# A type of new conserved quantity deduced from Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints*

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A type of new conserved quantity deduced from Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints is investigated. Nielsen equations and differential equations of motion for the holonomic mechanical system with unilateral constraints are established. The definition and the criterion of Mei symmetry for Nielsen equations in the holonomic systems with unilateral constraints under the infinitesimal transformations of Lie group are also given. The expressions of the structural equation and a type of new conserved quantity of Mei symmetry for Nielsen equations in the holonomic system with unilateral constraints are obtained. An example is given to illustrate the application of the results.

**Keywords:** holonomic system with unilateral constraints, Nielsen equation, structural equation, new type of conserved quantity

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## 1. Introduction

In 1918, Noether[1] revealed the potential relation between the symmetry and the conserved quantity. However, the scientific value of the Noether theory was not fully realized in the circles of mechanics until the 1970s.[2−7] Since then, the study of symmetry and conserved quantity has developed rapidly, and now it has become a developing direction of modern analytical mechanics.

In 2000, Mei[8] presented a type of new symmetry, which means that the dynamic function in equation of motion keeps the equation's primary form after the infinitesimal transformations, namely the form invariance, and it is now called Mei symmetry. From 2000 to 2007, some important results on Mei symmetry were gained.[5,9] Since 2008, Mei symmetry has been a hot research topic for the symmetry and conserved quantity in constrained mechanical systems.[10−23] The Nielsen system is one of the three major mechanical systems of analytical mechanics. Recently, the studies of Mei symmetry of Nielsen equations have received more and more attention.[24−29]

In the natural world and actual engineering technology, a great number of constraints belong to unilateral constraints. Hence, research on unilateral constraints have important actual significance. In recent years, the studies of symmetry and conserved quantity in holonomic systems with unilateral constraints have also made much progress. Zhang studied the form invariance in a holonomic system with unilateral constraints,[30] the symmetry and conserved quantity of phase space in a holonomic system with unilateral constraints,[31] and the non-Noether conserved quantity for a system with unilateral non-Chetaev nonholonomic constrains.[32] Li and Fang[33] studied Mei symmetry of a variable mass system with unilateral holonomic constraints. Jing *et al.*[34] studied perturbation of Lie symmetry and a type of generalized Hojman adiabatic invariant for a variable mass system with unilateral holonomic constraints. In this paper we study the structural equation and a type of new conserved quantity of Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints.

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## 2. Definition and criterion of Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints

Supposing that the system with ideal unilateral holonomic constraints is written as follows:

$$f_\beta(t,\boldsymbol{q}) \geq 0, \quad (\beta = 1,2,\ldots,g), \tag{1}$$

introducing the Nielsen arithmetic operators

$$N_s = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1,2,\ldots,n),$$

and supposing that the non-potential forces are $Q_s = Q_s(t,\boldsymbol{q},\dot{q})$, the Lagrange function of the system, Nielsen equation, and constraint equation can be written respectively as follows:

$$L = L(t,\boldsymbol{q},\dot{q}), \tag{2}$$

$$N_s(L) = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial q_s} = Q_s + \Lambda_s, \quad (s = 1,2,\ldots,n), \tag{3}$$

$$f_\beta = 0, \quad (\beta = 1,\ldots,g) \text{ (when } f_\beta = 0); \tag{4}$$

$$N_s(L) = Q_s, \quad (s = 1,2,\ldots,n), \tag{5}$$

$$f_\beta > 0, \quad (\beta = 1,\ldots,g) \text{ (when } f_\beta > 0). \tag{6}$$

For each $\beta$, we have $\lambda_\beta \geq 0$, $f_\beta \geq 0$, $\lambda_\beta f_\beta = 0$, $(\beta = 1,\ldots,g)$. Here we adopt Einstein's summation convention, and in Eq. (3), $\lambda_\beta$ are called the constraint multipliers, and

$$\Lambda_s = \Lambda_s(t,q,\dot{q}) = \lambda_\beta \frac{\partial f_\beta}{\partial q_s} \tag{7}$$

are called the constraint reaction forces corresponding to the generalized coordinates $q_s$. From Eqs. (3) and (5), we can solve all $\ddot{q}_s$ as follows:

$$\ddot{q}_s = A_s(t,\boldsymbol{q},\dot{q}),\ (s = 1,2,\ldots,n) \text{ (when } f_\beta = 0),$$

$$\ddot{q}_s = B_s(t,\boldsymbol{q},\dot{q}),\ (s = 1,2,\ldots,n) \text{ (when } f_\beta > 0). \tag{8}$$

Introducing the infinitesimal transformation of groups of time and quasi-coordinates or their expanded forms as follows:

$$t^* = t + \Delta t,\ q_s^*(t^*) = q_s(t) + \Delta q_s, \quad (s = 1,\ldots,n), \tag{9}$$

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{q},\dot{q}),$$

$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\dot{q}),\ (s = 1,\ldots,n), \tag{10}$$

where $\varepsilon$ is an infinitesimal parameter, $\xi_0$ and $\xi_s$ are infinitesimal generators. Introducing vector $X^{(0)}$ of the infinitesimal generator and its first-order expansion $X^{(1)}$

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \tag{11}$$

$$X^{(1)} = X^{(0)} + \left(\frac{\mathrm{d}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\mathrm{d}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial \dot{q}_s}, \tag{12}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + A_s\frac{\partial}{\partial \dot{q}_s}, \text{ (when } f_\beta = 0), \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + B_s\frac{\partial}{\partial \dot{q}_s}, \text{ (when } f_\beta > 0). \tag{14}$$

From Eq. (10), we obtain

$$\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} = \frac{\mathrm{d}q_s + \varepsilon\mathrm{d}\xi_s}{\mathrm{d}t + \varepsilon\mathrm{d}\xi_0} = \dot{q}_s + \varepsilon(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + O(\varepsilon^2),$$

$$\frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}} = \ddot{q}_s + \varepsilon[(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)^{\cdot} - \ddot{q}_s\dot{\xi}_0] + O(\varepsilon^2). \tag{15}$$

Supposing that under infinitesimal transformations (10), the dynamic function $L$ becomes $L^*$, $Q_s$ becomes $Q_s^*$, and $\Lambda_s$ becomes $\Lambda_s^*$. And then taking the Taylor expansions of $L^*$, $Q_s^*$, and $\Lambda_s^*$ at the point of $(t,\boldsymbol{q},\dot{q})$ along the trajectory of motion of the system, we have

$$L^* = L(t,\boldsymbol{q},\dot{q}) + \varepsilon X^{(1)}(L) + O(\varepsilon^2), \tag{16}$$

$$Q_s^* = Q_s(t,\boldsymbol{q},\dot{q}) + \varepsilon X^{(1)}(Q_s) + O(\varepsilon^2), \quad (s = 1,\ldots,n), \tag{17}$$

$$\Lambda_s^* = \Lambda_s(t,\boldsymbol{q},\dot{q}) + \varepsilon X^{(1)}(\Lambda_s) + O(\varepsilon^2), \quad (s = 1,\ldots,n), \tag{18}$$

$$f_\beta^* = f_\beta(t,\boldsymbol{q}) + \varepsilon X^{(0)}(f_\beta) + O(\varepsilon^2), \quad (\beta = 1,\ldots,g). \tag{19}$$

**Definition 1** If equation (3) keeps the equation's primary form when dynamic functions $L$, $Q_s$, and $\Lambda_s$, are substituted by the functions $L^*$, $Q_s^*$, and $\Lambda_s^*$ under infinitesimal transformations Eq. (10), that is,

$$N_s(L^*) = Q_s^* + \Lambda_s^*,\ (s = 1,\ldots,n), \tag{20}$$

then the invariance is called Mei symmetry of the Nielsen equation.

**Definition 2** If equation (4) keeps the equation's primary form when functions $f_\beta$ are substituted by functions $f_\beta^*$ under infinitesimal transformations Eq. (10), that is,

$$f_\beta^* = f_\beta(t^*,\boldsymbol{q}^*) = 0,\ (\beta = 1,\ldots,g) \text{ (when} f_\beta = 0), \tag{21}$$

then the invariance is called Mei symmetry of constraint equation (4).

**Definition 3** If equations (5) and (6) are of Mei symmetry, then the invariance is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints when $f_\beta = 0$.

**Definition 4** If equation (5) keeps the equation's primary form when dynamic functions $L$ and $Q_s$ are substituted by the functions $L^*$ and $Q_s^*$ under infinitesimal transformations Eq. (10), that is,

$$N_s(L^*) = Q_s^*,\ (s = 1, 2, \ldots, n), \tag{22}$$

then the invariance is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints when $f_\beta > 0$. Now, we have

$$f_\beta^* = f_\beta\left(t^*, \boldsymbol{q}^*\right) > 0,\ (\beta = 1, \ldots, g)\ \ (\text{when } f_\beta > 0)\,. \tag{23}$$

**Definition 5** If equations (3)–(5) are all of Mei symmetry, then the invariance is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints.

Substituting Eqs. (16)–(19) into Eqs. (20)–(23) and neglecting the terms that contain $\varepsilon^2$ or its higher order, by means of Eqs. (3)–(6), we obtain

$$N_s[X^{(1)}(L)] = X^{(1)}\left(Q_s + \Lambda_s\right), \quad (s = 1, \ldots, n), \tag{24}$$

$$X^{(0)}\left(f_\beta\right) = 0, \quad (\beta = 1, \ldots, g)\ \ (\text{when } f_\beta = 0)\,; \tag{25}$$

$$N_s[X^{(1)}(L)] = X^{(1)}\left(Q_s\right), \quad (s = 1, \ldots, n), \tag{26}$$

$$f_\beta\left(t, \boldsymbol{q}\right) + \varepsilon X^{(0)}\left(f_\beta\right) > 0, \quad (\beta = 1, \ldots, g)\ \ (\text{when } f_\beta > 0)\,. \tag{27}$$

Equations (24)–(27) are called the criterions of Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints.

**Criterion 1** If the infinitesimal generators $\xi_0$ and $\xi_s$ make the criterion equations (24) and (25) hold true, then the invariance of Eqs. (3) and (4) under the infinitesimal transformations (10) is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints when $f_\beta = 0$.

**Criterion 2** If the infinitesimal generators $\xi_0$ and $\xi_s$ make the criterion equation (26) hold true, then the invariance of Eq. (5) under the infinitesimal transformations (10) is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints when $f_\beta > 0$.

**Criterion 3** If the infinitesimal generators $\xi_0$ and $\xi_s$ make the criterion equations. (24)–(26) hold true, then the invariance of Eqs. (3)–(5) under the infinitesimal transformations (10) is called Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints.

## 3. A type of new conserved quantity of Mei symmetry for Nielsen equations in holonomic system with unilateral constraints

**Proposition 1** If the infinitesimal generators $\xi_0$ and $\xi_s$ and the gauge function $G_M = G_M(t, \boldsymbol{q}, \dot{q})$ of Nielsen equations in a holonomic system make the following structural equation hold true:

$$\frac{\partial X^{(1)}(L)}{\partial t} - X^{(1)}(Q_s + \Lambda_s)\dot{q}_s + \dot{G}_X = 0, \quad (\text{when } f_\beta = 0), \tag{28}$$

$$\frac{\partial X^{(1)}(L)}{\partial t} - X^{(1)}(Q_s)\dot{q}_s + \dot{G}_X = 0, \quad (\text{when } f_\beta > 0), \tag{29}$$

then the type of new conserved quantities of Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints is expressed as follows:

$$I_X = X^{(1)}(L) - \frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\dot{q}_s + G_X = \text{const}. \tag{30}$$

**Proof** When $f_\beta = 0$, noting Eq. (28) and taking the derivative of Eq. (30) with respect to $t$, we have

$$\begin{aligned}
\frac{\mathrm{d}I_X}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[X^{(1)}(L)\right] - \dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right] \\
&\quad - \ddot{q}_s\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s} + \dot{G}_X \\
&= \frac{\partial X^{(1)}(L)}{\partial t} + \dot{q}_s\frac{\partial X^{(1)}(L)}{\partial q_s} \\
&\quad - \dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial X^{(1)}(L)}{\partial \dot{q}_s}\right] + \dot{G}_X \\
&= \frac{\partial X^{(1)}(L)}{\partial t} - \dot{q}_s E_s\left[X^{(1)}(L)\right] + \dot{G}_X \\
&= \frac{\partial X^{(1)}(L)}{\partial t} - \dot{q}_s N_s\left[X^{(1)}(L)\right] + \dot{G}_X \\
&= \left\{X^{(1)}(Q_s + \Lambda_s) - N_s\left[X^{(1)}(L)\right]\right\}\dot{q}_s \\
&= 0,
\end{aligned}$$

where

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}$$

are called the Euler arithmetic operators.

In the same way, noting Eq. (29), it can also be proved when $f_\beta > 0$.

## 4. An illustrative example

The following example is used only to illustrate the application of the results in this paper.

A particle of unit mass moving in a plane perpendicular to the Earth's surface, the Lagrange function, generalized forces and constraint equation in a holonomic system with unilateral constraints are expressed as follows:

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - gq_2, \tag{31}$$

$$Q_1 = \ Q_2 = 0, \tag{32}$$

$$f = q_2 - q_1 \geq 0. \tag{33}$$

Now we come to study Mei symmetry and the new type of conserved quantity deduced from the Mei symmetry of the system.

Firstly, we study the Mei symmetry for the Nielsen equation in a holonomic system with unilateral constraints. When $f_\beta = 0$, substituting Eqs. (31)–(33) into Eq. (3), we obtain

$$\ddot{q}_1 = -\lambda, \ \ \ddot{q}_2 + g = \lambda,$$
$$f = q_2 - q_1 = 0. \tag{34}$$

So from Eq. (34), we have

$$\lambda = \frac{1}{2}g. \tag{35}$$

From Eqs. (34) and (35), differential equations of motion for the system, when $f_\beta = 0$, are written as follows:

$$A_1 = \ddot{q}_1 = -\frac{1}{2}g, \ A_2 = \ddot{q}_2 = -\frac{1}{2}g. \tag{36}$$

From Eqs. (5), (31), and (32), differential equations of motion for the system, when $f_\beta > 0$, are written as follows:

$$B_1 = \ddot{q}_1 = 0, \ B_2 = \ddot{q}_2 = -g. \tag{37}$$

From Eqs. (7) and (35), the constraint reaction forces can be written as follows:

$$\Lambda_1 = -\frac{1}{2}g, \ \Lambda_2 = \frac{1}{2}g. \tag{38}$$

Calculating Eqs. (11) and (12), we have

$$X^{(1)}(L) = -\xi_2 g + (\dot{\xi}_1 - \dot{\xi}_0\dot{q}_1)\dot{q}_1 + (\dot{\xi}_2 - \dot{\xi}_0\dot{q}_2)\dot{q}_2, \tag{39}$$

$$X^{(0)}(f) = \xi_2 - \xi_1. \tag{40}$$

Taking the infinitesimal generators

$$\xi_0 = 0, \ \xi_1 = \xi_2 = \ln(\dot{q}_1 + \dot{q}_2 + gt), \tag{41}$$

we have

$$X^{(1)}(L) = -g\ln(\dot{q}_1 + \dot{q}_2 + gt), \tag{42}$$

$$N_1[X^{(1)}(L)] = N_2[X^{(1)}(L)] = 0, \tag{43}$$

$$X^{(1)}(\Lambda_1) = X^{(1)}(\Lambda_2) = 0, \tag{44}$$

$$X^{(1)}(Q_1) = X^{(1)}(Q_2) = 0, \tag{45}$$

$$X^{(0)}(f) = 0. \tag{46}$$

After performing infinitesimal transformations (10), infinitesimal generators (41) satisfy Eqs. (24)–(26). Therefore, this illustrative example satisfies Criterion 3, which is the Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints.

Finally, we investigate the type of new conserved quantity deduced from the Mei symmetry of the system. Noting Eq. (36), substituting the infinitesimal generators in expression (41) into structural equations (28) and (29), we obtain

$$G_X = \frac{g^2 t}{\dot{q}_1 + \dot{q}_2 + gt}. \tag{47}$$

From Eq. (30), the type of new conserved quantities of Mei symmetry for the Nielsen equation in a holonomic system with unilateral constraints is expressed as follows:

$$I_X = g[1 - \ln(\dot{q}_1 + \dot{q}_2 + gt)] = \text{const}. \tag{48}$$

## 5. Conclusion

In this paper we investigated the structural equation and the type of new conserved quantity of Mei symmetry for Nielsen equations in a holonomic system with unilateral constraints in a brief way. The result obtained in this paper can be returned to the scenario of Mei symmetry for the Nielsen equation in a nonholonomic system with unilateral constraints. Therefore, the conclusion obtained in this paper is more universal.

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