An asymptotic solving method for the periodic solution of a class of disturbed nonlinear evolution equation*

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A class of disturbed evolution equation is considered using a simple and valid technique. We first introduce the periodic traveling-wave solution of a corresponding typical evolution equation. Then the approximate solution for an original disturbed evolution equation is obtained using the asymptotic method. We point out that the series of approximate solution is convergent and the accuracy of the asymptotic solution is studied using the fixed point theorem for the functional analysis.

Keywords: period solution, traveling wave, evolution equation asymptotic method

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1. Introduction

The evolution equation is one of the most important notions in nonlinear science, which has wide potential applications.[1−4] Recently, some new methods for studying the solution of the evolution equation have appeared, such as the homogeneous equilibrium method, the hyperbolic tangent function method, the Jacobi elliptic function method, the auxiliary equation method, etc. These methods have been applied to the study of the shock wave, scattering light wave, quantum mechanics, atmospheric physics, network of neurons, and so on.[5−10] The evolution equation describes a class of complicated natural phenomenon. We need to constitute and predigest it into a basal model, and sometimes, solve them using the asymptotic methods including the boundary layer method, the averaging method, the method of matched asymptotic expansion, and the method of multiple scales[11−13] Using the asymptotic method the authors have also investigated a class of generalized nonlinear equations.[14−20] In this paper, we consider a class of the disturbed evolution equation in the modern physics. Using special asymptotic method we obtain the approximate traveling-wave solution.

2. Disturbed evolution equation

Consider the following disturbed evolution equation:[8,9]

$$u_t - \alpha u u_x + \beta u_{xxx} = f(u), \tag{1}$$

where α and β are the constants and f is a disturbed term which is a bounded periodic analytic function. The disturbed evolution equation (1) possesses a variety of applications in theoretical physics, fluid mechanics, aeronautics, cosmonautics, applied chemistry, ecology environment, and metallurgical engineering. It also has wide applications in plasma physics, atomic physics, and solid physics.

We first introduce the traveling-wave transform

$$z = ax + ct, \tag{2}$$

where a and c are the undetermined constants. We have the following traveling-wave equation:

$$cu_z - a\alpha u u_z + a^3\beta u_{zzz} = f(u). \tag{3}$$

Now we consider a typical evolution equation as follows:

$$cu_z - a\alpha u u_z + a^3\beta u_{zzz} = 0. \tag{4}$$

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We can obtain the traveling-wave solution using the mapping method. Set

$$u(z) = k_0 + k_1 v(z) + k_2 v^2(z), \tag{5}$$

where k_0, k_1, k_2 are the undetermined constants, and $v(z)$ satisfies the following equation:

$$\frac{\mathrm{d}v}{\mathrm{d}z} = 1 + v^2. \tag{6}$$

Obviously, there is a following period solution:

$$v(z) = \tan z. \tag{7}$$

From Eqs. (5) and (6), we have

$$u_z = (k_1 + 2k_2 v)(1 + v^2), \tag{8}$$

$$uu_z = [k_0 k_1 + (k_1^2 + 2k_0 k_2)v + 3k_1 k_2 v^2 + 2k_2^2 v^3](1 + v^2), \tag{9}$$

$$u_{zz} = [2k_2 + 2k_1 v + 6k_2 v^2](1 + v^2), \tag{10}$$

$$u_{zzz} = [2k_1 + 16k_2 v + 6k_1 v^2 + 24k_2 v^3](1 + v^2). \tag{11}$$

Substituting Eqs. (8)–(11) into Eq. (4), packing the same species terms of $v^i(1 + v^2)$ $(i = 0, 1, 2, 3)$, and letting the coefficients of terms be zero, we have

$$\begin{aligned}
&k_1(c - a\alpha k_0 + 2a^3\beta) = 0,\\
&2ck_2 - a\alpha(k_1^2 + 2k_0 k_2) + 16a^3\beta k_2 = 0,\\
&3\alpha k_1^2 - k_2(c + 2k_0 + 8 \times 6^3\beta) = 0,\\
&-3ak_1(\alpha k_2 + 2a^3\beta) = 0,\\
&-2a(\alpha k_2 - 12a^2\beta) = 0.
\end{aligned}$$

Thus there is a solution

$$\begin{aligned}
&a = 6, \quad c = 6, \quad k_0 = \frac{72\beta - 1}{\alpha},\\
&k_1 = \frac{12}{\alpha}[3\beta(c + \frac{4\beta - 6}{\alpha} + 48\beta)]^{1/2},\\
&k_2 = \frac{12 \times 36\beta}{\alpha}.
\end{aligned}$$

Substituting the above expressions into Eq. (5), and from Eq. (7), we have the exact periodic solution of Eq. (4) as follows:

$$u(z) = \frac{72\beta - 1}{\alpha} + \frac{12}{\alpha}\left[3\beta(6 + \frac{4\beta - 6}{\alpha} + 48\beta)\right]^{1/2} \tan z + \frac{12 \times 36\beta}{\alpha} \tan^2 z, \tag{12}$$

where $z = 6(x + t)$.

As a simple example, we set $\alpha = 1$, $\beta = 2$, and $c = -19$. We have the surface (see Fig. 1) and curve (see Fig. 2 as $x = 0$) of the traveling-wave periodic solution for the typical evolution equation (4), respectively.

Because there is a disturbed term $f(u)$ in Eq. (3), which does not give an explicit exact solution generally, we need construct the approximate solution.

In order to obtain the approximate analytic solution of Eq. (3) with $a = c = 6$, we introduce a mapping $H(u, s) : R \times I \to R$[21]

$$H(u, s) = Lu - L\tilde{u}_0 + s[L\tilde{u}_0 - 6\alpha u u_z - f(u)], \tag{13}$$

where $R = (-\infty, +\infty)$, $I = [0, 1]$, s is an artificial parameter, $\tilde{u}_0$ is an initial approximate function of Eq. (3), which is determined below, and the linear operator L is

$$Lu = 6(u_z + 36\beta u_{zzz}). \tag{14}$$

Obviously, from expression (13), $H(u, 1) = 0$ is the same as Eq. (3) with $a = c = 6$. Therefore, the solution $u(z)$ of Eq. (3) with $a = c = 6$ is the solution of equation $H(u, s) = 0$ as $s \to 1$.

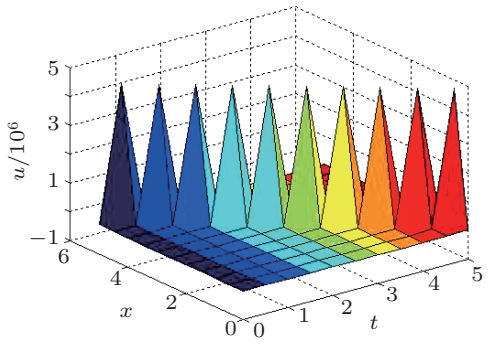


Fig. 1. (color online) Traveling-wave periodic solution $u(x, t)$.

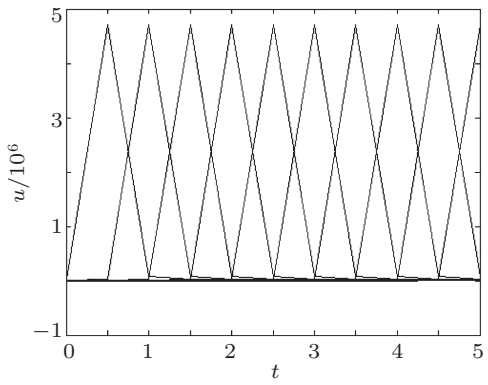


Fig. 2. Traveling-wave periodic solution $u(0, t)$.

3. Approximate solutions of the disturbed equation and its convergence

We set

$$u = \sum_{i=0}^{\infty} u_i(z) s^i. \tag{15}$$

From mapping (4), substituting Eq. (5) into $H(u,s)=0$, developing nonlinear terms in s and letting coefficients of the same powers of s be zero, we can obtain $u_n(z)$ $(n=0,1,2,\ldots)$ successively. Substituting $u_n(z)$ into Eq. (15), we obtain series (15).

Substituting Eq. (15) into $H(u,s)=0$, developing the nonlinear terms in s and equating coefficients of the same powers of s on both sides, for the zeroth-order powers in s, we obtain

$$L(u_0)=L(\tilde{u}_0). \tag{16}$$

We select an initial approximate solution $\tilde{u}_0$ as the periodic solution of Eq. (4). Using Eqs. (12) and (16), we have

$$\begin{aligned} u_0(z) =& \frac{72\beta-1}{\alpha} \\ &+\frac{12}{\alpha}\left[3\beta(6+\frac{4\beta-6}{\alpha}+48\beta)\right]^{1/2}\tan z \\ &+\frac{12\times 36\beta}{\alpha}\tan^2 z. \end{aligned} \tag{17}$$

In the $H(u,s)=0$, for the first-order powers in s, we obtain

$$L(u_1)=f(u_0). \tag{18}$$

It is easy to see that we can solve the solution of linear equation (18) as

$$u_1(z)=\frac{1}{6}\int_0^z\int_0^\xi f(u_0(\eta))\sin\frac{1}{6}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\xi. \tag{19}$$

Using $H(u,s)=0$, for the second-order powers in s, we obtain

$$L(u_2)=6\alpha(u_0u_{1z}+u_{0z}u_1)+F(u_0,u_1) \tag{20}$$

with

$$F(u_0,u_1)=\left[\frac{\partial}{\partial s}\left(f\left(\sum_{i=0}^{\infty}u_is^i\right)\right]_{s=0}.$$

Similarly, it is easy to obtain that the solution of Eq. (20) is

$$\begin{aligned} u_2(z) =& \frac{1}{6}\int_0^z\int_0^\xi(6\alpha(u_0(\eta)u_{1\eta}(\eta)+u_{0\eta}(\eta)u_1(\eta)) \\ &+F(u_0(\eta),u_1(\eta))\sin\frac{1}{6}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\xi. \end{aligned} \tag{21}$$

Thus, using Eqs. (15), (17), (19), and (21), the second-order approximate periodic solution $u_{2\,\mathrm{hom}}(z)$ of the generalized disturbed evolution equation (3) with $a=c=6$ is

$$\begin{aligned} &u_{2\,\mathrm{hom}}(z) \\ =& \frac{72\beta-1}{\alpha}+\frac{12}{\alpha}\left[3\beta\left(6+\frac{4\beta-6}{\alpha}+48\beta\right)\right]^{1/2}\tan z \\ &+\frac{12\times 36\beta}{\alpha}\tan^2 z \\ &+\frac{1}{6}\int_0^z\int_0^\xi f(u_0(\eta))\sin\frac{1}{6}(\xi_1-\eta)\mathrm{d}\eta\mathrm{d}\xi \\ &+\frac{1}{6}\int_0^z\int_0^\xi(6\alpha(u_0(\eta)u_{1\eta}(\eta)+u_{0\eta}(\eta)u_1(\eta)) \\ &+F(u_0(\eta),u_1(\eta))\sin\frac{1}{6}(\xi_1-\eta)\mathrm{d}\eta\mathrm{d}\xi, \end{aligned} \tag{22}$$

where u_0 and u_1 are expressed in Eqs. (17) and (19), respectively.

From the traveling-wave transform (2) with $a=c=6$ and Eq. (22), we obtain a second-order approximate periodic traveling-wave solution $u_{2\,\mathrm{hom}}(t,x)$ of the generalized disturbed evolution equation (1) as follows:

$$\begin{aligned} &u_{2\,\mathrm{hom}}(t,x) \\ =& \frac{72\beta-1}{\alpha} \\ &+\frac{12}{\alpha}\left[3\beta\left(c+\frac{4\beta-6}{\alpha}+48\beta\right)\right]^{1/2}\tan 6(x+t) \\ &+\frac{12\times 36\beta}{\alpha}\tan^2 6(x+t) \\ &+\frac{1}{6}\int_0^{6(x+t)}\int_0^\xi f(u_0(\eta))\sin\frac{1}{6}(\xi_1-\eta)\mathrm{d}\eta\mathrm{d}\xi \\ &+\frac{1}{6}\int_0^{6(x+t)}\int_0^\xi(6\alpha(u_0(\eta)u_{1\eta}(\eta)+u_{0\eta}(\eta)u_1(\eta)) \\ &+F(u_0(\eta),u_1(\eta))\sin\frac{1}{6}(\xi-\eta)\mathrm{d}\eta\mathrm{d}\xi. \end{aligned} \tag{23}$$

Using the above mapping (4), we can also obtain more higher-order approximate period travelling wave solutions of the disturbed evolution equation (1).

We know that under the condition $[H]$ and with selecting an initial approximate $u_0(t,x)(=u_0(z))$ as a solution of the nonlinear equation (4), we can have

$$u=\sum_{n=0}^{\infty}u_n(z)=\sum_{n=0}^{\infty}u_n(t,x)$$

on $s\in[0,1]$ using the fixed point theorem.[22]

Thus, further, we know that[22]

$$u_{\mathrm{exa}}(t,x)=\sum_{n=0}^{\infty}u_n(t,x),$$

which is constructed by using the asymptotic method, is an exact periodic traveling-wave solution for the generalized disturbed evolution equation (1).

4. Comparison between accuracies of approximate solution

Now we only consider a special weakly disturbed nonlinear evolution equation. Assume the disturbed term $f(u) = \varepsilon \sin u$, where ε is a small positive parameter then the generalized disturbed nonlinear evolution equation (1) will be

$$u_t - \alpha u u_x + \beta u_{xxx} = \varepsilon \sin u. \tag{24}$$

From Eq. (17), the zero$^{\text{th}}$-order approximate solution $u_{0\,\hom}(t,x)$ of Eq. (24) is

$$\begin{aligned} u_0(z) = {} & \frac{72\beta - 1}{\alpha} \\ & + \frac{12}{\alpha}\left[3\beta\left(6 + \frac{4\beta - 6}{\alpha} + 48\beta\right)\right]^{1/2} \tan z \\ & + \frac{12 \times 36\beta}{\alpha} \tan^2 z. \end{aligned} \tag{25}$$

And we have

$$u_1(z) = \frac{\varepsilon}{6}\int_0^z \int_0^\xi \sin(u_0(\eta)) \sin\frac{1}{6}(\xi_1 - \eta)\mathrm{d}\eta\mathrm{d}\xi. \tag{26}$$

From Eq. (25), the first-order approximate solution $u_{1\,\hom}(z)$ of Eq. (24) is

$$\begin{aligned} & u_{1\,\hom}(z) \\ = {} & \frac{72\beta - 1}{\alpha} + \frac{12}{\alpha}\left[3\beta\left(6 + \frac{4\beta - 6}{\alpha} + 48\beta\right)\right]^{1/2} \tan z \\ & + \frac{12 \times 36\beta}{\alpha} \tan^2 z \\ & + \frac{\varepsilon}{6}\int_0^z \int_0^\xi \sin(u_0(\eta)) \sin\frac{1}{6}(\xi_1 - \eta)\mathrm{d}\eta\mathrm{d}\xi. \end{aligned} \tag{27}$$

From Eqs. (21) and (27) we can obtain the second-order approximate periodic solution $u_{2\,\hom}(t,x)$ of Eq. (24)

$$\begin{aligned} & u_{2\,\hom}(z) \\ = {} & \frac{72\beta - 1}{\alpha} + \frac{12}{\alpha}\left[3\beta\left(6 + \frac{4\beta - 6}{\alpha} + 48\beta\right)\right]^{1/2} \tan z \\ & + \frac{12 \times 36\beta}{\alpha} \tan^2 z \\ & + \frac{\varepsilon}{6}\int_0^z \int_0^\xi \sin(u_0(\eta)) \sin\frac{1}{6}(\xi_1 - \eta)\mathrm{d}\eta\mathrm{d}\xi \\ & + \frac{\varepsilon}{6}\int_0^z \int_0^\xi (6\alpha(u_0(\eta)u_{1\eta}(\eta) + u_{0\eta}(\eta)u_1(\eta)) \\ & + u_1(\eta)\cos u_0(\eta)) \sin\frac{1}{6}(\xi - \eta)\mathrm{d}\eta\mathrm{d}\xi, \end{aligned} \tag{28}$$

where u_0 and u_1 are expressed in Eqs. (25) and (26), respectively.

From the travelling transform (2) with $a = c = 6$ we can obtain a second-order approximate periodic traveling-wave solution $u_{2\,\hom}(t,x)$ of the weakly disturbed nonlinear evolution equation (13) as follows:

$$\begin{aligned} & u_{2\,\hom}(t,x) \\ = {} & \frac{72\beta - 1}{\alpha} \\ & + \frac{12}{\alpha}\left[3\beta\left(6 + \frac{4\beta - 6}{\alpha} + 48\beta\right)\right]^{1/2} \tan 6(x+t) \\ & + \frac{12 \times 36\beta}{\alpha} \tan^2 6(x+t) \\ & + \frac{\varepsilon}{6}\int_0^{6(x+t)} \int_0^\xi \sin(u_0(\eta)) \sin\frac{1}{6}(\xi - \eta)\mathrm{d}\eta\mathrm{d}\xi \\ & + \frac{\varepsilon}{6}\int_0^{6(x+t)} \int_0^\xi (6\alpha(u_0(\eta)u_{1\eta}(\eta) + u_{0\eta}(\eta)u_1(\eta)) \\ & + u_1(\eta)\cos u_0(\eta)) \sin\frac{1}{6}(\xi - \eta)\mathrm{d}\eta\mathrm{d}\xi. \end{aligned}$$

Using mapping (13), we can also obtain more higher-order approximate periodic traveling-wave solutions of the weakly disturbed evolution equation (24).

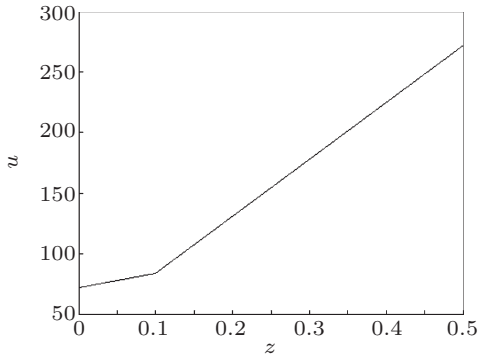


Fig. 3. Curve of solution $u_{2\,\hom}(z)$ with $\alpha = 1$, $\beta = 1$, and $\varepsilon = 0.01$.

Set $\alpha = 1$, $\beta = 1$, and $\varepsilon = 0.01$ for Eq. (24) then we will have the curves of solutions $u_{2\,\hom}(z)$ and $u_0(z)$, and be able to compare them, see Figs. 3–5. From Fig. 5, it is easy to see that as ε is a small parameter, the solutions $u_{2\,\hom}(z)$ and $u_0(z)$ are very close to each other. This behaviour is coincident with that for the approximate solution of the weakly disturbed evolution equation (24).

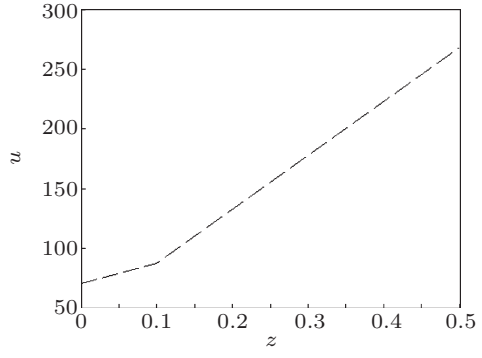


Fig. 4. Curve of solution $u_0(z)$ with $\alpha = 1$ and $\beta = 1$.

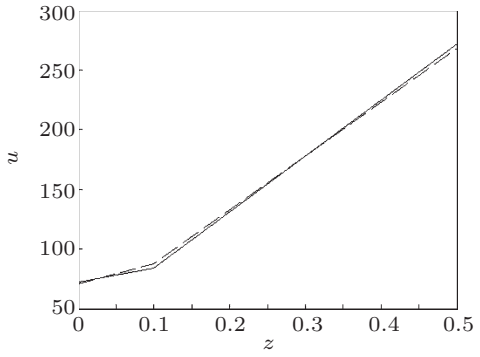


Fig. 5. Comparison between the curves of solutions $u_{2\,\mathrm{hom}}(z)$ (solid line) and $u_0(z)$ (dashed line) with $\alpha = 1$, $\beta = 1$, and $\varepsilon = 0.01$.

Set

$$u_{\mathrm{exa}}(t,x) = \sum_{i=0}^{\infty} u_i(t,x)$$

to be an exact solution of Eq. (24), which is determined by the asymptotic method. Now we estimate the error $u_{\mathrm{exa}} - u_{2\,\mathrm{hom}}$. We first estimate

$$\begin{aligned}
&L(u_{\mathrm{exu}} - u_{2\,\mathrm{hom}})\\
&= \frac{\partial(u_{\mathrm{exa}} - u_{2\,\mathrm{hom}})}{\partial t} + \beta(u_{\mathrm{exu}} - u_{2\,\mathrm{hom}})_{xxx}\\
&= \left(\frac{\partial u_{\mathrm{exa}}}{\partial t} + \beta(u_{\mathrm{exa}}\right)_{xxx} - \sum_{i=1}^{2}\left(\frac{\partial u_i}{\partial t} + \beta u_i\right)\\
&= \left(\alpha\sum_{n=0}^{\infty} u_n\right)\left(\sum_{n=0}^{\infty} u_n\right)_x + \varepsilon\sum_{j=2}^{\infty}\sin\left(\sum_{n=0}^{\infty} u_n\right)\\
&\quad - [\alpha u_0 u_{0x} + \varepsilon\sin u_0] + [\alpha(u_0 u_{1x} + u_{0x}u_1)\\
&\quad + 2\varepsilon u_0 u_1 \sin u_0)]\\
&= O(\varepsilon^3), \quad 0 < \varepsilon \ll 1.
\end{aligned}$$

Selecting an arbitrary constant such that $u_{\mathrm{exp}}(0) = u_{2\,\mathrm{hom}}(0)$, from the fixed point theorem,[22] we have

$$u_{\mathrm{exa}} - u_{2\,\mathrm{hom}} = O(\varepsilon^2), \quad 0 < \varepsilon \ll 1.$$

Therefore, from the above result, we know that the approximate periodic traveling-wave solution $u_{2\,\mathrm{hom}}(t,x)$, obtained by the asymptotic method, possesses better accuracy.

5. Summary

Nonlinear traveling-wave theory describes a class of complicated natural phenomenon. For solving a more complicated model, we need to use an approximate method. In this paper an asymptotic method is used, which is a simple and valid method in some cases. This method is an approximate analytic method, which is different from a general numerical method. The expansion of the solution via the asymptotic method can still be manipulated analytically. Thus, from Eq. (23) we can further study the qualitative and quantitative behaviours of the periodic traveling-wave solution. And we select the initial approximation of the disturbed evolution equation as a periodic traveling-wave solution of the corresponding typical evolution equation. It can quickly obtain an approximate solution with a required accuracy.

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