Approximate solutions of nonlinear PDEs by the invariant expansion*

Wu Jiang-Long(吴江龙)[a)†] and Lou Sen-Yue(楼森岳)[b)]

[a)] *Faculty of Science, Ningbo University, Ningbo* 315211, *China*

[b)] *Center of Nonlinear Science, Ningbo University, Ningbo* 315211, *China*

(Received 28 April 2012; revised manuscript received 20 June 2012)

It is difficult to obtain exact solutions of the nonlinear partial differential equations (PDEs) due to their complexity and nonlinearity, especially for non-integrable systems. In this paper, some reasonable approximations of real physics are considered, and the invariant expansion is proposed to solve real nonlinear systems. A simple invariant expansion with quite a universal pseudopotential is used for some nonlinear PDEs such as the Korteweg-de Vries (KdV) equation with a fifth-order dispersion term, the perturbed fourth-order KdV equation, the KdV–Burgers equation, and a Boussinesq-type equation.

Keywords: approximate solution, invariant expansion, Mobious transformation invariance

PACS: 02.30.Jr, 47.10.ab, 02.30.Ik **DOI:** 10.1088/1674-1056/21/12/120204

1. Introduction

There are some marvelous methods available for studying the integrability of a nonlinear partial differential equation (PDE). The Painlevé analysis developed by Weiss, Tabor, and Carnevale (WTC)[1] is one of the most effective approaches. Applying the WTC approach to nonlinear PDEs, one can obtain not only properties like the Painlevé property, Lax pair, bilinear form, and Bäcklund transformation of integrable models, but also solutions both for integrable and non-integrable models.[2−6]

Conte proposed a simplification of the WTC approach which is corresponds to the re-summation of the usual WTC approach so that the new expansion coefficients are all invariant under the Mobious transformation.[7] The truncated expansion approach in Conte's analysis is related to a special type of non-truncated summation in the usual WTC approach.[8] Pickering proposed a nonstandard truncation expansion approach based on Conte's Painlevé expansion.[9] Then some new nontrivial solutions may be obtained due to Pickering's nonstandard truncation approach. In Pickerinng's consideration, the truncation expansion will yield a new nontrivial solution only for those equations which possess more than one branch in the original WTC analysis. Similar to Conte's consideration, we may obtain some other types of expansions to study the Painlevé property. Perhaps, the different uses of the expansions may cause complexity in the study of the Painlevé analysis. However, it is useful to get different new solutions.

In this paper, we would like to use the invariant expansion approach to solve real nonlinear physical problems and find some Painlevé integrable models in the same dimensions which can be considered as reasonable approximations of real physical problems.

In the next section, we simply review the general aspects of the Conte's Painlevé expansion. In Section 3, due to the complexity and nonlinearity of some nonlinear PDEs, it is difficult to give some exact solutions. So we use the invariant expansion to solve real PDEs and find some reasonable approximations of real physics.[10−19] Taking the Korteweg-de Vries (KdV) equation with a fifth-order dispersion term as a simple example to show how the expansion yields some more reasonable approximate solutions. In some special cases, if the arbitrary constants are selected suitably, the approximate solutions may become exact. In Section 4, applying the same idea to

*Project supported by the National Natural Science Foundation of China (Grant No. 11175092), Scientific Research Fund of Zhejiang Provincial Education Department (Grant No. Y201017148), and K. C. Wong Magna Fund in Ningbo University.

†Corresponding author. E-mail: wjlazxm@sina.com or 375516508@qq.com

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three nonlinear equations, we find a quite universal expansion approach which is valid for various nonlinear PDEs. The last section is a short summary.

2. Invariant Painlevé expansions

For a given PDE, say

$$F(u, u_x, u_t, ..., u_{x^n}, u_{t^n}) = 0, \tag{1}$$

the invariant Painlevé expansion takes the form

$$u = \sum_{i=0}^{\infty} u_i \xi^{i-\alpha} \tag{2}$$

with the pseudopotential ξ of the given form

$$\xi_x = 1 + \frac{S\xi^2}{2}, \tag{3}$$

$$\xi_t = -C + C_x\xi - \frac{(SC + C_{xx})\xi^2}{2}. \tag{4}$$

The compatibility condition of Eqs. (3) and (4) has the following form

$$S_t + 2C_xS + C_{xxx} + CS_x = 0. \tag{5}$$

Equation 5 possesses a special solution

$$S = \frac{\phi_{xxx}}{\phi_x} - \frac{3}{2}\frac{{\phi_{xx}}^2}{{\phi_x}^2}, \quad C = -\frac{\phi_t}{\phi_x}, \tag{6}$$

and S, C satisfies with Mobious transformation invariance, $\phi \to (a + b\phi)/(c + d\phi)$, $ab \neq cd$.

In the next section, taking the KdV equation with a fifth-order dispersion term as an example, the invariant expansion is used to find some approximate solutions. In some special cases, the approximate solutions are just the exact solutions of the KdV equation with a fifth-order dispersion term.

3. Approximate solutions of the Korteweg-de Vries (KdV) equation with a fifth-order dispersion term

The KdV equation with fifth-order dispersion term[20] we discussed here has the form

$$u_t + \mu u_{xxx} + 6uu_x + \nu u_{xxxxx} = 0, \tag{7}$$

where μ, ν are arbitrary constants.

Using the leading order analysis first, substituting Eq. (7) into Eq. (2), we can find that equation (2) should have the form

$$u = \frac{u_0}{\xi^4} + \frac{u_1}{\xi^3} + \frac{u_2}{\xi^2} + \frac{u_3}{\xi} + u_4 + o(\xi). \tag{8}$$

In order to balance the effects of the nonlinearity, and those of the dispersion in negative directions. Substituting Eqs. (3), (4), and (8) into the KdV equation with a fifth-order dispersion term and canceling the coefficients of $\xi^i, (i = -9, -8, -7, -6, -5)$ yields five further constraints:

$$-24u_0(u_0 + 280\nu) = 0, \tag{9}$$

$$6u_0u_{0x} + 4200\nu u_{0x} - 2520\nu u_1 - 42u_0u_1 = 0, \tag{10}$$

$$\begin{aligned}-36u_0u_2 - 720\nu u_2 - 1200\nu u_{0xx} - 9600\nu u_0S \\ + 6u_0u_{1x} + 1800\nu u_{1x} - 12{u_0}^2S - 18{u_1}^2 \\ - 120\mu u_0 + 6u_1u_{0x} = 0,\end{aligned} \tag{11}$$

$$\begin{aligned}-30u_1u_2 - 120\nu u_3 + 600\nu u_{2x} + 60\mu u_{0x} - 60\mu u_1 \\ + 200\nu u_{0xxx} + 6u_0u_{2x} - 600\nu u_{1xx} \\ - 21u_0u_1S + 6u_2u_{0x} + 6u_1u_{1x} \\ - 30u_0u_3 - 3300\nu u_1S \\ + 2100\nu u_0S_x + 5200\nu u_{0x}S = 0,\end{aligned} \tag{12}$$

$$\begin{aligned}4u_0C + 6u_0u_{3x} - 1240\nu u_{0xx}S - 324\nu u_0S_{xx} \\ - 12\mu u_{0xx} + 120\nu u_{1xxx} - 240\nu u_{2xx} \\ - 20\nu u_{0xxxx} - 12{u_2}^2 - 4816\nu u_0S^2 \\ - 124\mu u_0S + 2040\nu u_1S + 780\nu u_1S_x \\ - 840\nu u_2S - 1060\nu u_{0x}S_x - 18u_0u_2S \\ + 6u_1u_{2x} + 6u_2u_{1x} + 6u_3u_{0x} + 36\mu u_{1x} \\ - 24\mu u_2 - 24u_1u_3 - 24u_4u_0 \\ + 120\nu u_{3x} - 9{u_1}^2S = 0.\end{aligned} \tag{13}$$

According to the above five further constraints, it is easy to find out all possible solutions of the overdetermined constraint equations. We consider only $u_0 \neq 0$ on these constraint equations. After some simplifications, the final result can be written as

$$u_0 = -280\nu, \tag{14}$$

$$u_1 = 0, \tag{15}$$

$$u_2 = -\frac{560\nu S}{3} - \frac{140\mu}{39}, \tag{16}$$

$$u_3 = \frac{140}{3}\nu S_x, \tag{17}$$

$$\begin{aligned}u_4 = -\frac{1}{3042\nu}(-507\nu C + 28392\nu^2S_{xx} \\ + 66248\nu^2S^2 + 3640\mu\nu S - 31\mu^2).\end{aligned} \tag{18}$$

So equation (8) is an approximate solution of the KdV equation with a fifth-order dispersion term, with u_0, u_1, u_2, u_3, u_4 being given by Eqs. (14), (16), (17), and (18) respectively, ξ is determined by Eqs. (3), (4), (5), (6), ϕ is arbitrary, and S, C satisfy Eqs. (5) and (6).

To get some more reasonable and concrete approximate solutions, we handle the approximations further as follows.

3.1. The first kind approximate solution

To fix the arbitrary function ϕ in the approximate solution (8), we take $u_4 = 0$. Substituting Eq. (6) into Eq. (18) with $u_4 = 0$, we get

$$\begin{aligned}&-507\nu\phi_t\phi_{xxx} - 28392\nu^2\phi_{xxxxx}\phi_{xxx}\\&+ 141960\nu^2\phi_{xxxx}\phi_{xx}\phi_x^2 - 283920\nu^2\phi_{xxx}\phi_{xx}^2\phi_x\\&+ 47320\nu^2\phi_{xxx}^2\phi_x^2 + 106470\nu^2\phi_{xxxx}\\&- 3640\mu\nu\phi_{xxx}\phi_x^3 + 5460\mu\nu\phi_x^2\phi_{xx}^2\\&+ 31\mu^2\phi_{xxxx} = 0.\end{aligned} \tag{19}$$

Obviously, $\phi = \mathrm{e}^{(kx+\omega t)}$ satisfies Eq. (19) if ω is fixed by

$$\omega = -\frac{(16562\nu^2k^4 - 1820\mu\nu k^2 - 31\mu^2)k}{507\nu}, \tag{20}$$

where k is an arbitrary constant.

The Mobious transformation of $\phi = \mathrm{e}^{(kx+\omega t)}$ has the form

$$\phi_1 = \frac{a + b\exp\left(kx - \dfrac{(16562\nu^2k^4 - 1820\mu\nu k^2 - 31\mu^2)kt}{507\nu}\right)}{c + d\exp\left(kx - \dfrac{(16562\nu^2k^4 - 1820\mu\nu k^2 - 31\mu^2)kt}{507\nu}\right)}. \tag{21}$$

Since the ϕ equation (19) satisfies the Mobious transformation invariance, ϕ_1 is also the solution of Eq. (19) above.

Now, u_0, u_1, u_2, u_3, u_4 are known to us, the only left thing is to find out ξ. It is not difficult to find that from Eqs. (3), (4), and (6),

$$\xi = \frac{2}{k}\coth\left[\frac{kx}{2} - \frac{(16562\nu^2k^4 - 1820\mu\nu k^2 - 31\nu^2)kt}{1014\nu} + t_0\right]. \tag{22}$$

Substituting u_0, u_1, u_2, u_3, u_4, ξ, Eqs. (6), (21) into Eq. (8), the approximate solution can be written as

$$\begin{aligned}u = &-\frac{35}{78}\tanh[(-507kx\nu - 31kt\mu^2 - 1820k^3t\mu\nu + 16562k^5t\nu^2 + 1014t_0\nu)/1014\nu]^2k^2\{2\mu + 39\nu\tanh[(-507kx\nu\\&- 31kt\mu^2 - 1820k^3t\mu\nu + 16562k^5t\nu^2 + 1014t_0\nu)/1014\nu]^2k^2 - 52\nu k^2\},\end{aligned} \tag{23}$$

where k, μ, ν, t_0 are arbitrary constants.

In order to understand approximate degree of the approximate solution we obtain, we should first try to find out the exact solution of the given model as soon as possible. Substituting Eq. (23) into Eq. (7), we know that

$$\begin{aligned}&-35\tanh[(-507kx\nu + 16562k^5t\nu^2 - 31kt\mu^2 - 1820k^3t\mu\nu + 1014t_0\nu)/1014\nu]k^3(\mu + 13\nu k^2)(1690\nu^2k^4\\&- 403\mu\nu k^2 + 31\mu^2)\{\tanh[(-507kx\nu + 16562k^5t\nu^2 - 31kt\mu^2 - 1820k^3t\mu\nu + 1014t_0\nu)/1014\nu] - 1\}\\&\times\{\tanh[(-507kx\nu + 16562k^5t\nu^2 - 31kt\mu^2 - 1820k^3t\mu\nu + 1014t_0\nu)/1014\nu] + 1\} = 0.\end{aligned} \tag{24}$$

It is easy to find out the relationship between μ, ν, and k,

$$k^2 = -\frac{\mu}{13\nu}. \tag{25}$$

Substituting Eq. (25) into Eq. (23), the result is the exact solution of the KdV equation with a fifth-order dispersion term, which reads

$$\begin{aligned}u = &-\frac{35}{2}\tanh\left(\frac{-507kx\nu + 34983k^5t\nu^2 + 1014t_0\nu}{1014\nu}\right)^2k^4\nu\\&\times\left[-2 + \tanh\left(\frac{-507kx\nu + 34983k^5t\nu^2 + 1014t_0\nu}{1014\nu}\right)^2\right],\end{aligned} \tag{26}$$

where k, μ, ν, t_0 are arbitrary constants.

Comparing the approximate solution and the exact solution of the KdV equation with a fifth-order dispersion term, we find that the closer $-\mu/13\nu$, the selected k is, the more precisely approximate solution we will get.

3.2. The second kind approximate solution

In the approximate solution (8) with Eqs. (14)–(18), the arbitrary function ϕ can be fixed in different ways. Say, we can take $u_3 = 0, u_4 \neq 0$. Substituting Eq. (6) into Eq. (17), yields

$$\phi_{xxxx}\phi_x^2 - 4\phi_{xxx}\phi_{xx}\phi_x + 3\phi_{xx}^3 = 0. \tag{27}$$

In accordance with the same principle of the above subsection, we take a special solution

$$\phi_2 = \frac{a + b\,\mathrm{e}^{(kx+\omega t)}}{c + d\,\mathrm{e}^{(kx+\omega t)}}, \tag{28}$$

$$\xi = \frac{2}{k}\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right). \tag{29}$$

Substituting u_0, u_1, u_2, u_3, u_4, ξ, Eqs. (6), (28) into Eq. (8), the final approximate solution can be written as

$$\begin{aligned} u = -\frac{1}{3042}\Bigg[& -70980k^5\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^2\nu^2 \\ & + 2730k^3\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^2\nu\mu \\ & -1820\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\mu\nu k^3 \\ & + 16562\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\nu^2k^5 \\ & + 53235\nu^2k^5 + 507\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\nu\omega \\ & - 31\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\mu^2k\Bigg] \\ & \Bigg/\left[\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4 k\nu\right], \end{aligned} \tag{30}$$

where k, μ, ν, t_0 are arbitrary constants.

Next we try to find out the exact solution, substituting Eq. (30) into Eq. (7), we know

$$\begin{aligned} & -\frac{35}{19773}k^3\left[\coth\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^2 - 1\right] \\ & \quad \times (31\mu^3 + 21970\nu^3k^6 - 3549k^4\nu^2\mu) = 0. \end{aligned} \tag{31}$$

So μ can be expressed with ν and k,

$$\mu = -13\nu k^2. \tag{32}$$

Substituting Eq. (32) into Eq. (30), the exact solution can be written as

$$\begin{aligned} u = \frac{1}{6}\Bigg[& 36\cosh\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\nu k^5 - 105\nu k^5 \\ & - \cosh\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\omega\Bigg] \\ & \Bigg/\left[k\cosh\left(\frac{kx}{2} + \frac{\omega t}{2} - t_0\right)^4\right]. \end{aligned} \tag{33}$$

From the approximate solution and the exact solution in this case, we get the same result with the above subsection about approximate degree.

4. Approximate solutions of some nonlinear PDEs

4.1. Perturbed fourth-order KdV equation

The idea used in the last section can also be used for other kinds of nonlinear systems. In this subsection, we used the idea to the perturbed fourth-order KdV equation[21]

$$u_t + \mu u_{xxx} + 6uu_x + \nu u_{xxxx} = 0. \tag{34}$$

Applying the invariant expansion to Eq. (34) leads to the results

$$u = \frac{u_0}{\xi^3} + \frac{u_1}{\xi^2} + \frac{u_2}{\xi} + u_3 + o(\xi), \tag{35}$$

$$u_0 = 20\nu, \tag{36}$$

$$u_1 = -\frac{5\mu}{2}, \tag{37}$$

$$u_2 = -\frac{5\mu^2}{152\nu} + 10\nu S, \tag{38}$$

$$u_3 = \frac{608\nu^2C - 9120\nu^3S_x - 13\mu^3 - 3040\mu\nu^2S}{3648\nu^2}, \tag{39}$$

$$\xi = \frac{2}{k}\coth\left(\frac{kx}{2} + \frac{5\mu k^3t}{4} - \frac{13\mu^3kt}{1216\nu^2} + t_0\right), \tag{40}$$

where k, μ, ν, t_0 are arbitrary constants, and S, C satisfy Eq. (5).

When $u_3 = 0$, we can get approximate solution of the perturbed fourth-order KdV equation in the form

$$u = \frac{5}{304}\tanh\left(\frac{608kx\nu^2 + 1520\mu k^3 t\nu^2 - 13\mu^3 kt + 1216t_0\nu^2}{1216\nu^2}\right)k \times \left[-38\mu\tanh\left(\frac{608kx\nu^2 + 1520\mu k^3 t\nu^2 - 13\mu^3 kt + 1216t_0\nu^2}{1216\nu^2}\right)k\nu - \mu^2 + 152k^2\nu^2\tanh\left(\frac{608kx\nu^2 + 1520\mu k^3 t\nu^2 - 13\mu^3 kt + 1216t_0\nu^2}{1216\nu^2}\right)^2 - 152k^2\nu^2\right]\Big/\nu. \quad (41)$$

4.2. KdV–Burgers equation

The KdV–Burgers equation[22]

$$u_t + \mu u_{xxx} + 6uu_x + \nu u_{xx} = 0, \quad (42)$$

may be one of the most important physical models of its wide applications. According to the invariant expansion, we may find the approximate solution in the form,

$$u = \frac{u_0}{\xi^2} + \frac{u_1}{\xi} + u_2 + o(\xi), \quad (43)$$

$$u_0 = -2\mu, \quad (44)$$

$$u_1 = \frac{2\nu}{5}, \quad (45)$$

$$u_2 = \frac{\nu^2}{150\mu} - \frac{2\mu S}{3} + \frac{C}{6}, \quad (46)$$

$$\xi = \frac{2}{k}\coth\left(\frac{kx}{2} + \mu k^3 t + \frac{kt\nu^2}{50\mu} + t_0\right), \quad (47)$$

where k, μ, ν, t_0 are arbitrary constants, and S, C satisfy Eq. (5).

When $u_2 = 0$, we can get a special approximate solution of the KdV–Burgers equation

$$u = -\frac{1}{2}\mu k^2\tanh\left(\frac{kx}{2} + \mu k^3 t + \frac{kt\nu^2}{50\mu} + t_0\right)^2 + \frac{1}{5}\nu k\tanh\left(\frac{kx}{2} + \mu k^3 t + \frac{kt\nu^2}{50\mu} + t_0\right). \quad (48)$$

Applying the idea in Section 3 to here, substituting Eq. (48) into the KdV–Burgers equation, we can get

$$k = \frac{\nu}{5\mu}. \quad (49)$$

Substituting Eq. (49) into Eq. 48), the KdV–Burgers equation has the following exact solution

$$u = -\frac{1}{50}\frac{\nu^2\tanh\left(\frac{\nu x}{10\nu} + \frac{3\nu^3 t}{250\mu^2} + t_0\right)^2}{\mu} + \frac{1}{25}\frac{\nu^2\tanh\left(\frac{\nu x}{10\nu} + \frac{3\nu^3 t}{250\mu^2} + t_0\right)}{\mu}, \quad (50)$$

where μ, ν, t_0 are arbitrary constants.

So the closer $\nu/5\mu$, the selected k is, the higher approximate degree we will obtain.

4.3. Boussinesq-type equation

The sixth-order Boussinesq-type equation[23]

$$u_{tt} + u_{xx} + 2u_x^2 + 2uu_{xx} + u_{xxxx} + \nu u_{xxxxxx} = 0. \quad (51)$$

Applying the invariant expansion to Eq. (51) leads to the results

$$u = \frac{u_0}{\xi^4} + \frac{u_1}{\xi^3} + \frac{u_2}{\xi^2} + \frac{u_3}{\xi} + u_4 + o(\xi), \quad (52)$$

$$u_0 = -840\nu, \quad (53)$$

$$u_1 = 0, \quad (54)$$

$$u_2 = -560\nu S - \frac{140}{13}, \quad (55)$$

$$u_3 = 140\nu S_x, \quad (56)$$

$$u_4 = -\frac{507\nu + 3640\nu S + 66248\nu^2 S^2 + 28392\nu^2 S_x x + 507\nu C - 31}{1014\nu}, \quad (57)$$

$$\xi = \frac{2}{k}\coth\left(\frac{kx}{2} + \frac{\sqrt{-3\nu(507\nu - 1820\nu k^2 + 16562k^4\nu^2 - 31)}kt}{78\nu} + t_0\right), \quad (58)$$

where k, μ, ν, t_0 are arbitrary constants, and S, C satisfy Eq. (5).

When $u_4 = 0$, we can get approximate solution of the sixth-order Boussinesq-type equation

$$u = -\frac{35}{26}\tanh[(39kx\nu + \sqrt{5460\nu^2k^2 - 49686k^4\nu^3 + 93\nu - 1521\nu^2}kt + 78t_0\nu)/78\nu]^2k^2\{-52\nu k^2 + 2 + 39\nu\tanh[(39kx\nu + \sqrt{5460\nu^2k^2 - 49686k^4\nu^3 + 93\nu - 1521\nu^2}kt + 78t_0\nu)/78\nu]^2k^2\}. \quad (59)$$

With the same point of view in the above subsection, substituting Eq. (48) into the sixth-order Boussinesq-type equation, we can obtain

$$k = \frac{1}{\sqrt{-13\nu}}. \tag{60}$$

Substituting Eq. (60) into Eq. (59), the sixth-order Boussinesq-type equation has the following exact solution

$$u = -\frac{105}{338}\tanh\left(\frac{13x\nu + \sqrt{-169\nu^2 - 69\nu}t + 26t_0\nu\sqrt{-13\nu}}{26\nu\sqrt{-13\nu}}\right)^2 \times \left[\tanh\left(\frac{13x\nu + \sqrt{-169\nu^2 - 69\nu}t + 26t_0\nu\sqrt{-13\nu}}{26\nu\sqrt{-13\nu}}\right)^2 - 2\right]\Big/\nu. \tag{61}$$

Obviously, if k infinitely closes to $1/\sqrt{-13\nu}$, the approximate solution becomes the exact solution.

5. Summary

In this paper, we have proposed the invariant expansion to approximately solve integrable or non-integrable nonlinear models. Taking the Korteweg-de Vries (KdV) equation with a fifth-order dispersion term as a concrete example, we have obtained two kinds of approximate solutions of this model. If the arbitrary constants are selected suitably, the approximate solutions may become exact. The same idea is also used to find approximate solutions of three other well-known nonlinear equations. The invariant expansion approach and other kinds of Painlevé expansions can be developed to find approximate or exact solutions.

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