Riemann theta function periodic wave solutions for the variable-coefficient mKdV equation*

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In this paper, a variable-coefficient modified Korteweg-de Vries (vc-mKdV) equation is considered. Bilinear forms are presented to explicitly construct periodic wave solutions based on a multidimensional Riemann theta function, then the one and two periodic wave solutions are presented, and it is also shown that the soliton solutions can be reduced from the periodic wave solutions.

Keywords: variable-coefficient mKdV equation, Riemann theta function, soliton solutions, periodic wave solutions

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1. Introduction

It is well known that finding exact soliton solutions of nonlinear evolution equations (NLEEs) is always one of the central themes in mathematics and physics.[1,2] A variety of effective methods for obtaining exact solutions of NLEEs has been presented. Such as the Hirota bilinear method,[3,4] the homogeneous balance method,[5] the inverse scattering method,[6,7] the Bäcklund transformation (BT),[8−11] the Wronskian technique,[12−14] and so on.

The bilinear method first proposed by Hirota provides us with a comprehensive approach to construct exact solutions. From bilinear forms, Nakamura has proposed a convenient way to construct a kind of quasi-periodic solution of a nonlinear equation by using the Riemann theta function,[15] it gives us a way to obtain periodic wave solutions of NLEEs conveniently. Some authors, such as Fan, Ma, and one of the authors, have extended this method to investigate the breaking soliton equation, ANNV, MNNV, generalized KdV, and the Boussinesq equation, respectively.[16−20]

In many physical situations, for example Bose–Einstein condensates and fluid dynamics, it is often preferable to have an equation with variable-coefficients, which can be used to describe the water waves propagating in a channel with an uneven bottom and deformed walls, and the trapped quasi-one-dimensional Bose–Einstein condensates with repulsive atom–atom interactions. It is obvious that variable-coefficient equations are often more complicated and difficult to be solved than constant-coefficient forms, and much attention has been paid to this subject.[21−26] The variable coefficient of the KdV equation provides a description of surface waves in a more realistic situation than the KdV equation itself, and the variable coefficients can also control the appearance of the soliton wave solutions.

In this paper, we will consider the variable-coefficient modified Korteweg-de Vries (vc-mKdV) equation,[1,2] which can be read as

$$u_t + g(t)u_{xxx} + l(t)u + f(t)u^2u_x + q(t)u_x = 0, \quad (1)$$

where $g(t)$, $f(t)$, $l(t)$, and $q(t)$ are all analytic functions and relevant to the background density and shear flow stratification, and the $l(t)u$ represents the perturbed term, or linear loss/gain. Equation (1) has attracted considerable attention in many different physical fields including ocean dynamics, fluid mechanics, and plasma physics.[27] It is shown that the constraints which the variable-coefficient functions must satisfy with Eq. (1) to pass the Painlevé tests for complete

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integrability, are

$$f(t) = \frac{6}{{c_0}^2} g(t)\, \mathrm{e}^{2\int l(t)\,\mathrm{d}t}.$$

We have investigated the exact soliton solutions and integrable properties (BT, Lax pair),[1] but the periodic wave solutions have not been studied. In the present work we will show that the periodic wave solutions and the soliton solutions of Eq. (1) can be obtained by the limits of these periodic wave solutions.

This paper is organized as follows. In Section 2, we briefly present necessary notations on the Riemann theta function. In Section 3, we construct one-periodic wave and two-periodic wave solutions of the vc-mKdV equation through bilinear forms, and demonstrate that the periodic wave solutions can be reduced to the known soliton solutions. Concluding remarks are given in the last section.

2. The Riemann theta function

Now we introduce the notations of the Riemann theta function. The multi-dimensional Riemann theta function is defined by

$$\begin{aligned}\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{s}|\boldsymbol{\tau}) = \sum_{n\in Z^N} & \exp\{2\pi\mathrm{i}\langle\boldsymbol{\xi}+\boldsymbol{\varepsilon}, \boldsymbol{n}+\boldsymbol{s}\rangle \\ & - \pi\langle\boldsymbol{\tau}(\boldsymbol{n}+\boldsymbol{s}), \boldsymbol{n}+\boldsymbol{s}\rangle\},\end{aligned} \tag{2}$$

where $\boldsymbol{n}$ is an integer value vector, $\boldsymbol{s}$ and $\boldsymbol{\varepsilon}$ is a complex parameter vector, $\boldsymbol{\xi}$ are complex phase variables, $\langle\cdot\rangle$ is inner product of two vectors, and the matrix $\boldsymbol{\tau} = (\tau_{ij})$ is a positive-definite and real-valued symmetric $N\times N$ matrix.

Suppose that $\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon'}, \boldsymbol{0}|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{0}|\boldsymbol{\tau})$ are two Riemann theta functions, in which $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$,

$$\xi_j = k_j x + l_j y + \omega_j t + \xi_j^{(0)}, \quad j = 1, 2, \ldots, N.$$

Then operators D_x, D_y, D_t exhibit the following perfect properties when they act on a pair of theta functions

$$\begin{aligned}& D_x\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon'}, \boldsymbol{0}|\boldsymbol{\tau}) \cdot \vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{0}|\boldsymbol{\tau}) \\ &= \sum_{\boldsymbol{\mu}} \partial_x\vartheta\left(2\boldsymbol{\xi}, \boldsymbol{\varepsilon'} - \boldsymbol{\varepsilon}, -\frac{\boldsymbol{\mu}}{2}\middle|2\boldsymbol{\tau}\right)\Big|_{\boldsymbol{\xi}=0} \\ &\quad \times \vartheta\left(2\boldsymbol{\xi}, \boldsymbol{\varepsilon'} + \boldsymbol{\varepsilon}, \frac{\boldsymbol{\mu}}{2}|2\boldsymbol{\tau}\right).\end{aligned} \tag{3}$$

In general, for a polynomial operator $H(D_x, D_y, D_t)$ with respect to D_x, D_y, and D_t, we have the following useful formula

$$\begin{aligned}& H(D_x, D_y, D_t)\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon'}, \boldsymbol{0}|\boldsymbol{\tau}) \cdot \vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{0}|\boldsymbol{\tau}) \\ &= \sum_{\mu} C(\boldsymbol{\varepsilon'}, \boldsymbol{\varepsilon}, \boldsymbol{\mu})\vartheta(2\boldsymbol{\xi}, \boldsymbol{\varepsilon'} + \boldsymbol{\varepsilon}, \frac{\boldsymbol{\mu}}{2}|2\boldsymbol{\tau}),\end{aligned} \tag{4}$$

in which, explicitly

$$\begin{aligned}C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = & \sum_{\boldsymbol{n}\in Z^N} H(\mathscr{M}) \\ & \times \exp\left[-2\pi\left\langle\boldsymbol{\tau}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right\rangle\right. \\ & \left. + 2\pi\mathrm{i}\left\langle\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{\varepsilon'} - \boldsymbol{\varepsilon}\right\rangle\right],\end{aligned} \tag{5}$$

where we denote $\mathscr{M} = (4\pi\mathrm{i}\langle\boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{k}\rangle, 4\pi\mathrm{i}\langle\boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{l}\rangle, 4\pi\mathrm{i}\langle\boldsymbol{n} - \boldsymbol{\mu}/2, \boldsymbol{\omega}\rangle)$.

Considering Eqs. (4) and (5), if the following equations are satisfied

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu}) = 0, \tag{6}$$

for all possible combinations $\mu_1 = 0, 1$; $\mu_2 = 0, 1$; …; $\mu_n = 0, 1$, in other words, all such combinations are solutions of Eq. (5), then $\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon'}, 0|\boldsymbol{\tau})$ and $\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, 0|\boldsymbol{\tau})$ are N-periodic wave solutions of the bilinear equation

$$H(D_x, D_y, D_t)\vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon'}, \boldsymbol{0}|\boldsymbol{\tau}) \cdot \vartheta(\boldsymbol{\xi}, \boldsymbol{\varepsilon}, \boldsymbol{0}|\boldsymbol{\tau}) = 0.$$

Let $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ and $H(D_x, D_y, D_t)$ be given in Eq. (6), and make a choice such that $\boldsymbol{\varepsilon'_j} - \boldsymbol{\varepsilon_j} = 1/2$, $j = 1, 2, \ldots, N$.

(i) If $H(D_x, D_y, D_t)$ is a symmetric operator, i.e.,

$$H(-D_x, -D_y, -D_t) = H(D_x, D_y, D_t), \tag{7}$$

then $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^N \mu_j$ is an odd number, namely

$$\begin{aligned}& C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})|_{\boldsymbol{\mu}} = 0 \\ & \quad \text{for} \sum_{j=1}^N \mu_j = 1, \ \mathrm{mod}\ 2.\end{aligned} \tag{8}$$

(ii) If $H(D_x, D_y, D_t)$ is a skew-symmetric operator, i.e.,

$$H(-D_x, -D_y, -D_t) = -H(D_x, D_y, D_t), \tag{9}$$

then $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})$ vanishes automatically for the case when $\sum_{j=1}^N \mu_j$ is an even number, namely

$$\begin{aligned}& C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon'}, \boldsymbol{\mu})|_{\boldsymbol{\mu}} = 0 \\ & \quad \text{for} \sum_{j=1}^N \mu_j = 0, \ \mathrm{mod}\ 2.\end{aligned} \tag{10}$$

Let $\boldsymbol{\varepsilon'_j} - \boldsymbol{\varepsilon_j} = 1/2, j = 1, 2, \ldots, N$. Assume $H(D_x, D_y, D_t)$ is a linear combination of even and odd functions

$$H(-D_x, -D_y, -D_t)$$

$$= H_1(D_x, D_y, D_t) + H_2(D_x, D_y, D_t), \quad (11)$$

where H_1 is even and H_2 is odd. In addition, $C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu})$ corresponding to Eq. (5) is given by

$$C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) = C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) + C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}), \quad (12)$$

where

$$\begin{aligned} C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) = & \sum_{\boldsymbol{n} \in Z^N} H_1(\mathscr{M}) \\ & \times \exp\left[-2\pi \left\langle \boldsymbol{\tau}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2} \right\rangle \right. \\ & \left. + 2\pi \mathrm{i} \left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon} \right\rangle \right], \\ C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) = & \sum_{\boldsymbol{n} \in Z^N} H_2(\mathscr{M}) \\ & \times \exp\left[-2\pi \left\langle \boldsymbol{\tau}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2} \right\rangle \right. \\ & \left. + 2\pi \mathrm{i} \left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{\varepsilon}' - \boldsymbol{\varepsilon} \right\rangle \right]. \end{aligned} \quad (13)$$

Then

$$\begin{aligned} & C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) = C_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}), \\ & \quad \text{for} \sum_{j=1}^{N} \mu_j = 1, \ \mathrm{mod}\ 2, \\ & C(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}) = C_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}', \boldsymbol{\mu}), \\ & \quad \text{for} \sum_{j=1}^{N} \mu_j = 0, \ \mathrm{mod}\ 2. \end{aligned} \quad (14)$$

3. The variable-coefficient mKdV equation

In this section, we have the necessity to write out the bilinear forms and the one-, two-soliton solutions of Eq. (1).

Under the transformation defined by

$$u = I c_0 \, \mathrm{e}^{-\int l(t)\,\mathrm{d}t} \frac{\partial}{\partial_x} \ln \frac{F^*}{F} \ (I = \sqrt{-1}), \quad (15)$$

where F^* is the conjugation of F, we can write Eq. (1) in the bilinear forms

$$\begin{aligned} & D_x^2 F \cdot F^* = 0, \\ & [D_t + g(t) D_x^3 + q(t) D_x] F \cdot F^* = 0, \end{aligned} \quad (16)$$

where the well-known Hirota bilinear operator D is defined by

$$\begin{aligned} D_x^m D_t^n f \cdot g = & (\partial_x - \partial_{x'})(\partial_t - \partial_{t'})^n \\ & \times f(x,t) g(x',t')|_{x'=x, t'=t}. \end{aligned}$$

Equation (1) admits a one-soliton solution and two-soliton solutions

$$\begin{aligned} u_1 = & I c_0 \, \mathrm{e}^{-\int l(t)\,\mathrm{d}t} \frac{\partial}{\partial_x} \ln \frac{1 - I\,\mathrm{e}^{\xi_1}}{1 + I\,\mathrm{e}^{\xi_1}} \\ = & c_0 \, \mathrm{e}^{-\int l(t)\,\mathrm{d}t} k_1 \mathrm{sech}\, \xi_1, \end{aligned} \quad (17)$$

$$\begin{aligned} u_2 = & I c_0 \, \mathrm{e}^{-\int l(t)\,\mathrm{d}t} \\ & \times \frac{\partial}{\partial_x} \ln \frac{1 - I\,\mathrm{e}^{\xi_1} - I\,\mathrm{e}^{\xi_2} - \mathrm{e}^{\xi_1+\xi_2+A_{12}}}{1 + I\,\mathrm{e}^{\xi_1} + I\,\mathrm{e}^{\xi_2} - \mathrm{e}^{\xi_1+\xi_2+A_{12}}}, \end{aligned} \quad (18)$$

where phase variable

$$\begin{aligned} \xi_j = & \ k_j x - k_j^3 \int g(t)\,\mathrm{d}t - k_j \\ & \times \int q(t)\,\mathrm{d}t + \xi_j^{(0)}, \\ \mathrm{e}^{A_{12}} = & \ \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2, \quad (j = 1, 2). \end{aligned}$$

3.1. One-periodic wave solution of the vc-mKdV equation

Let us first construct one-periodic wave solutions of the vc-mKdV equation (1) by using bilinear forms (16). As a simple case of the Riemann theta function (2) with $N = 1$, $s = 0$, here we choose F and F^* as follows:

$$\begin{aligned} F = & \ \vartheta\left(\xi, \frac{1}{4}, 0|\tau\right) \\ = & \sum_{n \in Z} \exp\left(2\pi \mathrm{i} n \left(\xi + \frac{1}{4}\right) - \pi n^2 \tau\right) \\ = & \sum_{n \in Z} (I)^n \exp(2\pi \mathrm{i} n \xi - \pi n^2 \tau), \\ F^* = & \ \vartheta\left(\xi, \frac{3}{4}, 0|\tau\right) \\ = & \sum_{n \in Z} \exp\left(2\pi \mathrm{i} n \left(\xi + \frac{3}{4}\right) - \pi n^2 \tau\right) \\ = & \sum_{n \in Z} (-I)^n \exp(2\pi \mathrm{i} n \xi - \pi n^2 \tau), \end{aligned} \quad (19)$$

where $\xi = kx + \int \omega\,\mathrm{d}t + \xi^{(0)}$ is the phase variable, and $\tau > 0$ is a positive parameter.

According to formulae (7) and (8), the operator $H_1 = D_x^2$ in bilinear equation (16) is symmetric, and its corresponding constraint equation in formula (6) automatically vanishes for $\mu = 1$. Meanwhile,

$$H_2 = D_t + g(t) D_x^3 + q(t) D_x$$

is skew symmetric, and its corresponding constraint equation automatically vanishes for $\mu = 0$. Therefore, the Riemann theta function (19) is a solution of

the bilinear equations (16), substituting Eqs. (19) into Eqs. (16), we have

$$\begin{aligned}
&\sum_{n\in Z}\left\{\left[4\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right]^2k^2\right\}\\
&\times\exp\left\{-2\pi\tau\left(n-\frac{\mu}{2}\right)^2+\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right\}\Big|_{\mu=0}\\
=&\ 0,\\
&\sum_{n\in Z}\left\{\left[4\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right]\omega+g(t)\left[4\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right]^3k^3\right.\\
&\left.+q(t)\left[4\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right]k\right\}\exp\left\{-2\pi\tau\left(n-\frac{\mu}{2}\right)^2\right.\\
&\left.+\pi\mathrm{i}\left(n-\frac{\mu}{2}\right)\right\}\Big|_{\mu=1}=0. \qquad (20)
\end{aligned}$$

Let us introducing the quantities

$$\begin{aligned}
\rho=&\ \mathrm{e}^{-\pi\tau/2},\\
\vartheta_1(\xi,\rho)=&\ \vartheta\left(2\xi,\frac{1}{2},-\frac{1}{2}|2\tau\right)\\
=&\sum_{n\in Z}\rho^{(2n-1)^2}\exp\left[4\mathrm{i}\pi\left(n-\frac{1}{2}\right)\left(\xi+\frac{1}{4}\right)\right],\\
\vartheta_2(\xi,\rho)=&\ \vartheta\left(2\xi,\frac{1}{2},0|2\tau\right)\\
=&\sum_{n\in Z}\rho^{4n^2}\exp\left[4\mathrm{i}\pi n\left(\xi+\frac{1}{4}\right)\right], \qquad (21)
\end{aligned}$$

then, we can write Eqs. (20) as a linear system about ω

$$\begin{aligned}
&\vartheta_2^{''}k^2=0,\\
&\vartheta_1^{'}\omega+g(t)\vartheta_1^{'''}k^3+q(t)\vartheta_1^{'}k=0, \qquad (22)
\end{aligned}$$

where the derivative value of $\vartheta_j(\xi,\rho)$ at $\xi=0$ is denoted by simple notations

$$\vartheta_j'=\vartheta_j'(0,\rho)=\frac{\mathrm{d}\vartheta_j(\xi,\rho)}{\mathrm{d}\xi}\Big|_{\xi=0},\quad j=1,2, \qquad (23)$$

solving this system, we obtain

$$\omega=-\frac{g(t)\vartheta_1^{'''}k^3+q(t)\vartheta_1^{'}k}{\vartheta_1^{'}}. \qquad (24)$$

We thus we arrive at one-periodic wave solution of Eq. (1) as follows:

$$u=Ic_0\,\mathrm{e}^{-\int l(t)\,\mathrm{d}t}\frac{\partial}{\partial_x}\ln\frac{\vartheta(\xi,3/4,0|\tau)}{\vartheta(\xi,1/4,0|\tau)}, \qquad (25)$$

where $\xi=kx+\int\omega\mathrm{d}t+\xi^{(0)}$ and parameter ω is given by Eq. (24), while other parameters k,τ are arbitrary. Among the three parameters, the two ones k and τ completely dominate a one-periodic wave.

In the following, we will see that the one-periodic wave solution (25) can be broken into soliton solution (17) under a long time limit and their relation can be established as follows.

Here we use system (22) to analyze asymptotic properties of the one-periodic solution (25). Setting

$$\xi=\frac{\xi'}{2\pi i}+\frac{\tau}{2},\quad \xi'=k'x+\int\omega'\mathrm{d}t+\xi^{(0)'}, \qquad (26)$$

where $k'=2\pi\mathrm{i}k$, $\omega'=2\pi\mathrm{i}\omega$, $\xi^{(0)'}=2\pi\mathrm{i}\xi^{(0)}-\pi\tau$.

Let us expand the coefficients of system (22) as follows:

$$\begin{aligned}
&\vartheta_1'=-4\pi\rho+12\pi\rho^9+\cdots,\\
&\vartheta_1''=-8\mathrm{i}\pi^2\rho+24\mathrm{i}\pi^2\rho^9+\cdots,\\
&\vartheta_1'''=16\pi^3\rho+432\pi^3\rho^9+\cdots,\\
&\vartheta_2=1-2\rho^4+\cdots,\\
&\vartheta_2''=32\pi^2\rho^4-128\pi\rho^{16}+\cdots, \qquad (27)
\end{aligned}$$

suppose that the solution of system (22) has the following form

$$\omega=\omega_0+\omega_1\rho^1+\omega_2\rho^2+\cdots=\omega_0+o(\rho), \qquad (28)$$

substituting expansions (27) and (28) into system (22) and letting $\rho\to0$, we can obtain the following relation immediately

$$\omega_0=4g(t)\pi^2k^3-q(t)k, \qquad (29)$$

combining Eqs. (26) and (28), then equation (29) leads to

$$\begin{aligned}
\omega'=&\ 2\pi\mathrm{i}\omega\to8\pi^3g(t)\mathrm{i}k^3-2\pi\mathrm{i}q(t)k\\
=&\ -g(t)k'^3-q(t)k'. \qquad (30)
\end{aligned}$$

It remains to verify that the one-periodic wave (25) has the same form as the one-soliton solution (17) under the limit $\rho\to0$. Let us expand the function F and F^* in the following form

$$\begin{aligned}
F=&\ 1+\rho^2(I\,\mathrm{e}^{2\pi\mathrm{i}\xi}-I\,\mathrm{e}^{-2\pi\mathrm{i}\xi})\\
&+\rho^8(-\mathrm{e}^{4\pi\mathrm{i}\xi}-\mathrm{e}^{-4\pi\mathrm{i}\xi})+\cdots,\\
F^*=&\ 1-\rho^2(I\,\mathrm{e}^{2\pi\mathrm{i}\xi}-I\,\mathrm{e}^{-2\pi\mathrm{i}\xi})\\
&+\rho^8(-\mathrm{e}^{4\pi\mathrm{i}\xi}-\mathrm{e}^{-4\pi\mathrm{i}\xi})+\cdots. \qquad (31)
\end{aligned}$$

It follows from Eqs. (26) and (31) that

$$\begin{aligned}
F=&\ 1+I\,\mathrm{e}^{\xi'}+\rho^4(-I\,\mathrm{e}^{-\xi'}-\mathrm{e}^{2\xi'})\\
&+\rho^{12}(\mathrm{e}^{-2\xi'}+\mathrm{e}^{3\xi'})+\cdots\\
\to&\ 1+I\,\mathrm{e}^{\xi'}
\end{aligned}$$

$$
\begin{aligned}
&\to 1 + I\,\mathrm{e}^{\xi},\\
F^* &= 1 - I\,\mathrm{e}^{\xi'} + \rho^4(-I\,\mathrm{e}^{-\xi'} - \mathrm{e}^{2\xi'})\\
&\quad + \rho^{12}(\,\mathrm{e}^{-2\xi'} + \mathrm{e}^{3\xi'}) + \cdots\\
&\to 1 - I\,\mathrm{e}^{\xi'}\\
&\to 1 - I\,\mathrm{e}^{\xi},\\
&\text{as } \rho \to 0. \qquad (32)
\end{aligned}
$$

Thus, we conclude that the one-periodic solution (25) may go to one-soliton solutions (17) as the amplitude $\rho \to 0$.

3.2. Two-periodic wave solutions of the vc-mKdV equation

Let us now consider two-periodic wave solutions to the vc-mKdV equation (1). For the case of $N = 2$, $\boldsymbol{\varepsilon} = \mathbf{1/4} = (1/4, 1/4), \boldsymbol{\varepsilon} = \mathbf{3/4} = (3/4, 3/4)$ in the Riemann theta function (2), we choose F and F^* as follows:

$$
\begin{aligned}
F &= \vartheta\left(\boldsymbol{\xi}, \frac{\mathbf{1}}{\mathbf{4}}, \mathbf{0}|\boldsymbol{\tau}\right)\\
&= \sum_{n\in Z^2} \exp\left\{2\pi\mathrm{i}\left\langle \boldsymbol{\xi} + \frac{\mathbf{1}}{\mathbf{4}}, \boldsymbol{n}\right\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\right\}\\
&= \sum_{n\in Z^2} (I)^{n_1+n_2} \exp\{2\pi\mathrm{i}\langle\boldsymbol{\xi}, \boldsymbol{n}\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\},\\
F^* &= \vartheta\left(\boldsymbol{\xi}, \frac{\mathbf{3}}{\mathbf{4}}, \mathbf{0}|\boldsymbol{\tau}\right)\\
&= \sum_{n\in Z^2} \exp\left\{2\pi\mathrm{i}\left\langle \boldsymbol{\xi} + \frac{\mathbf{3}}{\mathbf{4}}, \boldsymbol{n}\right\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\right\}\\
&= \sum_{n\in Z^2} (-I)^{n_1+n_2}\\
&\quad \times \exp\{2\pi\mathrm{i}\langle\boldsymbol{\xi}, \boldsymbol{n}\rangle - \pi\langle\boldsymbol{\tau n}, \boldsymbol{n}\rangle\}, \qquad (33)
\end{aligned}
$$

where $\boldsymbol{n} = (n_1, n_2) \in Z^2$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathscr{C}^2$,

$$
\xi_j = k_j x + \int \omega_j\,\mathrm{d}t + \xi_j^{(0)}, \quad j = 1, 2
$$

and $\boldsymbol{k} = (k_1, k_2) \in Z^2$, $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathscr{C}^2$, which is the phase variable, and $\boldsymbol{\tau} > 0$ is a positive parameter, where $\boldsymbol{\tau} = (\tau_{i,j})_{2\times 2}$, $\tau_{12} = \tau_{21}$, $\tau_{11} > 0$, $\tau_{22} > 0$, $\tau_{11}\tau_{22} - \tau_{12}^2 > 0$.

According to formulae (8) and (9), the operator $H_1 = D_x^2$ in bilinear equation (16) is symmetric, and its corresponding constraint equation in formula (6) automatically vanishes for $(\mu_1, \mu_2) = (0, 1), (1, 0)$. Meanwhile, $H_2 = D_t + g(t)D_x^3 + q(t)D_x$ is skew-symmetric, and its corresponding constraint equation automatically vanishes for $(\mu_1, \mu_2) = (0, 0), (1, 1)$. Therefore, the Riemann theta function (33) is a solution of the bilinear forms (16), provided the following equations

$$
\begin{aligned}
&\sum_{n_1,n_2\in Z^2} \left[-16\pi^2 \left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, k\right\rangle^2\right]\\
&\quad \times \exp\left\{ - 2\pi \left\langle \boldsymbol{\tau}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right\rangle\right.\\
&\quad \left. + \pi\mathrm{i}(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2})\right\}\Big|_{\mu=(\mu_1,\mu_2)} = 0,
\end{aligned}
$$

where $(\mu_1, \mu_2) = (0, 0), (1, 1)$,

$$
\begin{aligned}
&\sum_{n_1,n_2\in Z^2} \left\{ \left[4\pi\mathrm{i}\left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{\omega}\right\rangle\right]\right.\\
&\quad + g(t)\left[-64\pi^3\mathrm{i}\left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{k}\right\rangle^3\right]\\
&\quad + q(t)\left[4\pi\mathrm{i}\left\langle \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}, \boldsymbol{k}\right\rangle\right]\\
&\quad \times \exp\left\{ - 2\pi \left\langle \boldsymbol{\tau}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right\rangle\right.\\
&\quad \left.\left. + \pi\mathrm{i}\left(\boldsymbol{n} - \frac{\boldsymbol{\mu}}{2}\right)\right\}\right|_{\mu=(\mu_1,\mu_2)} = 0, \qquad (34)
\end{aligned}
$$

where $(\mu_1, \mu_2) = (0, 1), (1, 0)$. We introduce the notations by

$$
\begin{aligned}
\rho_{kl} &= \exp\left(-\frac{\pi\tau_{kl}}{2}\right), \quad k, l = 1, 2,\\
\rho &= (\rho_{11}, \rho_{12}, \rho_{22}),\\
\vartheta_j(\boldsymbol{\xi}, \boldsymbol{\rho}) &= \vartheta\left(2\boldsymbol{\xi}, \frac{\mathbf{1}}{\mathbf{2}}, -\frac{\boldsymbol{s_j}}{2}|2\tau\right)\\
&= \sum_{n_1,n_2\in Z} \exp\left[2\mathrm{i}\pi\left\langle 2\boldsymbol{\xi} + \frac{\mathbf{1}}{\mathbf{2}}, \boldsymbol{n} - \frac{\boldsymbol{s_j}}{\mathbf{2}}\right\rangle\right.\\
&\quad \left. - \pi\left\langle \tau\left(\boldsymbol{n} - \frac{\boldsymbol{s_j}}{\mathbf{2}}\right), \boldsymbol{n} - \frac{\boldsymbol{s_j}}{\mathbf{2}}\right\rangle\right]\\
&= \sum_{n_1,n_2\in Z} \exp\left[4\mathrm{i}\pi\left\langle \boldsymbol{\xi} + \frac{\mathbf{1}}{\mathbf{4}}, \boldsymbol{n} - \frac{\boldsymbol{s_j}}{\mathbf{2}}\right\rangle\right]\\
&\quad \times \prod_{k,l=1}^{2} \rho_{kl}^{(2n_k - s_{j,k})(2n_l - s_{j,l})},\\
\boldsymbol{s_j} &= (\boldsymbol{s_{j,1}}, \boldsymbol{s_{j,2}}), \quad j = 1, 2, 3, 4, \quad \boldsymbol{s}_1 = (0, 1),\\
\boldsymbol{s}_2 &= (1, 0), \quad \boldsymbol{s}_3 = (0, 0), \quad \boldsymbol{s}_4 = (1, 1), \qquad (35)
\end{aligned}
$$

then equations (34) can be written as a linear system about ω

$$
\begin{aligned}
&(\boldsymbol{k}\cdot\boldsymbol{\nabla})^2\vartheta_j = 0, \quad j = 3, 4,\\
&(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\vartheta_j + g(t)(\boldsymbol{k}\cdot\boldsymbol{\nabla})^3\vartheta_j + q(t)(\boldsymbol{k}\cdot\boldsymbol{\nabla})\vartheta_j = 0,\\
&\quad j = 1, 2, \qquad (36)
\end{aligned}
$$

where ϑ_j represent the derivative value of $\vartheta_j(\boldsymbol{\xi}, \boldsymbol{\rho})$ at $\xi_1 = 0$, $\xi_2 = 0$, the operators $\boldsymbol{\nabla} = (\partial_{\xi_1}, \partial_{\xi_2}, \ldots, \partial_{\xi_n})$ and

$$
\boldsymbol{k}\cdot\boldsymbol{\nabla} = k_1\partial_{\xi_1} + k_1\partial_{\xi_2} + \cdots + k_1\partial_{\xi_n}, \qquad (37)
$$

solve system (36), we are able to gain a two-periodic wave solution to the variable-coefficient mKdV equation (1)

$$u = Ic_0\,\mathrm{e}^{-\int l(t)\,\mathrm{d}t}\frac{\partial}{\partial_x}\ln\frac{\vartheta(\boldsymbol{\xi},\mathbf{3/4},\mathbf{0}|\boldsymbol{\tau})}{\vartheta(\boldsymbol{\xi},\mathbf{1/4},\mathbf{0}|\boldsymbol{\tau})}, \tag{38}$$

where $\boldsymbol{\xi} = \boldsymbol{k}x + \int \boldsymbol{\omega}\,\mathrm{d}t + \boldsymbol{\xi}^{(\mathbf{0})}$ and parameter $\boldsymbol{\omega}$ is given by Eqs. (36), and other parameters k_1, k_2, $\boldsymbol{\tau}$ are arbitrary.

In the following, we will see that the two-periodic wave solution (38) can be broken into soliton solution (18) under a long time limit and their relation can be established.

Here we use system (36) to analyze asymptotic properties of the two-periodic solutions (38). Setting

$$\xi_j = \frac{\xi_j'}{2\pi i} + \frac{\tau_{jj}}{2},\ \xi_j' = k_j'x + \int \omega_j'\,\mathrm{d}t + \xi_j^{(0)'}, \tag{39}$$

where $k_j' = 2\pi i k_j$, $\xi_j^{(0)'} = 2\pi\mathrm{i}\xi_j^{(0)} - \pi\tau_{jj}$, $A_{12} = -2\pi\tau_{12}$, $j = 1,2$, and A_{12} being the same as given in Eqs. (18).

Let us expand the function F and F^* in the following form

$$\begin{aligned}
F &= 1 + (I\,\mathrm{e}^{2\pi\mathrm{i}\xi_1} - I\,\mathrm{e}^{-2\pi\mathrm{i}\xi_1})\,\mathrm{e}^{-\pi\tau_{11}} \\
&\quad + (I\,\mathrm{e}^{2\pi\mathrm{i}\xi_2} - I\,\mathrm{e}^{-2\pi\mathrm{i}\xi_2})\,\mathrm{e}^{-\pi\tau_{22}} \\
&\quad + (-\,\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)} - \mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)}) \\
&\quad \times \mathrm{e}^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots. \\
F^* &= 1 - (I\,\mathrm{e}^{2\pi\mathrm{i}\xi_1} - I\,\mathrm{e}^{-2\pi\mathrm{i}\xi_1})\,\mathrm{e}^{-\pi\tau_{11}} \\
&\quad - (I\,\mathrm{e}^{2\pi\mathrm{i}\xi_2} - I\,\mathrm{e}^{-2\pi\mathrm{i}\xi_2})\,\mathrm{e}^{-\pi\tau_{22}} \\
&\quad + (-\,\mathrm{e}^{2\pi\mathrm{i}(\xi_1+\xi_2)} - \mathrm{e}^{-2\pi\mathrm{i}(\xi_1+\xi_2)}) \\
&\quad \times \mathrm{e}^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \cdots.
\end{aligned} \tag{40}$$

Furthermore, adopting Eq. (39) and making a transformation we infer that

$$\begin{aligned}
F &= 1 + I\,\mathrm{e}^{\xi_1'} + I\,\mathrm{e}^{\xi_2'} - \mathrm{e}^{\xi_1'+\xi_2'-2\pi\tau_{12}} - I\rho_{11}^4\,\mathrm{e}^{-\xi_1'} \\
&\quad - I\rho_{22}^4\,\mathrm{e}^{-\xi_2'} - \rho_{11}^4\rho_{22}^4\,\mathrm{e}^{-\xi_1'-\xi_2'-2\pi\tau_{12}} + \cdots \\
&\to 1 + I\,\mathrm{e}^{\xi_1'} + I\,\mathrm{e}^{\xi_2'} - \mathrm{e}^{\xi_1'+\xi_2'+A_{12}}, \\
F^* &= 1 - I\,\mathrm{e}^{\xi_1'} - I\,\mathrm{e}^{\xi_2'} - \mathrm{e}^{\xi_1'+\xi_2'-2\pi\tau_{12}} + I\rho_{11}^4\,\mathrm{e}^{-\xi_1'} \\
&\quad + I\rho_{22}^4\,\mathrm{e}^{-\xi_2'} - \rho_{11}^4\rho_{22}^4\,\mathrm{e}^{-\xi_1'-\xi_2'-2\pi\tau_{12}} + \cdots \\
&\to 1 + I\,\mathrm{e}^{\xi_1'} + I\,\mathrm{e}^{\xi_2'} - \mathrm{e}^{\xi_1'+\xi_2'+A_{12}}, \\
&\text{as } \rho_{11}, \rho_{22} \to 0,
\end{aligned} \tag{41}$$

where $\xi_j' = k_j' + \int \omega_j'\,\mathrm{d}t + \xi_j^{(0)}$, and $\omega_j' = 2\pi i\omega_j, j = 1,2$. Thus the two-periodic wave solutions can be reduced to two-soliton solutions (18) under the limit $\rho_{11}, \rho_{22}\to 0$, we only need to prove that

$$\omega_j' \to -g(t)k_j'^3 - q(t)k_j',\quad j = 1,2. \tag{42}$$

We can expand the solution of system (36) to the following form

$$\begin{aligned}
\omega_1 &= \omega_{1,0} + \omega_{1,1}\rho^{11} + \omega_{2,2}\rho^{22} + o(\rho^{11}, \rho^{22}), \\
\omega_2 &= \omega_{2,0} + \omega_{2,1}\rho^{11} + \omega_{2,2}\rho^{22} + o(\rho^{11}, \rho^{22}).
\end{aligned} \tag{43}$$

Expanding functions $\vartheta_j, j = 1,2,3,4$ in Eqs. (36) and (39), substituting the expansions (43) into system (36), and letting $\rho_{11}, \rho_{22} \to 0$, we obtain

$$\begin{aligned}
&-64g(t)\pi^3\mathrm{i}\left(-\frac{1}{2}\right)^3 k_1^3 + 4\pi\mathrm{i}\left(-\frac{1}{2}\right)\omega_{1,0} \\
&\quad + 4q(t)\pi\mathrm{i}\left(-\frac{1}{2}\right)k_1 = 0, \\
&-64g(t)\pi^3\mathrm{i}\left(-\frac{1}{2}\right)^3 k_2^3 + 4\pi\mathrm{i}\left(-\frac{1}{2}\right)\omega_{2,0} \\
&\quad + 4q(t)\pi\mathrm{i}\left(-\frac{1}{2}\right)k_2 = 0,
\end{aligned} \tag{44}$$

using Eqs. (43) and (44), we have

$$\omega_j = 4g(t)\pi^2 k_j^3 - q(t)k_j,\quad j = 1,2. \tag{45}$$

Thus, we conclude that the two-periodic solution (38) may go to two-soliton solutions (18) as the amplitude $\rho_{11}, \rho_{22} \to 0$.

4. Concluding remarks

In this study, the vc-mKdV equation is discussed. Based on bilinear representation, employing the theory of Riemann theta function, we have derived the one periodic wave solution, which can be reduced to the soliton solutions under small amplitude $\rho \to 0$ limits; under small amplitude $\rho_{11}, \rho_{22} \to 0$ limits, and the calculation of two-periodic wave solutions reduced to the soliton solutions is given in Appendix A.

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Appendix A: Proof of the two-periodic wave solution broken into two-soliton solutions

In the following, we prove that system (34) can be converted to system (36), and

$$\omega_j = 4g(t)\pi^2 k_j^3 - q(t)k_j,\quad j = 1,2,$$

as $\rho_{11}, \rho_{22} \to 0$.

First, we demonstrate that system (34) can be converted to system (36). For example, when $j = 3$, $\mu = (\mu_1, \mu_2) = (0, 0)$, therefore

$$\begin{aligned}
&\vartheta_3(\boldsymbol{\xi}, \boldsymbol{\rho}) \\
&= \vartheta\left(2\boldsymbol{\xi}, \frac{\mathbf{1}}{\mathbf{2}}, -\frac{\boldsymbol{s_3}}{2}|2\tau\right) \\
&= \sum_{n_1,n_2\in Z^2} \exp\left\{4\mathrm{i}\pi\left\langle\begin{pmatrix}\xi_1+1/4\\ \xi_2+1/4\end{pmatrix}, \begin{pmatrix}n_1\\ n_2\end{pmatrix}\right\rangle\right\} \\
&\quad\times\ \rho_{11}^{(2n_1)^2}\rho_{12}^{2(4n_1n_2)}\rho_{22}^{(2n_2)^2} \\
&= \sum_{n_1,n_2\in Z^2} \exp\left\{4\mathrm{i}\pi\left[n_1\xi_1 + n_2\xi_2 + \frac{1}{4}(n_1+n_2)\right]\right\} \\
&\quad\times\ \rho_{11}^{(2n_1)^2}\rho_{12}^{2(4n_1n_2)}\rho_{22}^{(2n_2)^2}, \qquad \text{(A1)}
\end{aligned}$$

and the first equation of system (34) follows that

$$\begin{aligned}
&\sum_{n_1,n_2\in Z^2}\left[-16\pi^2\left\langle \boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}, \boldsymbol{k}\right\rangle^2\right] \\
&\qquad\times\ \exp\left\{-2\pi\left\langle\boldsymbol{\tau}\left(\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}\right\rangle\right. \\
&\qquad\left.+\ \pi\mathrm{i}\left(\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}\right)\right\}\Big|_{\mu=(\mu_1,\mu_2)=(0,0)} \\
&= \sum_{n_1,n_2\in Z^2}\left[-16\pi^2\left\langle\begin{pmatrix}n_1\\ n_2\end{pmatrix}, \begin{pmatrix}k_1\\ k_2\end{pmatrix}\right\rangle^2\right] \\
&\qquad\times\ \exp\left\{-2\pi\left\langle\begin{pmatrix}\tau_{11}&\tau_{12}\\ \tau_{21}&\tau_{22}\end{pmatrix}\begin{pmatrix}n_1\\ n_2\end{pmatrix}, \begin{pmatrix}n_1\\ n_2\end{pmatrix}\right\rangle\right. \\
&\qquad\left.+\ \pi\mathrm{i}(n_1+n_2)\right\} \\
&= \sum_{n_1,n_2\in Z^2}[-16\pi^2(n_1k_1+n_2k_2)^2] \\
&\qquad\times\ \exp\{-2\pi[\tau_{11}n_1^2 + 2\tau_{12}(n_1n_2) + \tau_{22}n_2^2] \\
&\qquad+\ \pi\mathrm{i}(n_1+n_2)\} \\
&= \sum_{n_1,n_2\in Z^2}[-16\pi^2(n_1k_1+n_2k_2)^2] \\
&\qquad\times\ \rho_{11}^{(2n_1)^2}\rho_{12}^{2(4n_1n_2)}\rho_{22}^{(2n_2)^2}\,\mathrm{e}^{\pi\mathrm{i}(n_1+n_2)} \\
&= 0, \qquad \text{(A2)}
\end{aligned}$$

comparing Eq. (A2) together with Eqs. (37), then equation (A2) can be converted to

$$(\boldsymbol{k}\cdot\boldsymbol{\nabla})^2\vartheta_3 = 0. \qquad \text{(A3)}$$

Second, we demonstrate $\omega_j = 4g(t)\pi^2k_j^3 - q(t)k_j$, $(j = 1, 2)$ as $\rho_{11}, \rho_{22} \to 0$.

For example, when $j = 1$, and $\mu = (\mu_1, \mu_2) = (0, 1)$, then

$$\begin{aligned}
\vartheta_1(\boldsymbol{\xi}, \boldsymbol{\rho}) &= \vartheta\left(2\boldsymbol{\xi}, \mathbf{1/2}, -\frac{\boldsymbol{s_1}}{2}|2\tau\right) \\
&= \sum_{n_1,n_2\in Z^2}\exp\left\{4\mathrm{i}\pi\left\langle\begin{pmatrix}\xi_1+1/4\\ \xi_2+1/4\end{pmatrix}, \begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}\right\rangle\right\}\rho_{11}^{(2n_1)^2}\rho_{12}^{2(2n_1)(2n_2-1)}\rho_{22}^{(2n_2-1)^2} \\
&= \sum_{n_1,n_2\in Z^2}\exp\left\{4\mathrm{i}\pi\left[n_1\xi_1 + n_2\xi_2 + \frac{1}{4}(n_1+n_2) - 1/2\xi_2 - \frac{1}{8}\right]\right\}\rho_{11}^{(2n_1)^2}\rho_{12}^{2(2n_1)(2n_2-1)}\rho_{22}^{(2n_2-1)^2}, \qquad \text{(A4)}
\end{aligned}$$

and we have the second equation of system (34)

$$\begin{aligned}
&\sum_{n_1,n_2\in Z^2}\left\{\left[4\pi\mathrm{i}\left\langle\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}, \boldsymbol{\omega}\right\rangle\right] + \left[-64\pi^3\mathrm{i}\left\langle\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}, \boldsymbol{k}\right\rangle^3\right] + \left[4\pi\mathrm{i}\left\langle\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}, \boldsymbol{k}\right\rangle\right]\right\} \\
&\quad\times\ \exp\left\{-2\pi\left\langle\boldsymbol{\tau}\left(\boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}\right), \boldsymbol{n}-\frac{\boldsymbol{\mu}}{2}\right\rangle + \pi\mathrm{i}\left(n_j-\frac{\mu_j}{2}\right)\right\}\Big|_{\mu=(\mu_1,\mu_2)=(0,1)} \\
&= \sum_{n_1,n_2\in Z^2}\left\{\left[-64g(t)\pi^3\mathrm{i}\left\langle\begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}, \begin{pmatrix}k_1\\ k_2\end{pmatrix}\right\rangle^3 + \left[4\pi\mathrm{i}\left\langle\begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}, \begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right\rangle\right]\right.\right. \\
&\quad\left.\left.+\ \left[4q(t)\pi\mathrm{i}\left\langle\begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}, \begin{pmatrix}k_1\\ k_2\end{pmatrix}\right\rangle\right]\right\}\exp\left\{-2\pi\left\langle\begin{pmatrix}\tau_{11}&\tau_{12}\\ \tau_{21}&\tau_{22}\end{pmatrix}\begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}, \begin{pmatrix}n_1\\ n_2-1/2\end{pmatrix}\right\rangle\right. \\
&\quad\left.+\ \pi\mathrm{i}\left(n_1+n_2-1/2\right)\right\} \\
&= \sum_{n_1,n_2\in Z^2}\left\{\left[-64g(t)\pi^3\mathrm{i}\left(n_1k_1 + \left(n_2-1/2\right)k_2\right)^3\right] + \left[4\pi\mathrm{i}\left(n_1\omega_1 + \left(n_2-1/2\right)\omega_2\right)\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+\ [4q(t)\pi \mathrm{i}\,(n_1k_1+(n_2-1/2)\,k_2)]\}\exp\{-2\pi[\tau_{11}n_1^2+2\tau_{12}n_1\,(n_2-1/2)+\tau_{22}\,(n_2-1/2)^2]\\
&+\ \pi \mathrm{i}\,(n_1+n_2-1/2)\}\\
=&\sum_{n_1,n_2\in Z^2}\{[-64g(t)\pi^3\mathrm{i}\,(n_1k_1+(n_2-1/2)\,k_2)^3]+[4\pi \mathrm{i}\,(n_1\omega_1+(n_2-1/2)\,\omega_2)]\\
&+\ [4q(t)\pi \mathrm{i}\,(n_1k_1+(n_2-1/2)\,k_2)]\}\rho_{11}^{(2n_1)^2}\rho_{12}^{2(2n_1)(2n_2-1)}\rho_{22}^{(2n_2-1)^2}\,\mathrm{e}^{\pi \mathrm{i}(n_1+n_2-1/2)}\\
=&\,0, \qquad \text{(A5)}
\end{aligned}
$$

combining Eqs. (37) and (A5) leads to

$$g(t)(\boldsymbol{k}\cdot\boldsymbol{\nabla})^3\vartheta_1+(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\vartheta_1+q(t)(\boldsymbol{k}\cdot\boldsymbol{\nabla})\vartheta_1=0, \qquad \text{(A6)}$$

as the amplitude $\rho_{11},\rho_{22}\to 0$, from Eq. (A6) we can obtain

$$
\begin{aligned}
&-64g(t)\pi^3\mathrm{i}\,(-1/2)^3+4\pi\mathrm{i}\,(-1/2)\,\omega_2\\
&+\ 4q(t)\pi\mathrm{i}\,(-1/2)\,k_2=0, \qquad \text{(A7)}
\end{aligned}
$$

substituting Eq. (43) into Eq. (A7), we obtain

$$\omega_{2,0}=4g(t)\pi^2k_2^3-q(t)k_2. \qquad \text{(A8)}$$

In a similar procedure, when $j=2$, and $\mu=(\mu_1,\mu_2)=(1,0)$, let the amplitude $\rho_{11},\rho_{22}\to 0$, we have

$$\omega_{1,0}=4g(t)\pi^2k_1^3-q(t)k_1, \qquad \text{(A9)}$$

which implies Eq. (45), thus the proof is completed.

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