant. On the other hand, the system with the same Hamiltonian and additional constraints introduced in such a way that the relevant manifold of coherent states is invariant also preserves the equivalence classes of quantum states. This constrained Hamiltonian system when restricted on the manifold of coherent states generates by definition the dynamics of the coarse-grained reduced quantum system. Since the constrained manifold of the relevant coherent states Γ is known to be symplectic, the constrained system reduced on Γ is Hamiltonian with the Hamilton's function is given simply by $H(Q, P) = \langle Q, P|\hat{H}|Q, P\rangle$, $(Q, P) \in \Gamma$ where (Q, P) parameterize the relevant coherent states.[10,11]

The two coarse-grained systems corresponding to systems (3) and (5) define Hamiltonian dynamical systems on the manifolds of the respective coherent states, $\Gamma = \mathbb{R}^4$ for the oscillators and $\Gamma = S^2$ for the spin. For the oscillator system, the evolutions of the coarse-grained states, or their representative points in Γ, are generated by $H(q_1, q_2, p_1, p_2) = \langle q_1, p_1|\langle q_2, p_2|\hat{H}|q_1, p_1\rangle|q_2, p_2\rangle$,[10] given by

$$\begin{aligned} & H(q_1, q_2, p_1, p_2) \\ &= \epsilon_1 \frac{q_1^2 + p_1^2}{2} + \epsilon_2 \frac{q_2^2 + p_2^2}{2} - \delta(q_1 q_2 + p_1 p_2) \\ &\quad + \frac{c}{2}\left(\frac{(q_1^2 + p_1^2)^2}{4} + \frac{(q_2^2 + p_2^2)^2}{4}\right). \end{aligned} \quad (8)$$

It can be checked that the quantity $q_1^2 + p_1^2 + q_2^2 + p_2^2$ is preserved during the evolution governed by Eq. (8). Also the evolution by Eq. (8) preserves the minimal sum of dispersions of the relevant dynamical variables $F_{q,p} \equiv \Delta^2 \hat{q}_1 + \Delta^2 \hat{q}_2 + \Delta^2 \hat{p}_1 + \Delta^2 \hat{p}_2$.

The evolution equations in the case of the spin are given in terms of the canonical coordinates $(q, p) \in S^2$ by the Hamilton's function

$$\begin{aligned} H(q, p) = & \frac{\epsilon_2 - \epsilon_1}{2}(q^2 + p^2 - 2j) - \delta q(4j - q^2 - p^2)^{1/2} \\ & + \frac{c}{4}(q^2 + p^2 - 2j)^2 \\ & + \frac{c}{8j}(q^2 + p^2)(4j - q^2 - p^2). \end{aligned} \quad (9)$$

The canonical coordinates (q, p) are related to the coherent state averages $\langle \hat{J}_\alpha \rangle(q, p) \equiv \langle q, p|\hat{J}_\alpha|q, p\rangle$ $(\alpha = x, y, z)$ by

$$\langle \hat{J}_x \rangle(p, q) = \frac{q}{2}(4j - q^2 - p^2)^{1/2}, \quad (10a)$$

$$\langle \hat{J}_y \rangle(p, q) = -\frac{p}{2}(4j - q^2 - p^2)^{1/2}, \quad (10b)$$

$$\langle \hat{J}_z \rangle(p, q) = \frac{1}{2}(q^2 + p^2 - 2j). \quad (10c)$$

We have also used the following relation

$$\langle \hat{J}_z^2 \rangle(q, p) = \langle \hat{J}_z \rangle^2(p, q) + \frac{1}{8j}(q^2 + p^2)(4j - p^2 - q^2), \quad (11)$$

where due to quantum fluctuations $\langle \hat{J}_z^2 \rangle(q, p) \neq \langle \hat{J}_z \rangle^2(q, p)$. The evolution by Eq. (9) preserves the minimal sum of dispersions of the relevant dynamical variables $F_j \equiv \Delta^2 \hat{J}_x + \Delta^2 \hat{J}_y + \Delta^2 \hat{J}_z$.

The coarse-graining Eqs. (6) and (7) are interpreted as different approximate and incomplete descriptions upon which different specific types of quantum states and the related observables are considered as physically distinguishable. The requirement that the choice of relevant variables and states is preserved during the evolution determines the appropriate Hamilton's function (8) or (9). From the form of the Hamilton's function, one then reads the physical interpretation of the coarse-grained description. In the two coarse-grained models different physical quantities will have minimal quantum fluctuations. Let us stress that the coarse-grained evolution with Hamiltonians (8) or (9) preserves the minimal sum of dispersions of the relevant basic variables, i.e., $F_{q,p}$ with Eq. (8) and F_j with Eq. (9). On the other hand, the sum of dispersions F_j is not preserved by Eq. (8) and the sum of dispersions $F_{q,p}$ is not preserved by Eq. (9). The coarse-grained systems preserve, in modified forms, some of the quantum features of the original models. In coarse-grained models (6) and (8), the quantum fluctuations of the relevant variables q_1, q_2, p_1, p_2 are constrained to remain minimal, but on the other hand the fluctuations F_j can be arbitrary large. Analogously, in models (7) and (9) the

variables J_x, J_y, J_z are all as classical as it is consistent with the corresponding indeterminacy relations, but the quantum fluctuations of $F_{q,p}$ can be arbitrary large. Only in the macro-limit are the quantum features of the two coarse-grained systems negligible.

The Hamiltonian (8) and the corresponding system with R^4 as its phase space are interpreted as a pair of nonlinear interacting oscillators. A non-classical feature of the model is the form of the interaction containing the term p_1p_2, which typically does not occur in classical mechanical models. Furthermore, the important property of the interaction is that the quantity $q_1^2+p_1^2+q_2^2+p_2^2$ is conserved, but this conservation is not incorporated explicitly in the formulation of the model. On the other hand, the Hamilton's function (9) and the system on S^2 as its phase space are interpreted as a Hamiltonian system constrained to evolve in such a way that the quantity $J_x^2+J_y^2+J_z^2=j^2$ is conserved. The corresponding phase space S^2 is not a tangent bundle of some configuration manifold, contrary to the phase spaces of standard mechanical models.

A possible alternative physical interpretation of the basic quantities occurring in coarse-grained models (8) and (9) can be inferred by relating these quantities to the original physical picture behind the quantum model (1). For example the canonical coordinate q in the model (9), related to $SU(2)$ coherent state expectations $\langle\theta,\psi|\hat{J}_{x,y,z}|\theta,\psi\rangle$ by Eq. (10), represents the relative phase $q=\psi$ of the two modes in system (1) and the canonical momentum $p=\cos^2(\theta/2)$ gives the population of the second mode. Here the angles θ,ψ provide an alternative parametrization of S^2, i.e., of the $SU(2)$ coherent states. Similarly, the quantity J_z gives the population imbalance between the two modes, while the polar angle coordinate of the $SU(2)$ coherent state, i.e., the polar coordinate of the classical state in the phase space S^2, is the relative phase of the two modes. Therefore, coarse-grained systems (7) and (9) represent the Hamiltonian model of the original quantum system in which the quantum fluctuations of the stated quantities are minimal, but still consistent with the corresponding indeterminacy relations.

In order that the two coarse-grained systems display fully classical properties, the corresponding macro-limits are necessary. The macro-limit of the 2mBH system amounts to the limit of a large total number of atoms $N\to\infty$. In the case of the two oscillator coarse-grained systems (6) and (8), the states of the system in the corresponding macro-limit belong to the hypersurface of a large $q_1^2+p_1^2+q_2^2+p_2^2\to\infty$. Bounded and small quantum fluctuations $F_{q,p}$ that occur in the coarse-grained system are negligible in this limit. The macro-limit of $SU(2)$ coarse-grained systems (7) and (9) is achieved with $j\to\infty$. The phase space S^2 in the macro-limit asymptotically approaches R^2 and small and bounded quantum fluctuations F_j become negligible. All features of the quantum behavior are still present in the two coarse-grained systems for relatively small values of N become negligible in the macro-limit $N\to\infty$.

There is a difference between the oscillators and the spin representations of the 2mBH system in the treatment of the macro-limit. In the case of the oscillators' representation, the coarse-grained Hamiltonian system has an additional constant of motion $q_1^2+p_1^2+q_2^2+p_2^2$, and is therefore integrable. Due to this additional integral, the difference between the coherent state expectation of the Hamiltonian expressed in terms of the basic operators (8) and the Hamiltonian expression in terms of the coherent state expectations of the basic operators is a constant term that can be neglected in the Hamiltonian. The only manifestation of the quantum fluctuations in the coherent states is in the overlap between the initial values. Therefore, the macro-limit of the oscillator system is given by the same Hamiltonian (8) and the condition that only the phase-space hyper-surfaces with $q_1^2+p_1^2+q_2^2+p_2^2\gg 1$ are considered. In the case of the spin system, the system is already reduced on an invariant manifold and there is no additional invariance. Therefore the quantum fluctuations in a spin coherent state imply $\langle\hat{J}_z^2\rangle(q,p)\neq\langle\hat{J}_z\rangle^2(q,p)$ and are manifested dynamically in the last term in Eq. (9).

4. Comparison of the coarse-grained systems

The two equivalent quantum models (3) and (5) generate the same evolution of the averages: $\langle\hat{J}_x\rangle$, $\langle\hat{J}_y\rangle$, $\langle\hat{J}_z\rangle$ or $\langle a_1^\dagger a_2+a_2^\dagger a_1\rangle$, $\langle \mathrm{i}(a_1^\dagger a_2-a_2^\dagger a_1)\rangle$, $\langle a_2^\dagger a_1-a_2^\dagger a_1\rangle$ from any initial state. It is of interest to compare the dynamics of the corresponding coarse-grained models for a finite total number of atoms. The coarse-grained system corresponding to the two oscillators algebra is defined on the phase space identified with the parameter space $(q_1,q_2,p_1,p_2)\in\boldsymbol{R}^4$ of the corresponding coherent states $|q_1,p_1\rangle|q_2,p_2\rangle$, and analogously the spin coarse-grained system is defined on

the sphere $(q,p) \in S^2$. However, there are points in the phase spaces of the two systems which give the same value of the quantities $\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle$. We use such states as initial states for the comparison of the time dependence of $\langle \hat{J}_x \rangle, \langle \hat{J}_y \rangle, \langle \hat{J}_z \rangle$ generated by the dynamics of Eqs. (8) and (9).

The dynamics generated by the two coarse-grained systems is illustrated in Fig. 1.

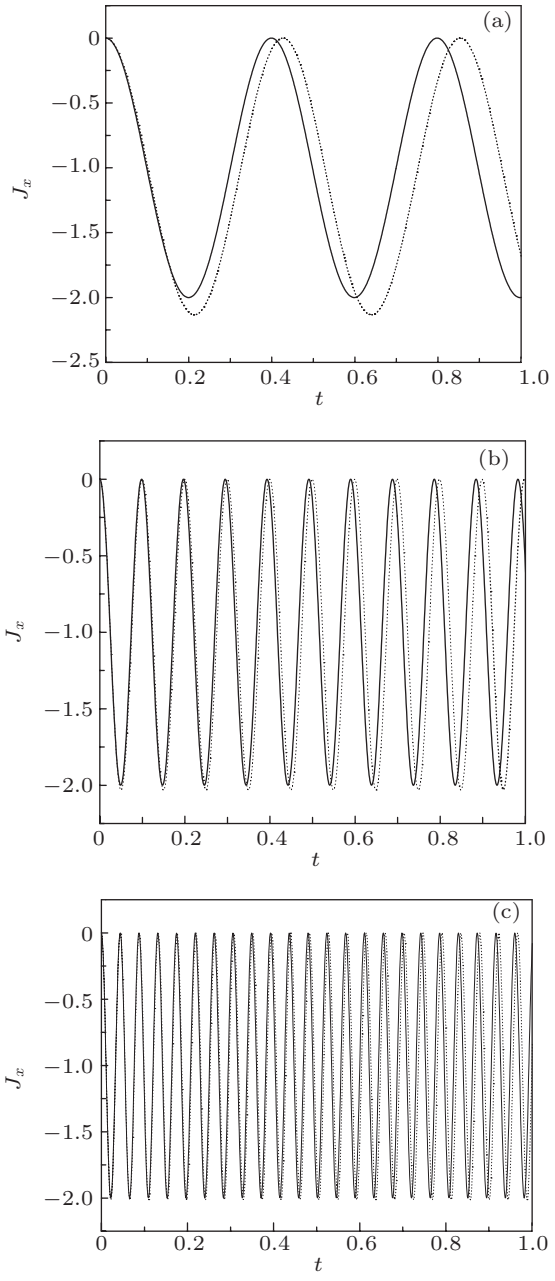


Fig. 1. Figures illustrate the evolution of $J_x = \langle \hat{J}_x \rangle$ as generated by the two coarse-grained systems (8) (full line) and (9) (dotted line). The initial states are as explained in the main text resulting in (a) $j = 8$, (b) $j = 32$, and (c) $j = 72$. The values of parameters are $\epsilon_1 = \epsilon_2 = 1$, $\delta = c = 1$.

The initial state for both systems gives the same initial value of $\langle \hat{J}_x \rangle$, $\langle \hat{J}_y \rangle$, $\langle \hat{J}_z \rangle$. In Fig. 1 the initial values of the canonical coordinates (q,p) for the spin systems are $(q,p) = (0,0)$, implying $\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$, and the size of the spin was in Fig. 1(a) $j = 8$, in Fig. 1(b) $j = 32$, and in Fig. 1(c) $j = 72$ implying $\langle \hat{J}_z \rangle = 8$, $\langle \hat{J}_z \rangle = 32$, and $\langle \hat{J}_z \rangle = 72$, respectively. The initial points for the oscillators' coarse-grained dynamics that imply the same values of $\langle \hat{J}_x \rangle = 0$, $\langle \hat{J}_y \rangle = 0$, $\langle \hat{J}_z \rangle = j$, are always of the same form $(q_1, q_2, p_1, p_2) = (0, q_2, 0, p_2)$ with $(q_2^2 + p_2^2)/4 = j$. For example, in Fig. 1(a) $(q_1, q_2, p_1, p_2) = (0, 4, 0, 4)$, in Fig. 1(b) $(q_1, q_2, p_1, p_2) = (0, 8, 0, 8)$, and in Fig. 1(c) $(q_1, q_2, p_1, p_2) = (0, 12, 0, 12)$. The evolution of $\langle \hat{J}_x \rangle$ by the two systems (8) and (9) are clearly different, even for quite small periods of time or for a small number of characteristic oscillations. The difference between the two evolutions depends on the size of the spin, i.e., on the total number of atoms. The difference is larger for smaller values of j and decreases as j is increased. In Fig. 1 the oscillator evolution starts from the oscillators coherent states of the same type $|0,0\rangle|q_2,p_2\rangle$ corresponding to the initial spin coherent state $|j,j\rangle$ with $j = (q_2^2 + p_2^2)/4$. Qualitatively the same conclusion is obtained (not illustrated) by considering the oscillators' initial coherent states of the form $|q_1,p_1\rangle|q_2,p_2\rangle$, where both component states are at some distance from the ground state $|0,0\rangle$, and the corresponding (q,p) for the spin are nonzero implying that the initial averages $\langle \hat{J}_x \rangle \neq 0$, $\langle \hat{J}_y \rangle \neq 0$, $\langle \hat{J}_z \rangle \neq 0$ are all nonzero.

Our results imply that the two coarse-grained descriptions, corresponding to the equivalent quantum models, are not equivalent for finite values of the total number of atoms. Let us stress that each of the two coarse-grained descriptions is natural from the point of view of the corresponding dynamical algebra. Only in the macro-limit do the two coarse-grained dynamical systems generate indistinguishable evolution corresponding to the unique classical limit. However, for a moderate number of trapped atoms the difference between the two coarse-grained descriptions might be detected.

5. Summary

We have studied alternative coarse-grained descriptions that can be associated with a given quantum system. The following general picture emerges from our analyses of the 2mBH example. A quantum system is described by the Hamiltonian expressed in

terms of some basic operators. The commutation relations between the basic operators define the dynamical algebra g_1 of the system and the system's Hilbert space appears as the space of an irreducible representation of the dynamical algebra. However, it might occur that the Hilbert space is in fact a direct sum of spaces that are invariant for the considered Hamiltonian and which generate irreducible representations of some other algebra g_2. In this case the Hamiltonian can be written in terms of the operators satisfying the commutation relations of the algebra g_2. Nevertheless, the two ways of writing the Hamiltonian are equivalent and the dynamical evolution of the physical observables is the same whichever form of the Hamiltonian is used. However, the appropriate coarse-grained description is determined by the system's dynamical algebra. The coarse-grained system is defined as a Hamiltonian system on the manifold of the coherent states, determined by the algebra, given by the Hamiltonian which is equal to the coherent state expectation of the original Hamilton's operator. If the original quantum system is consistent with two dynamical algebras, the above procedure leads to two natural coarse-grained systems. We have shown that the two coarse-grained systems generate different evolutions of the same dynamical variables, which become canonically equivalent only in the corresponding macro-limits.

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