Robust stabilization of state delayed discrete-time Takagi–Sugeno fuzzy systems with input saturation via an anti-windup fuzzy design*

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We consider the robust stabilization problem for discrete-time Takagi–Sugeno (T–S) fuzzy systems with time-varied delays subjected to input saturation. We design static and dynamic anti-windup fuzzy controllers to ensure the convergence of all admissible initial states within the domain of attraction. Based on the project lemma and classical sector conditions, the conditions for the existence of solutions to this problem are obtained and expressed in terms of a set of linear matrix inequalities. Finally, a numerical example is provided to illustrate the effectiveness of the proposed design approach.

Keywords: anti-windup controller, Takagi–Sugeno fuzzy model, time delay, input saturation

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1. Introduction

The Takagi–Sugeno (T–S) fuzzy system is one of the most popular fuzzy systems in model-based fuzzy control.[1−5] T–S fuzzy models are nonlinear systems described by a set of if-then rules. It has been shown that T–S fuzzy models can approximate any smooth nonlinear function to any specified accuracy within any compact set. Thus, it is expected that T–S fuzzy systems can be used to represent a large class of nonlinear systems. Therefore, many stability and control issues related to T–S fuzzy systems have been studied in the past two decades, see, e.g., Ref. [6], and the references therein.

On the other hand, time delays are frequently encountered in many practical engineering systems, such as chemical processes and long transmission lines in pneumatic systems. It has been shown that the presence of time delay in a dynamical system is often a primary source of instability and performance degradation. Therefore, much research related to the time-delay system has been carried out.[7] The studies on the analysis and the synthesis of control systems with input saturation using the anti-windup compensation have followed two main approaches in general. The first one is the static anti-windup schemes. The second approach is the dynamic anti-windup schemes. However, compared to the static anti-windup schemes, the dynamical anti-windup compensators offer more degrees of freedom in the synthesis procedure.[8] This fact in general leads to less conservative solutions for the performance of the closed-loop system. In Refs. [9] and [10], the static anti-windup compensators for time-delay systems were addressed, while the dynamic anti-windup compensation for time-delay systems was considered in Refs. [11] and [12].

Recently, the controller design of T-S fuzzy systems with time delays subjected to input saturation has attracted lots of attention.[13,14] In particular, in a continuous-time T–S fuzzy system with time-varying delays, the fuzzy control was developed in Ref. [14] via the static anti-windup approach. Nevertheless, stability analysis and the synthesis of discrete-time is uncertain in T–S fuzzy systems with time-varying delays subjected to input saturation via static and dynamic anti-windup methods, and they have seldom been considered which is the aim of the present study.

2. Main results

Consider a discrete-time T–S fuzzy system with time-varying delays and input saturation represented by the following if-then rules. Plant rule i: if $s_1(k)$ is

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$\mu_{i1},\ldots$, and $s_p(k)$ is μ_{ip}, then

$$\begin{aligned}\boldsymbol{x}(k+1) &= [\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(k)]\boldsymbol{x}(k) \\ &\quad + [\boldsymbol{A}_{\mathrm{d}i} + \Delta\boldsymbol{A}_{\mathrm{d}i}(k)]\boldsymbol{x}(k-h(k)) \\ &\quad + [\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(k)]\boldsymbol{\nu}(k), \quad (1)\\ \boldsymbol{y}(k) &= \boldsymbol{C}_i\boldsymbol{x}(k), \quad (2)\\ \boldsymbol{x}(k) &= \boldsymbol{\phi}(k), \quad \forall k \in [-\tau, 0], \quad i = 1, 2, \ldots, r, \quad (3)\end{aligned}$$

where μ_{ij} is the fuzzy set, r is the number of if-then rules, and $s_1(k)$, …, $s_p(k)$ are the premise variables. Throughout this paper, it is assumed that the premise variables do not depend on the control variables; $\boldsymbol{x}(k) \in \mathbf{R}^n$ represents the state; $\nu(k) \in \mathbf{R}^m$ is the control input; and $\boldsymbol{y}(k) \in \mathbf{R}^p$ is the measured output. The time-varying delay $h(k)$ satisfies $h_1 \le h(k) \le h_2$, where h_1 and h_2 are real positive constants presenting the lower and the upper bounds of the time-varying delay. The $\boldsymbol{A}_i$, $\boldsymbol{A}_{\mathrm{d}i}$, $\boldsymbol{B}_i$, and $\boldsymbol{C}_i$ are known real constant matrices; $\Delta\boldsymbol{A}_i(k)$, $\Delta\boldsymbol{A}_{\mathrm{d}i}(k)$, and $\Delta\boldsymbol{B}_i(k)$ are real-valued unknown matrices representing the time-varying parameter uncertainties, and are assumed to be of the form

$$\begin{aligned}&[\Delta\boldsymbol{A}_i(k)\ \Delta\boldsymbol{A}_{\mathrm{d}i}(k)\ \Delta\boldsymbol{B}_i(k)] \\ &= \boldsymbol{M}_i\boldsymbol{F}_i(k)[\boldsymbol{N}_{\mathrm{a}i}\ \boldsymbol{N}_{\mathrm{ad}i}\ \boldsymbol{N}_{\mathrm{b}i}] \quad i = 1, 2, \ldots, r, \quad (4)\end{aligned}$$

where $\boldsymbol{M}_i$, $\boldsymbol{N}_{\mathrm{a}i}$, $\boldsymbol{N}_{\mathrm{ad}i}$, and $\boldsymbol{N}_{\mathrm{b}i}$ are known real constant matrices, and $\boldsymbol{F}_i : \mathbf{N} \to \mathbf{R}^{l_1 \times l_2}$ is an unknown time-varying matrix function satisfying

$$\boldsymbol{F}_i(k)^{\mathrm{T}}\boldsymbol{F}_i(k) \le \boldsymbol{I}, \quad \forall k. \quad (5)$$

The parameter uncertainties $\Delta\boldsymbol{A}_i(k)$, $\Delta\boldsymbol{A}_{\mathrm{d}i}(k)$, and $\Delta\boldsymbol{B}_i(k)$ are considered to be admissible if both relations (4) and (5) hold.

The constrained input $\boldsymbol{\nu}(k)$ is denoted as $\boldsymbol{\nu}(k) = \boldsymbol{\sigma}(\boldsymbol{u}(k))$, where $\boldsymbol{u}(k)$ is the designed controller, and $\boldsymbol{\sigma} : \mathbf{R}^m \to \mathbf{R}^m$ represents a saturation function of the form

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{u}) &= [\sigma(u_1)\ \sigma(u_2)\ \cdots\ \sigma(u_m)]^{\mathrm{T}}, \\ \sigma(u_i) &= \mathrm{sign}(u_i)\min(u_0, \mid u_i \mid).\end{aligned}$$

Then, the final output of the fuzzy system is

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(s(k))\{[\boldsymbol{A}_i + \Delta\boldsymbol{A}_i(k)]\boldsymbol{x}(k) \\ &\quad + [\boldsymbol{A}_{\mathrm{d}i} + \Delta\boldsymbol{A}_{\mathrm{d}i}(k)]\boldsymbol{x}(t-h(k)) \\ &\quad + [\boldsymbol{B}_i + \Delta\boldsymbol{B}_i(k)]\boldsymbol{\nu}(k)\}, \quad (6)\\ \boldsymbol{y}(k) &= \sum_{i=1}^{r} h_i(s(k))\boldsymbol{C}_i\boldsymbol{x}(k), \quad (7)\end{aligned}$$

where for all k,

$$\begin{aligned}&h_i(s(k)) \ge 0, \quad i = 1, \cdots, r, \\ &\sum_{i=1}^{r} h_i(s(k)) = 1, \quad \forall k.\end{aligned}$$

2.1. Static anti-windup design

According to the parallel distributed compensation (PDC), the dynamic output feedback fuzzy controller, which consists of the anti-windup terms, is given by

$$\begin{aligned}\boldsymbol{x}_{\mathrm{c}}(k+1) &= \sum_{i=1}^{r} h_i(s(k))[\boldsymbol{A}_{\mathrm{c}i}x_{\mathrm{c}}(k) + \boldsymbol{B}_{\mathrm{c}i}y(k) \\ &\quad + \boldsymbol{E}_{\mathrm{c}i}(\boldsymbol{\sigma}(u) - \boldsymbol{u}(k))], \quad (8)\\ \boldsymbol{u}(k) &= \sum_{i=1}^{r} h_i(s(k))[\boldsymbol{C}_{\mathrm{c}i}\boldsymbol{x}_{\mathrm{c}}(k) + \boldsymbol{D}_{\mathrm{c}i}\boldsymbol{y}(k)], \quad (9)\end{aligned}$$

where $\boldsymbol{x}_{\mathrm{c}}(k) \in \mathbf{R}^{n_{\mathrm{c}}}$, $\boldsymbol{y}(k) \in \mathbf{R}^p$, and $\boldsymbol{u}(k) \in \mathbf{R}^m$ are the controller state, the input, and the output vectors, respectively, $\boldsymbol{A}_{\mathrm{c}i}$, $\boldsymbol{B}_{\mathrm{c}i}$, $\boldsymbol{C}_{\mathrm{c}i}$, and $\boldsymbol{D}_{\mathrm{c}i}$ are real constant matrices of appropriate dimensions, and $\boldsymbol{E}_{\mathrm{c}i}$ is the anti-windup gain.

Define an extended state vector

$$\boldsymbol{\xi}(k) = \begin{bmatrix}\boldsymbol{x}(k) \\ \boldsymbol{x}_{\mathrm{c}}(k)\end{bmatrix} \in \mathbf{R}^{n+n_{\mathrm{c}}}.$$

From Eqs. (6)–(9), the closed-loop dynamics can be written as

$$\begin{aligned}\boldsymbol{\xi}(k+1) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t)) \\ &\quad \times \{[\boldsymbol{A}_{ijl} + \Delta\boldsymbol{A}_{ijl}(k)]\boldsymbol{\xi}(k) \\ &\quad + [\bar{\boldsymbol{A}}_{\mathrm{d}i} + \Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k)]\boldsymbol{\xi}(k-h(k)) \\ &\quad - [\bar{\boldsymbol{B}}_i + \Delta\bar{\boldsymbol{B}}_i(k)]\boldsymbol{\Psi}(\boldsymbol{u}(k))\}, \quad (10)\end{aligned}$$

where

$$\boldsymbol{A}_{ijl} = \begin{bmatrix}\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{D}_{\mathrm{c}j}\boldsymbol{C}_l & \boldsymbol{B}_i\boldsymbol{C}_{\mathrm{c}j} \\ \boldsymbol{B}_{\mathrm{c}i}\boldsymbol{C}_j & \boldsymbol{A}_{\mathrm{c}i}\end{bmatrix},$$

$$\Delta\boldsymbol{A}_{ijl}(k) = \begin{bmatrix}\Delta\boldsymbol{A}_i(k) + \Delta\boldsymbol{B}_i(k)\boldsymbol{D}_{\mathrm{c}j}\boldsymbol{C}_l & \Delta\boldsymbol{B}_i(k)\boldsymbol{C}_{\mathrm{c}j} \\ 0 & 0\end{bmatrix},$$

$$\bar{\boldsymbol{A}}_{\mathrm{d}i} = \begin{bmatrix}\boldsymbol{A}_{\mathrm{d}i} & 0 \\ 0 & 0\end{bmatrix}, \quad \Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k) = \begin{bmatrix}\Delta\boldsymbol{A}_{\mathrm{d}i}(k) & 0 \\ 0 & 0\end{bmatrix},$$

$$\bar{\boldsymbol{B}}_i = \hat{\boldsymbol{B}}_i + \boldsymbol{R}\boldsymbol{E}_{\mathrm{c}i}, \quad \Delta\bar{\boldsymbol{B}}_i(k) = \begin{bmatrix}\Delta\boldsymbol{B}_i(k) \\ 0\end{bmatrix},$$

$$\hat{\boldsymbol{B}}_i = \begin{bmatrix}\boldsymbol{B}_i \\ 0\end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix}0 \\ \boldsymbol{I}_{nc}\end{bmatrix},$$

$\boldsymbol{\psi}(\boldsymbol{u}(t)) \triangleq \boldsymbol{u}(t) - \mathrm{sat}(\boldsymbol{u}(t)) \in \mathbf{R}^m$.

The controller output can be expressed as

$$\boldsymbol{u}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(s(t))h_j(s(t))\boldsymbol{K}_{ij}\boldsymbol{\xi}(t), \qquad (11)$$

where

$$\boldsymbol{K}_{ij} = \begin{bmatrix} \boldsymbol{D}_{\mathrm{c}i}\boldsymbol{C}_j & \boldsymbol{C}_{\mathrm{c}i} \end{bmatrix}.$$

We first give the following results, which will be used in the proof of our main results.

Lemma 1[15] Consider matrices $\boldsymbol{K}_{ij}$, $i,j = 1,\ldots,r$ in Eq. (11). Given the corresponding matrices $\boldsymbol{L}_i \in \mathbf{R}^{m\times 2(n+n_{\mathrm{c}})}$, if vector $\boldsymbol{\xi}(k)$ belongs to the polyhedral set $\boldsymbol{D}(\boldsymbol{u}_0)$ is defined as

$$\begin{aligned} &\boldsymbol{D}(\boldsymbol{u}_0) \\ &\triangleq \{\boldsymbol{\xi}(k) \in \mathbf{R}^{2(n+n_{\mathrm{c}})}; \\ &\quad -\boldsymbol{u}_{0(z)} \le (\boldsymbol{K}_{ij(z)} - \boldsymbol{L}_{i(z)})\boldsymbol{\xi}(k) \le \boldsymbol{u}_{0(z)}, \\ &\quad \boldsymbol{u}_{0(z)} > 0, \quad i,j = 1,\ldots,r, \quad z = 1,\ldots,m\}, \end{aligned} \qquad (12)$$

then the following relationship holds:

$$\boldsymbol{\psi}(\boldsymbol{u}(k))^{\mathrm{T}}\boldsymbol{\Gamma}\left(\boldsymbol{\psi}(\boldsymbol{u}(k)) - \sum_{i=1}^{r} h_i(s(k))\boldsymbol{L}_i\boldsymbol{\xi}(k)\right) \le 0, \qquad (13)$$

for any diagonal positive definite matrix $\boldsymbol{\Gamma} \in \mathbf{R}^{m\times m}$.

Lemma 2[16] For any matrices $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ with appropriate dimensions, and $\mathcal{Y} > 0$, we have

$$\mathcal{X}^{\mathrm{T}}\mathcal{Z} + \mathcal{Z}^{\mathrm{T}}\mathcal{X} \le \mathcal{X}^{\mathrm{T}}\mathcal{Y}\mathcal{X} + \mathcal{Z}^{\mathrm{T}}\mathcal{Y}^{-1}\mathcal{Z}.$$

Theorem 1 Consider the discrete-time fuzzy system (10). Given h_2 and h_1, if there exist symmetric positive definite matrices $\boldsymbol{W}$, $\boldsymbol{Q}$, $\boldsymbol{S}$ and matrices $\boldsymbol{Y}$, $\boldsymbol{Z}_i$ $(i = 1,\ldots,r)$, such that the following linear matrix inequalities (LMIs) hold:

$$\boldsymbol{\Omega}_{1iil} < 0, \ 1 \le i \le r, \qquad (14)$$

$$\boldsymbol{\Omega}_{1ijl} + \boldsymbol{\Omega}_{1jil} < 0, \ 1 \le i < j \le r, \qquad (15)$$

$$\begin{bmatrix} \boldsymbol{W} & * \\ \boldsymbol{K}_{ij(z)}\boldsymbol{W} - \boldsymbol{Y}_{(z)} & \boldsymbol{u}_{0(z)}^2 \end{bmatrix} \ge 0, \quad i,j = 1,\ldots,r; \ z = 1,\ldots,m, \qquad (16)$$

where

$$\boldsymbol{\Omega}_{1ijl} = \begin{bmatrix} -\boldsymbol{W} & 0 & \boldsymbol{Y}^{\mathrm{T}} & -\boldsymbol{W}\boldsymbol{A}_{ijl}^{\mathrm{T}} & 0 & \boldsymbol{W}\boldsymbol{N}_{\mathrm{a}ijl}^{\mathrm{T}} & \boldsymbol{W} \\ * & -\boldsymbol{Q} & 0 & -\bar{\boldsymbol{A}}_{\mathrm{d}i}^{\mathrm{T}} & 0 & \bar{\boldsymbol{N}}_{\mathrm{ad}i}^{\mathrm{T}} & 0 \\ * & * & -2\boldsymbol{S} & \boldsymbol{S}\hat{\boldsymbol{B}}_i^{\mathrm{T}} + \boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}} & 0 & -\boldsymbol{S}\boldsymbol{N}_{\mathrm{b}i}^{\mathrm{T}} & 0 \\ * & * & * & -\boldsymbol{W} & -\bar{\boldsymbol{M}}_i & 0 & 0 \\ * & * & * & * & -\boldsymbol{I} & 0 & 0 \\ * & * & * & * & * & -\boldsymbol{I} & 0 \\ * & * & * & * & * & * & -h_1 \end{bmatrix} < 0,$$

$$\bar{\boldsymbol{M}}_i = \begin{bmatrix} \boldsymbol{M}_i \\ 0 \end{bmatrix}, \ \boldsymbol{N}_{\mathrm{a}ijl} = \begin{bmatrix} \boldsymbol{N}_{\mathrm{a}i} + \boldsymbol{N}_{\mathrm{b}i}\boldsymbol{D}_{\mathrm{cj}}\boldsymbol{C}_l & \boldsymbol{N}_{\mathrm{b}i}\boldsymbol{C}_{\mathrm{cj}} \end{bmatrix},$$

$$\bar{\boldsymbol{N}}_{\mathrm{ad}i} = \begin{bmatrix} \boldsymbol{N}_{\mathrm{ad}i} & 0 \end{bmatrix}, \ h_1 = \bar{h}^{-1}, \ \bar{h} = h_2 - h_1 + 1,$$

then the closed-loop system with the initial states located in the ellipse $\varepsilon(\boldsymbol{P},1)$, with $\boldsymbol{P} = \boldsymbol{W}^{-1}$, is guaranteed to be asymptotically stable by the static anti-windup fuzzy controller (10a) and (10b). Furthermore, the gain matrix is $\boldsymbol{E}_{\mathrm{c}i} = \boldsymbol{Z}_i\boldsymbol{S}^{-1}$.

Proof The satisfaction of relationship (16) implies that the set $\varepsilon(\boldsymbol{P}) = \{\xi(t) \in \mathbf{R}^{n+n_{\mathrm{c}}}; \boldsymbol{\xi}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\xi}(t) \le 1\}$ is included in the polyhedral set $\boldsymbol{D}$ defined as in Eq. (12) with $\boldsymbol{G} = \boldsymbol{Y}\boldsymbol{P}$.

To establish the robust stability of system (10), we define the following Lyapunov function candidate:

$$V(k) = \boldsymbol{\xi}(k)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\xi}(k) + \sum_{i=k-h(k)}^{k-1} \boldsymbol{\xi}(k)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\xi}(k) + \sum_{j=-h_2+2}^{-h_1+1}\sum_{l=k+j-1}^{k-1} \boldsymbol{\xi}(l)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\xi}(l),$$

then

$$\begin{aligned} &\Delta V(k) \\ &= \boldsymbol{\xi}(k+1)^{\mathrm{T}}\boldsymbol{P}\dot{\boldsymbol{\xi}}(k+1) \\ &\quad - \boldsymbol{\xi}(k)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\xi}(k) + \boldsymbol{\xi}(k)^{\mathrm{T}}\bar{h}\boldsymbol{\xi}(k) \\ &\quad - \boldsymbol{\xi}(k-h(k))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\xi}(k-h(k)) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(s(t))h_j(s(t))h_l(s(t)) \\ &\quad \times \{\boldsymbol{\xi}(k)^{\mathrm{T}}[(\boldsymbol{A}_{ijl} + \Delta\boldsymbol{A}_{ijl}(k))^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{A}_{ijl} + \Delta\boldsymbol{A}_{ijl}(k)) \end{aligned}$$

$$
\begin{aligned}
&-\boldsymbol{P}+\bar{h}]\boldsymbol{\xi}(k)-\boldsymbol{\xi}(k-h(k))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\xi}(k-h(k))\\
&+\boldsymbol{\xi}(k-h(k))^{\mathrm{T}}(\bar{\boldsymbol{A}}_{\mathrm{d}i}+\Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k))^{\mathrm{T}}\\
&\times\boldsymbol{P}(\bar{\boldsymbol{A}}_{\mathrm{d}i}+\Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k))\boldsymbol{\xi}(k-h(k))\\
&+\boldsymbol{\Psi}(\boldsymbol{u}(k))^{\mathrm{T}}(\bar{\boldsymbol{B}}_i+\Delta\bar{\boldsymbol{B}}_i(k))^{\mathrm{T}}\\
&\times\boldsymbol{P}(\bar{\boldsymbol{B}}_i+\Delta\bar{\boldsymbol{B}}_i(k))\boldsymbol{\Psi}(\boldsymbol{u}(k))\\
&+2\boldsymbol{\xi}(k)(\boldsymbol{A}_{ijl}+\Delta\boldsymbol{A}_{ijl}(k))^{\mathrm{T}}\\
&\times\boldsymbol{P}(\bar{\boldsymbol{A}}_{\mathrm{d}i}+\Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k))\boldsymbol{\xi}(k-h(k))\\
&-2\boldsymbol{\xi}(k)(\boldsymbol{A}_{ijl}+\Delta\boldsymbol{A}_{ijl}(k))^{\mathrm{T}}\\
&\times\boldsymbol{P}(\bar{\boldsymbol{B}}_i+\Delta\bar{\boldsymbol{B}}_i(k))\boldsymbol{\Psi}(\boldsymbol{u}(k))\\
&-2\boldsymbol{\xi}(k-h(k))^{\mathrm{T}}(\bar{\boldsymbol{A}}_{\mathrm{d}i}+\Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k))^{\mathrm{T}}\\
&\times\boldsymbol{P}(\bar{\boldsymbol{B}}_i+\Delta\bar{\boldsymbol{B}}_i(k))\boldsymbol{\Psi}(\boldsymbol{u}(k))\}.
\end{aligned}
$$

Based on Lemma 1,

$$\Delta V(k)<\Delta V(k)-2\boldsymbol{\Psi}(\boldsymbol{u}(k))^{\mathrm{T}}\boldsymbol{T}(\boldsymbol{\Psi}(\boldsymbol{u}(k))-\boldsymbol{G}\boldsymbol{\xi}(k)).$$

Then $\Delta V(k)<0$, if the following matrix inequality is satisfied:

$$\boldsymbol{\Omega}_{2iil}<0,\ 1\le i\le r, \tag{17}$$

$$\boldsymbol{\Omega}_{2ijl}+\boldsymbol{\Omega}_{2jil}<0,\ 1\le i<j\le r, \tag{18}$$

where

$$\boldsymbol{\Omega}_{2ijl}=\begin{bmatrix}-\boldsymbol{P}+\bar{h} & 0 & \boldsymbol{G}^{\mathrm{T}}\boldsymbol{T} & -\boldsymbol{A}_{ijl}^{\mathrm{T}}-\Delta\boldsymbol{A}_{ijl}(k)^{\mathrm{T}}\\ * & -\boldsymbol{Q} & 0 & -\bar{\boldsymbol{A}}_{\mathrm{d}i}^{\mathrm{T}}-\Delta\bar{\boldsymbol{A}}_{\mathrm{d}i}(k)^{\mathrm{T}}\\ * & * & -2\boldsymbol{T} & \bar{\boldsymbol{B}}_i^{\mathrm{T}}+\Delta\bar{\boldsymbol{B}}_i(k)^{\mathrm{T}}\\ * & * & * & -\boldsymbol{P}^{-1}\end{bmatrix}.$$

Now using Lemma 2, and pre- and post-multiplying the obtained inequality by $\mathrm{diag}(\boldsymbol{W},\boldsymbol{I},\boldsymbol{S},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I})$ and its transpose, we get the LMI (26) by Schur's complement.

Using a similar method, we can obtain LMI (15) from LMI (18).

Hence, if LMIs (14)–(16) are satisfied, it follows that $\dot{V}(t)<0$ is verified and $\varepsilon(\boldsymbol{P},1)$ is the domain of attraction. All initial states located in the ellipse $\varepsilon(\boldsymbol{P},1)$ converge to the origin as time goes by. Consequently, the closed-loop system is asymptotically stable. This completes the proof.

2.2. Dynamic anti-windup design

It is supposed that controllers (19) and (20) are computed to stabilize the system (6) and (7) without taking into account the saturation of the system input

$$\boldsymbol{x}_{\mathrm{c}}(k+1)=\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{A}_{\mathrm{c}i}x_{\mathrm{c}}(k)+\boldsymbol{B}_{\mathrm{c}i}\boldsymbol{y}(k)], \tag{19}$$

$$\boldsymbol{u}(k)=\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{C}_{\mathrm{c}i}\boldsymbol{x}_{\mathrm{c}}(k)+\boldsymbol{D}_{\mathrm{c}i}\boldsymbol{y}(k)]. \tag{20}$$

In order to cope with such a disregard, according to the PDC, an anti-windup compensator is then considered to minimize the effect of the saturation of the closed-loop performance

$$\begin{aligned}\boldsymbol{x}_{\mathrm{a}}(k+1)=&\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{A}_{\mathrm{a}i}\boldsymbol{x}_{\mathrm{a}}(k)\\&+\boldsymbol{B}_{\mathrm{a}i}\boldsymbol{\psi}(\boldsymbol{u}(k))],\end{aligned} \tag{21}$$

$$\boldsymbol{u}_{\mathrm{a}}(k)=\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{C}_{\mathrm{a}i}\boldsymbol{x}_{\mathrm{a}}(k)+\boldsymbol{D}_{\mathrm{a}i}\boldsymbol{\psi}(\boldsymbol{u}(k))], \tag{22}$$

where $\boldsymbol{x}_{\mathrm{a}}(k)\in\mathbf{R}^{n+n_{\mathrm{c}}}$ and $\boldsymbol{\psi}(\boldsymbol{u}(k))\triangleq\mathrm{sat}(\boldsymbol{u}(k))-\boldsymbol{u}(k)\in\mathbf{R}^{m}$ are the anti-windup compensator state and input vectors, respectively, $\boldsymbol{A}_{\mathrm{a}i}$, $\boldsymbol{B}_{\mathrm{a}i}$, $\boldsymbol{C}_{\mathrm{a}i}$, and $\boldsymbol{D}_{\mathrm{a}i}$ are real constant matrices of appropriate dimensions, and vector $\boldsymbol{u}_{\mathrm{a}}(k)\in\mathbf{R}^{n_{\mathrm{c}}}$ is the compensating signal of the controller dynamics. Hence, the compensated controllers (19) and (20) become

$$\begin{aligned}\boldsymbol{x}_{\mathrm{c}}(k+1)=&\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{A}_{\mathrm{c}i}\boldsymbol{x}_{\mathrm{c}}(k)\\&+\boldsymbol{B}_{\mathrm{c}i}\boldsymbol{y}(k)+\boldsymbol{u}_{\mathrm{a}}(k)],\end{aligned} \tag{23}$$

$$\boldsymbol{u}(k)=\sum_{i=1}^{r}h_i(s(k))[\boldsymbol{C}_{\mathrm{c}i}\boldsymbol{x}_{\mathrm{c}}(k)+\boldsymbol{D}_{\mathrm{c}i}\boldsymbol{y}(k)]. \tag{24}$$

Define vector $\boldsymbol{\eta}(k)=\begin{bmatrix}\boldsymbol{x}(k)^{\mathrm{T}} & \boldsymbol{x}_{\mathrm{c}}(k)^{\mathrm{T}} & \boldsymbol{x}_{\mathrm{a}}(k)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$, then the closed-loop system becomes

$$\begin{aligned}\boldsymbol{\eta}(k+1)=&\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}h_i(s(t))h_j(s(t))h_l(s(t))\\&\times\{[\mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V}]\boldsymbol{\eta}(k)\\&+[\mathcal{A}_{\mathrm{d}i}+\Delta\mathcal{A}_{\mathrm{d}i}(k)]\eta(k-h(k))\\&+[\mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l}]\boldsymbol{\psi}(\boldsymbol{u}(k))\},\end{aligned} \tag{25}$$

where

$$\mathcal{A}_{ijl}=\begin{bmatrix}\hat{\boldsymbol{A}}_{ijl} & 0\\ 0 & 0\end{bmatrix},$$

$$\Delta\mathcal{A}_{ijl}(k)=\begin{bmatrix}\Delta\boldsymbol{A}_{ijl}(k) & 0\\ 0 & 0\end{bmatrix},$$

$$\mathcal{B}_1=\begin{bmatrix}0 & \boldsymbol{B}_1\\ \boldsymbol{I}_{n_{\mathrm{a}}} & 0\end{bmatrix},\quad \mathcal{K}_{1l}=\begin{bmatrix}\boldsymbol{A}_{\mathrm{a}l}\\ \boldsymbol{C}_{\mathrm{a}l}\end{bmatrix},$$

$$\mathcal{V}=\begin{bmatrix}0 & 0 & \boldsymbol{I}_{n_{\mathrm{a}}}\end{bmatrix},\quad \mathcal{A}_{\mathrm{d}i}=\begin{bmatrix}\hat{\boldsymbol{A}}_{\mathrm{d}i} & 0\\ 0 & 0\end{bmatrix},$$

$$\Delta\mathcal{A}_{\mathrm{d}i}(t)=\begin{bmatrix}\Delta\hat{\boldsymbol{A}}_{\mathrm{d}i}(k) & 0\\ 0 & 0\end{bmatrix},$$

$$\mathcal{B}_i = \begin{bmatrix} \hat{\mathcal{B}}_i \\ 0 \end{bmatrix}, \quad \Delta\mathcal{B}_i(k) = \begin{bmatrix} \Delta\hat{\mathcal{B}}_i(k) \\ 0 \end{bmatrix},$$

and

$$\hat{\boldsymbol{A}}_{ijl} = \begin{bmatrix} \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{D}_{\rm cj}\boldsymbol{C}_l & \boldsymbol{B}_i\boldsymbol{C}_{\rm cj} \\ \boldsymbol{B}_{\rm ci}\boldsymbol{C}_j & \boldsymbol{A}_{\rm ci} \end{bmatrix},$$

$$\Delta\hat{\boldsymbol{A}}_{ijl}(k) = \begin{bmatrix} \Delta\boldsymbol{A}_i(k) + \Delta\boldsymbol{B}_i(k)\boldsymbol{D}_{\rm cj}\boldsymbol{C}_l & \Delta\boldsymbol{B}_i(k)\boldsymbol{C}_{\rm cj} \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 0 \\ \boldsymbol{I}_{n_{\rm c}} \end{bmatrix}, \quad \hat{\boldsymbol{A}}_{{\rm d}i} = \begin{bmatrix} \boldsymbol{A}_{{\rm d}i} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Delta\hat{\boldsymbol{A}}_{{\rm d}i}(k) = \begin{bmatrix} \Delta\boldsymbol{A}_{{\rm d}i}(k) & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{\mathcal{B}}_i = \begin{bmatrix} -\boldsymbol{B}_i \\ 0 \end{bmatrix}, \quad \Delta\hat{\mathcal{B}}_i(k) = \begin{bmatrix} -\Delta\boldsymbol{B}_i(k) \\ 0 \end{bmatrix}.$$

The controller output can be expressed as

$$\boldsymbol{u}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(s(t))h_j(s(t))\hat{\boldsymbol{K}}_{ij}\boldsymbol{\xi}(t),$$

where

$$\hat{\boldsymbol{K}}_{ij} = \begin{bmatrix} \boldsymbol{D}_{\rm ci}\boldsymbol{C}_j & \boldsymbol{C}_{\rm ci} & 0 \end{bmatrix}.$$

Theorem 2 Consider the time-delay uncertain fuzzy system (11). If there exist symmetric positive definite matrices $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{S}_1$, $\boldsymbol{H}$ and matrices $\boldsymbol{F}$, $\boldsymbol{G}_{11}$, $\boldsymbol{A}_{{\rm aw}l}$, $\boldsymbol{B}_{{\rm aw}l}$, $\boldsymbol{C}_{{\rm aw}l}$, $\boldsymbol{D}_{{\rm aw}l}$ $(l = 1, \ldots, r)$ such that the following LMIs hold:

$$\Upsilon_{iil} < 0, \ 1 \le i \le r, \tag{26}$$

$$\Upsilon_{ijl} + \Upsilon_{jil} < 0, \ 1 \le i < j \le r, \tag{27}$$

$$\begin{bmatrix} \boldsymbol{X} & * & * \\ \boldsymbol{I}_{n+n_{\rm c}} & \boldsymbol{Y} & * \\ \boldsymbol{K}_{ij(z)}\boldsymbol{X} - \boldsymbol{F}_{(z)} & \boldsymbol{K}_{ij(z)} - \boldsymbol{G}_{11(z)} & \boldsymbol{u}_{0(z)}^2 \end{bmatrix} > 0,$$
$$i, j = 1, \ldots, r; \ z = 1, \ldots, m, \tag{28}$$

where

$$\Upsilon_{ijl} = \begin{bmatrix} \boldsymbol{\Delta}_1 & * & * & * \\ \boldsymbol{\Delta}_{2ijl} & -\boldsymbol{\Delta}_{3i} & * & * \\ \boldsymbol{\Delta}_4 & 0 & -h_1 & * \\ \boldsymbol{\Delta}_{6ijl} & \boldsymbol{\Delta}_7 & 0 & -\boldsymbol{I} \end{bmatrix},$$

$$\boldsymbol{\Delta}_1 = \begin{bmatrix} -\hat{\boldsymbol{R}} & * & * \\ 0 & -\boldsymbol{H} & * \\ \hat{\boldsymbol{G}}_1 & 0 & -2\boldsymbol{S}_1 \end{bmatrix},$$

$$\boldsymbol{\Delta}_{2ijl} = \begin{bmatrix} \hat{\boldsymbol{\Sigma}}_{1ijl} & \hat{\boldsymbol{R}}_{\boldsymbol{A}{\rm d}i} & \hat{\boldsymbol{\Sigma}}_{2il} \\ 0 & 0 & 0 \\ \tilde{\boldsymbol{N}}_{{\rm a}ijl} & \bar{\boldsymbol{N}}_{{\rm ad}i} & -\boldsymbol{N}_{{\rm b}i}\boldsymbol{S}_1 \end{bmatrix},$$

$$\boldsymbol{\Delta}_{3i} = \begin{bmatrix} \hat{\boldsymbol{R}} & * & * \\ -\hat{\boldsymbol{M}}_i^{\rm T} & \boldsymbol{I} & * \\ 0 & 0 & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{\Delta}_4 = \begin{bmatrix} \boldsymbol{\Pi} & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\Delta}_{6ijl} = \begin{bmatrix} 0 & 0 & 0 \\ \boldsymbol{A}_{Xijl} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{\mathcal{B}}_i S_1 \end{bmatrix}, \quad \boldsymbol{\Delta}_7 = \begin{bmatrix} \hat{\boldsymbol{Y}} & 0 & 0 \\ 0 & 0 & 0 \\ \hat{\boldsymbol{Y}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\hat{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{I}_{n+n_{\rm c}} \\ \boldsymbol{I}_{n+n_{\rm c}} & \boldsymbol{Y} \end{bmatrix}, \quad \hat{\boldsymbol{G}}_1 = \begin{bmatrix} \boldsymbol{F} & \boldsymbol{G}_{11} \end{bmatrix},$$

$$\hat{\boldsymbol{\Sigma}}_{1ijl} = \begin{bmatrix} \hat{\boldsymbol{A}}_{ijl}\boldsymbol{X} + \boldsymbol{B}_1\boldsymbol{C}_{{\rm aw}l} & \hat{\boldsymbol{A}}_{ijl} \\ \boldsymbol{A}_{{\rm aw}l} & \boldsymbol{Y}\hat{\boldsymbol{A}}_{ijl} \end{bmatrix},$$

$$\hat{\boldsymbol{M}}_i = \begin{bmatrix} \bar{\boldsymbol{M}}_i \\ 0 \end{bmatrix}, \quad \hat{\boldsymbol{R}}_{\boldsymbol{A}{\rm d}i} = \begin{bmatrix} \hat{\boldsymbol{A}}_{{\rm d}i} & 0 \\ Y\hat{\boldsymbol{A}}_{{\rm d}i} & 0 \end{bmatrix},$$

$$\hat{\boldsymbol{\Sigma}}_{2il} = \begin{bmatrix} \hat{\mathcal{B}}_i\boldsymbol{S}_1 + \boldsymbol{B}_1\boldsymbol{D}_{{\rm aw}l} \\ \boldsymbol{B}_{{\rm aw}l} \end{bmatrix},$$

$$\tilde{\boldsymbol{N}}_{{\rm a}ijl} = \begin{bmatrix} \boldsymbol{N}_{{\rm a}ijl}X & \boldsymbol{N}_{{\rm a}ijl} \end{bmatrix}, \quad \hat{\boldsymbol{Y}} = \begin{bmatrix} 0 & \boldsymbol{Y} \end{bmatrix},$$

$$\boldsymbol{A}_{Xijl} = \begin{bmatrix} \hat{\boldsymbol{A}}_{ijl}\boldsymbol{X} & 0 \end{bmatrix},$$

then the closed-loop system with the initial states located in the ellipse $\varepsilon(\boldsymbol{P}, 1)$ is guaranteed to be asymptotically stable by the anti-windup fuzzy controllers (23) and (24) with the following parameters:

$$\boldsymbol{C}_{\rm al} = \boldsymbol{C}_{{\rm aw}l}\boldsymbol{M}^{-{\rm T}}, \tag{29}$$

$$\boldsymbol{D}_{\rm al} = \boldsymbol{D}_{{\rm aw}l}\boldsymbol{S}_1^{-1}, \tag{30}$$

$$\boldsymbol{A}_{\rm al} = \boldsymbol{N}^{-1}(\boldsymbol{A}_{{\rm aw}l} - \boldsymbol{Y}\boldsymbol{B}_1\boldsymbol{C}_{{\rm aw}l})\boldsymbol{M}^{-{\rm T}}, \tag{31}$$

$$\boldsymbol{B}_{\rm al} = \boldsymbol{N}^{-1}(\boldsymbol{B}_{{\rm aw}l} - \boldsymbol{Y}\boldsymbol{B}_1\boldsymbol{D}_{{\rm aw}l})\boldsymbol{S}_1^{-1}. \tag{32}$$

The nonsingular matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ satisfy $\boldsymbol{N}\boldsymbol{M}^{\rm T} = \boldsymbol{I}_{n+n_{\rm c}} - \boldsymbol{Y}\boldsymbol{X}$, which guarantees the region $\varepsilon(\boldsymbol{R})$ with

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{Y} & \boldsymbol{N} \\ \boldsymbol{N}^{\rm T} & \bullet \end{bmatrix}, \quad \boldsymbol{R}^{-1} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{M} \\ \boldsymbol{M}^{\rm T} & \bullet \end{bmatrix}.$$

Proof The satisfaction of relationsip (28) implies that the set $\varepsilon(\boldsymbol{R}) = \{\boldsymbol{\xi}(t) \in \mathbf{R}^{n+n_{\rm c}}; \boldsymbol{\xi}(t)^{\rm T}\boldsymbol{R}\boldsymbol{\xi}(t) \le 1\}$ is included in the polyhedral set $\boldsymbol{D}$ as defined in Eq. (12).

To establish the robust stability of system (25), we define the following Lyapunov function candidate:

$$\begin{aligned}\boldsymbol{V}(\eta(k)) =\ & \boldsymbol{\eta}(k)^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\eta}(k)\\ &+\sum_{i=k-h(k)}^{k-1}\boldsymbol{\eta}(k)^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\eta}(k)\\ &+\sum_{j=-h_2+2}^{-h_1+1}\sum_{l=k+j-1}^{k-1}\boldsymbol{\eta}(l)^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\eta}(l).\end{aligned}$$

Then

$$\begin{aligned}&\Delta V(\boldsymbol{\eta}(k))\\ &=\boldsymbol{\eta}(k+1)^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\eta}(k+1)-\boldsymbol{\eta}(k)^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\eta}(k)\\ &\quad+\boldsymbol{\eta}(k)^{\mathrm{T}}\bar{h}\boldsymbol{\eta}(k)-\boldsymbol{\eta}(k-h(k))^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\eta}(k-h(k))\\ &=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}h_i(s(t))h_j(s(t))h_l(s(t))\\ &\quad\times\{\boldsymbol{\eta}(k)^{\mathrm{T}}[(\mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V})^{\mathrm{T}}\boldsymbol{R}\\ &\quad\times(\mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V})-\boldsymbol{R}+\bar{h}]\boldsymbol{\eta}(k)\\ &\quad-\boldsymbol{\eta}(k-h(k))^{\mathrm{T}}\boldsymbol{H}\boldsymbol{\eta}(k-h(k))\\ &\quad+\boldsymbol{\eta}(k-h(k))^{\mathrm{T}}(\mathcal{A}_{\mathrm{d}i}+\Delta\mathcal{A}_{\mathrm{d}i}(k))^{\mathrm{T}}\boldsymbol{R}\mathcal{A}_{\mathrm{d}i}\\ &\quad+\Delta\mathcal{A}_{\mathrm{d}i}(k))\boldsymbol{\eta}(k-h(k))\\ &\quad+\boldsymbol{\Psi}(\boldsymbol{u}(k))^{\mathrm{T}}(\mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l})^{\mathrm{T}}\\ &\quad\times\boldsymbol{R}(\mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l})\boldsymbol{\Psi}(\boldsymbol{u}(k))\\ &\quad+2\boldsymbol{\eta}(k)(\mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V})^{\mathrm{T}}\\ &\quad\times\boldsymbol{R}(\mathcal{A}_{\mathrm{d}i}+\Delta\mathcal{A}_{\mathrm{d}i}(k))\boldsymbol{\eta}(k-h(k))\\ &\quad-2\boldsymbol{\eta}(k)(\mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V})^{\mathrm{T}}\\ &\quad\times\boldsymbol{R}(\mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l})\boldsymbol{\Psi}(\boldsymbol{u}(k))\\ &\quad-2\boldsymbol{\eta}(k-h(k))^{\mathrm{T}}(\mathcal{A}_{\mathrm{d}i}+\Delta\mathcal{A}_{\mathrm{d}i}(k))^{\mathrm{T}}\\ &\quad\times\boldsymbol{R}(\mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l})\boldsymbol{\Psi}(\boldsymbol{u}(k))\}.\end{aligned}$$

Based on Lemma 1,

$$\Delta V(k)<\Delta V(k)-2\boldsymbol{\Psi}(\boldsymbol{u}(k))^{\mathrm{T}}T_1(\boldsymbol{\Psi}(\boldsymbol{u}(k))-\boldsymbol{G}_1\boldsymbol{\eta}(k)).$$

Then, $\Delta V(k)<0$, if the following matrix inequality is satisfied:

$$\begin{bmatrix}-\boldsymbol{R}+\bar{\boldsymbol{h}} & * & * & *\\ 0 & -\boldsymbol{H} & * & *\\ \boldsymbol{T}_1\boldsymbol{G}_1 & 0 & -2\boldsymbol{T}_1 & *\\ \mathcal{A}_{ijl}+\Delta\mathcal{A}_{ijl}(k)+\mathcal{B}_1\mathcal{K}_{1l}\mathcal{V} & \mathcal{A}_{\mathrm{d}i}+\Delta\mathcal{A}_{\mathrm{d}i}(k) & \mathcal{B}_i+\Delta\mathcal{B}_i(k)+\mathcal{B}_1\mathcal{K}_{2l} & -\boldsymbol{R}^{-1}\end{bmatrix}<0. \tag{33}$$

Noting

$$\begin{aligned}&[\Delta\mathcal{A}_{ijl}(k)\quad\Delta\mathcal{A}_{\mathrm{d}i}(k)\quad\Delta\mathcal{B}_i(k)]\\ &=\hat{\boldsymbol{M}}_i\boldsymbol{F}_i(k)[\hat{\boldsymbol{N}}_{\mathrm{a}ijl}\quad\hat{\boldsymbol{N}}_{\mathrm{ad}i}\quad-\boldsymbol{N}_{\mathrm{b}i}],\end{aligned}$$

where

$$\hat{\boldsymbol{M}}_i=\begin{bmatrix}\bar{\boldsymbol{M}}_i\\ 0\end{bmatrix},\ \hat{\boldsymbol{N}}_{\mathrm{a}ijl}=\begin{bmatrix}\boldsymbol{N}_{\mathrm{a}ijl}\ 0\end{bmatrix},\ \hat{\boldsymbol{N}}_{\mathrm{ad}i}=\begin{bmatrix}\bar{\boldsymbol{N}}_{\mathrm{ad}i}\ 0\end{bmatrix},$$

and using Lemma 2, we have a new inequality.

Define matrix

$$\boldsymbol{\Pi}=\begin{bmatrix}\boldsymbol{X} & \boldsymbol{I}_{n+n_{\mathrm{c}}}\\ \boldsymbol{M}^{\mathrm{T}} & 0\end{bmatrix}.$$

Pre- and post-multiplying the obtained new inequality by $\mathrm{diag}(\boldsymbol{\Pi}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{S}_1,\boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{R},\boldsymbol{I},\boldsymbol{I})$ and its transpose, respectively, with $\boldsymbol{S}_1=\boldsymbol{T}_1^{-1}$ and Eqs. (29)–(32), we obtain

$$\boldsymbol{\Sigma}_{ijl}=\begin{bmatrix}-\hat{\boldsymbol{R}}+\boldsymbol{\Pi}^{\mathrm{T}}\bar{h}\boldsymbol{\Pi} & * & * & * & * & *\\ 0 & -\boldsymbol{H} & * & * & * & *\\ \hat{\boldsymbol{G}}_1 & 0 & -2\boldsymbol{S}_1 & * & * & *\\ \boldsymbol{\Sigma}_{1ijl} & \hat{\boldsymbol{R}}_{\boldsymbol{A}\mathrm{d}i} & \boldsymbol{\Sigma}_{2il} & -\hat{\boldsymbol{R}} & * & *\\ 0 & 0 & 0 & \hat{\boldsymbol{M}}_i^{\mathrm{T}} & -\boldsymbol{I} & *\\ \tilde{\boldsymbol{N}}_{\mathrm{a}ijl} & \hat{\boldsymbol{N}}_{\mathrm{ad}i} & -\boldsymbol{N}_{\mathrm{b}i}\boldsymbol{S}_1 & 0 & 0 & -\boldsymbol{I}\end{bmatrix}<0, \tag{34}$$

where

$$\boldsymbol{\Sigma}_{1ijl}=\begin{bmatrix}\hat{\boldsymbol{A}}_{ijl}\boldsymbol{X}+\boldsymbol{B}_1\boldsymbol{C}_{\mathrm{awl}} & \hat{\boldsymbol{A}}_{ijl}\\ \boldsymbol{Y}\hat{\boldsymbol{A}}_{ijl}\boldsymbol{X}+\boldsymbol{A}_{\mathrm{awl}} & \boldsymbol{Y}\hat{\boldsymbol{A}}_{ijl}\end{bmatrix},$$

$$\boldsymbol{\Sigma}_{2il}=\begin{bmatrix}\hat{\mathcal{B}}_i\boldsymbol{S}_1+\boldsymbol{B}_1\boldsymbol{D}_{\mathrm{awl}}\\ \boldsymbol{Y}\hat{\mathcal{B}}_i\boldsymbol{S}_1+\boldsymbol{B}_{\mathrm{awl}}\end{bmatrix}.$$

Using Lemma 2 and applying the Schur's complement formula to LMI (26), we have

$$\boldsymbol{\Sigma}_{ijl}<0. \tag{35}$$

On the other hand, by pre and post-multiplying inequality (28) by $\begin{bmatrix}\boldsymbol{\Pi}^{-\mathrm{T}} & 0\\ 0 & 1\end{bmatrix}$ and its transpose, re-

spectively, it is easy to see that LMI (28) ensures $\varepsilon(\boldsymbol{P},1) \subset \boldsymbol{D}(\boldsymbol{u}_0)$.

Hence, if LMIs (26)–(28) are satisfied, $\dot{V}(t) < 0$ is verified, and $\varepsilon(\boldsymbol{P},1)$ is the domain of attraction. All initial states located in the ellipse $\varepsilon(\boldsymbol{P},1)$ converge to the origin as time goes by. Consequently, the closed-loop system is asymptotically stable. This completes the proof.

3. Simulation example

Consider the uncertain fuzzy system with time-varying delays and saturated inputs (6) and (7) with the following parameters:

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 0.1 & -2 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0.2 & 1 \\ 0.1 & -0.5 - \frac{0.015}{\pi} \end{bmatrix},$$
$$\boldsymbol{A}_{\mathrm{d}1} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.2 \end{bmatrix}, \quad \boldsymbol{A}_{\mathrm{d}2} = \begin{bmatrix} -0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix},$$
$$\boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$
$$\boldsymbol{C}_1 = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}, \quad \boldsymbol{C}_2 = \boldsymbol{C}_1,$$

and $\Delta\boldsymbol{A}_1(k)$ $\Delta\boldsymbol{A}_{21}(k)$ $\Delta\boldsymbol{B}_1(k)$ and $\Delta\boldsymbol{A}_2(k)$ $\Delta\boldsymbol{A}_{22}(k)$ $\Delta\boldsymbol{B}_2(k)$ can be represented in the form of Eq. (4) with

$$\boldsymbol{M}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{N}_{\mathrm{a}1} = \begin{bmatrix} -0.01 & 0.01 \end{bmatrix},$$
$$\boldsymbol{N}_{\mathrm{ad}1} = \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}, \quad \boldsymbol{N}_{b1} = 0.01,$$
$$\boldsymbol{M}_2 = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix},$$
$$\boldsymbol{N}_{\mathrm{a}2} = \boldsymbol{N}_{\mathrm{a}1}, \quad \boldsymbol{N}_{\mathrm{ad}2} = \boldsymbol{N}_{\mathrm{ad}1}, \quad \boldsymbol{N}_{b2} = \boldsymbol{N}_{b1}.$$

The membership functions are set as

$$h_1(x_1(k)) = \begin{cases} \frac{1}{3}, & \text{for } x_1 < -1, \\ \frac{2}{3} + \frac{1}{3}x_1, & \text{for } |x_1| \le 1, \\ 1, & \text{for } x_1 > 1. \end{cases}$$
$$h_2(x_1(k)) = 1 - h_1(x_1(k)). \tag{36}$$

Now, the dynamic controller is given as

$$\boldsymbol{A}_{\mathrm{c}1} = \begin{bmatrix} -1 & -3 \\ -4 & -10 \end{bmatrix}, \quad \boldsymbol{A}_{\mathrm{c}2} = \begin{bmatrix} -0.5 & -3 \\ -1 & -10 \end{bmatrix},$$
$$\boldsymbol{B}_{\mathrm{c}1} = \begin{bmatrix} -10 \\ -20 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{c}2} = \begin{bmatrix} -15 \\ -40 \end{bmatrix},$$
$$\boldsymbol{C}_{\mathrm{c}1} = \begin{bmatrix} -3 & -2 \end{bmatrix}, \quad \boldsymbol{C}_{\mathrm{c}2} = \begin{bmatrix} -1 & -3 \end{bmatrix},$$
$$\boldsymbol{D}_{\mathrm{c}1} = \boldsymbol{D}_{\mathrm{c}2} = 0.$$

In this example, the initial condition is $\boldsymbol{x}(0) = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}^{\mathrm{T}}$, the control bound is given by $u_0 = 1$. We assume that the time-varying delay $h(k)$ is given in Fig. 1. For such a delay, we should choose $h_2 = 11$ and $h_1 = 2$. By solving the LMIs (14)–(16), the static anti-windup compensator with the parameters $\boldsymbol{E}_{\mathrm{c}1}$ and $\boldsymbol{E}_{\mathrm{c}2}$ can be obtained. Then by solving LMIs (26)–(28), the dynamic anti-windup compensator with parameters $\boldsymbol{A}_{\mathrm{a}1}$, $\boldsymbol{A}_{\mathrm{a}2}$, $\boldsymbol{B}_{\mathrm{a}1}$, $\boldsymbol{B}_{\mathrm{a}2}$, $\boldsymbol{C}_{\mathrm{a}1}$, $\boldsymbol{C}_{\mathrm{a}2}$ can be obtained.

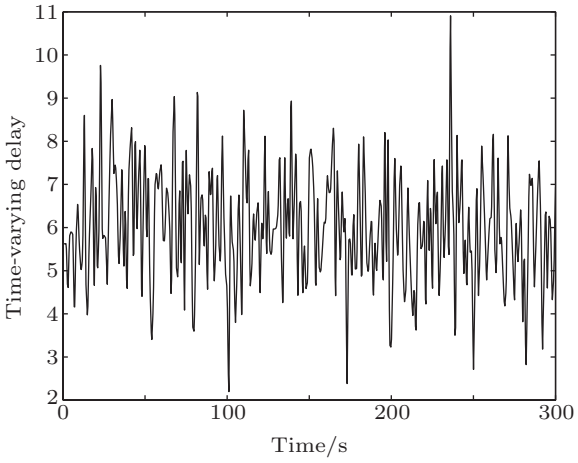


Fig. 1. Time-varying delay.

With the anti-windup output feedback controller, the simulation result of the state response of fuzzy systems (6) and (7) is given in Fig. 2. The simulation result clearly demonstrates that the proposed static and dynamic anti-windup methods are effective. Also, as expected, the state response with the dynamic anti-windup presents a smaller settling time.

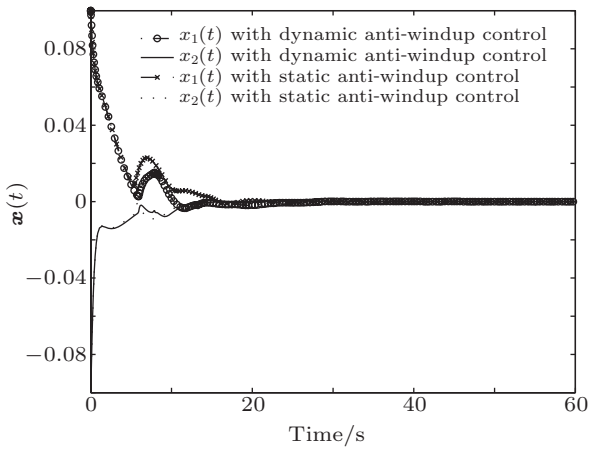


Fig. 2. State response $\boldsymbol{x}(t)$.

4. Conclusion

The anti-windup synthesis problem for T–S fuzzy systems with uncertain parameters and time-varying delays subjected to a saturation input has been addressed. Based on the project lemma and a classical sector condition, sufficient conditions for the existences of static and dynamic fuzzy controllers, which robustly stabilize the uncertain T–S fuzzy time-delay systems, have been obtained. An example has been provided to show the effectiveness of the proposed method.

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