Model of tunnelling through periodic array of quantum dots in a magnetic field*

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(Received 19 March 2012; revised manuscript received 26 July 2012)

A two-dimensional periodic array of quantum dots with two laterally coupled leads in a magnetic field is considered. The model of electron transport through the system based on the theory of self-adjoint extensions of symmetric operators is suggested. We obtain the formula for the transmission coefficient and investigate its dependence on the magnetic field.

Keywords: tunnelling, nanostructure, operator extension theory

PACS: 73.23.Ad, 02.30.Tb **DOI:** 10.1088/1674-1056/21/11/117306

1. Introduction

Surprising spectral properties of 2D periodic systems in a magnetic field attract great attention from physicists and mathematicians. Fractal structure of the spectrum known as "Hofstadter butterfly"[1] is interesting not only from the mathematical point of view,[2,3] but also from the physical one. It was shown[4] that one can observe this spectrum in experiments with ultra cold neutral atoms. There are few theoretical approaches to the description of the periodic arrays of quantum dots. In this paper, we use the zero-range potential model.[5,6] It is an advantage of the model that it is explicitly solvable. In spite of the fact that it is a rather rough approximation of realistic potential, the model gives one an effective instrument for describing many of the interesting physical properties of periodic arrays of quantum dots in a magnetic field.[7–9]

Due to the high performance of the crystal structure of nanodevices, one can observe the high free path of electron (10^{-5} m) and large Fermi wave length (4×10^{-8} m). It leads to a ballistic regime of electron transport. In this case one can use Landauer–Buttiker formalism and can derive the conductivity σ for the nanostructure with several leads from the transmission coefficient $T(E)|_{E=E_{\rm F}}$. Particularly, for the case of one incoming channel and one outgoing channel, the formula takes the following form:[10]

$$\sigma = \frac{e^2}{h}T(1-T)^{-1}. \tag{1}$$

Abstract mathematical formula for the transmission coefficient in the case of tunnelling through a quantum dot with two leads was obtained in Ref. [11] by constructing the scattering problem solution. The consideration was based on the operator extension theory.[12] In this paper, we apply this approach to describe tunnelling in the system consisting of a 2D periodic array of quantum dots with two laterally coupled semi-infinite leads orthogonal to the plane of the array (see Fig. 1). We use the zero-range potential model for

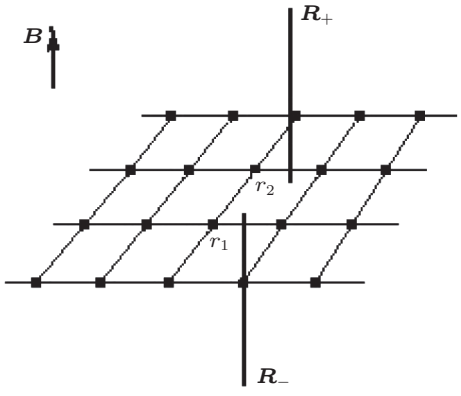


Fig. 1. Geometry of the system.

*Project supported by the Federal Targeted Program "Scientific and Educational Human Resources for Innovation-Driven Russia" (Grant Nos. P689 NK-526P, 14.740.11.0879, 16.740.11.0030, and 2012-1.2.2-12-000-1001-047), the Russian Foundation for Basic Researches (Grant No. 11-08-00267), and the Federal Targeted Program "Researches and Development in the Priority Directions Developments of a Scientific and Technological Complex of Russia 2007-2013" (Grant No. 07.514.11.4146).

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the array. It is similar to that used in Ref. [8]. We study the influence of the magnetic field on the transmission coefficient and observe an interesting phenomenon: for some values of the magnetic field, one has zero transmission.

2. Model of array

Let us start from the Hamiltonian of a charged spinless particle of mass m in constant homogeneous magnetic field $\boldsymbol{B}$ in the space $\mathrm{L}^2(\boldsymbol{R}^3)$. We choose the standard basis $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ in $\boldsymbol{R}^3$, and assume that $\boldsymbol{B}$ is parallel to $\boldsymbol{k}$: $\boldsymbol{B} = B\boldsymbol{k}$ ($B \neq 0$). After separating of free motion along the z axis, one obtains the Hamiltonian H_0 in the space $\mathrm{L}^2(\boldsymbol{R}^2)$, where $\boldsymbol{R}^2$ is a plane orthogonal to $\boldsymbol{B}$ and having a basis $\boldsymbol{i}$ and $\boldsymbol{j}$. The Hamiltonian H_0 can be given by

$$H_0 = (2m)^{-1}(\boldsymbol{p} - ec^{-1}\boldsymbol{A})^2. \tag{2}$$

Here $\boldsymbol{p} = -\mathrm{i}\hbar\boldsymbol{\nabla}$ is the two-dimensional momentum operator, e is the particle charge, c is the speed of light, $\boldsymbol{A}(\boldsymbol{r})$ is a vector potential of the field ($\boldsymbol{B} = \mathrm{curl}\boldsymbol{A}$). The choice of this potential is not unique. Usually, one uses symmetric or Landau gauge, which can be described by $\boldsymbol{A} = -\boldsymbol{B} \times \boldsymbol{r}$ and $\boldsymbol{A} = (-B\boldsymbol{y}, 0)$, respectively.

In each case, the operator H_0 is essentially self-adjoint in $\mathrm{L}^2(R^2)$. The change of gauge leads to the replacement of H_0 by a unitary equivalent operator.[13] We shall use the symmetric gauge.

Let us introduce the following standard notations: cyclotron frequency, $\omega = |eB|/cm$; quantum of the magnetic flux, $\Phi_0 = 2\pi\hbar c/|e|$. Let us make ξ equal with the following ratio: $\xi = \pm B/\Phi_0$. Here, the sign ($\pm$) is chosen in such a way that $\xi > 0$. The value of $|\xi|$ equals the number of the flux quanta through the unit area in R^2. To simplify formulas, we shall use the system of units: $e = \hbar = c = 1$. Then the Hamiltonian H_0 takes the following form:

$$H_0 = -\frac{1}{2}\left[\left(\frac{\partial}{\partial x} + \pi\xi\mathrm{i}y\right)^2 + \left(\frac{\partial}{\partial y} - \pi\xi\mathrm{i}x\right)^2\right]. \tag{3}$$

The spectrum of H_0 is pure point and consists of infinitely degenerate eigenvalues (Landau levels)[7]

$$\varepsilon_l = \left(l + \frac{1}{2}\right)\omega. \tag{4}$$

Then we shall use the Green function of the operator H_0 (i.e., the resolvent kernel of $R^0(\boldsymbol{z}) = (H_0 - \boldsymbol{z})^{-1}$), which has the following form:[14]

$$G_0(\boldsymbol{r}_i, \boldsymbol{r}_j; z) = \frac{1}{2\pi}\Gamma\left(\frac{1}{2} - \frac{z}{\omega}\right)\exp\left[-\pi\mathrm{i}\boldsymbol{r}_i \wedge \boldsymbol{r}_j - \frac{\pi\xi(\boldsymbol{r}_i - \boldsymbol{r}_j)^2}{2}\right] \times \Psi\left(\frac{1}{2} - \frac{z}{\omega}, 1, \pi\xi(\boldsymbol{r}_i - \boldsymbol{r}_j)^2\right). \tag{5}$$

Here $\Gamma(x)$ is the Euler function, and $\Psi(a, c; x)$ is the confluent hypergeometric function of the second type.[15]

Consider a two-dimensional periodic array of quantum dots Λ. The centers of the quantum dots coincide with the nodes of the square lattice Λ. Let us fix two basic vectors a_1 and a_2 from Λ such that for any $\lambda \in \Lambda$, we have a unique representation $\lambda = \lambda_1 a_1 + \lambda_2 a_2$, where λ_1 and λ_2 are integers. The Hamiltonian of the array of quantum dots can be expressed in the following form:

$$H = H_0 + \sum_{\gamma} V(\boldsymbol{r} - \gamma). \tag{6}$$

We use the zero-range potential model based on the operator extension theory. In this way, the potential V is replaced by the δ-like one. The Hamiltonian H of the array in this case is constructed in the following way.[8] First, we restrict the initial self-adjoint operator H_0 onto the set of smooth functions vanishing at the nodes of the lattice, and construct its closure. It is a symmetric non-self-adjoint operator with infinite deficiency indices. The model Hamiltonian is given by a self-adjoint extension H of this symmetric restriction. The Green function G of the operator H can be obtained by Krein's resolvent formula[8,16]

$$\begin{aligned} &G(r, r'; E) \\ &= G_0(r, r'; E) \\ &\quad - \sum_{\gamma,\mu}[\boldsymbol{Q}(E) + \boldsymbol{A}]^{-1}_{\lambda,\mu} G_0(r, \lambda; E) G_0(\mu, r'; E), \end{aligned} \tag{7}$$

where $\boldsymbol{Q}(E)$ is the Krein matrix having the diagonal form in our model: $Q_{ij}(E) = q(E)\delta_{ij}$,

$$q(E) = \left(-\frac{m}{2\hbar}\right)\left[\psi\left(\frac{1}{2} - \frac{E}{\Omega}\right) + \log\left(\frac{m\Omega}{2\pi\hbar}\right) + 2C_{\mathrm{E}}\right], \tag{8}$$

$\psi(x)$ is the logarithmic derivative of the Euler function $\Gamma(x)$, $\Omega = \sqrt{\omega^2 + 4\omega_0^2}$, and C_{E} is the Euler constant.

The Hermitian matrix $\boldsymbol{A}$ parameterizes the self-adjoint extension of the operator H_0. Due to the periodicity of the system, its Hamiltonian should be invariant with respect to the magnetic translation group of transformations.[7] As a consequence, one obtains the terms of A satisfying the following property:

$$A_{\lambda-\gamma,\mu-\gamma} = \exp\left[\pi\mathrm{i}\xi(\gamma \times (\lambda - \mu))\boldsymbol{e}_z\right]A_{\lambda,\mu}. \tag{9}$$

We only take into account the nearest-neighbouring interaction. Hence, the matrix inputs have the following form:

$$A_{\lambda,0} = g[\delta(\lambda_1,0)(\delta(\lambda_2,1)+\delta(\lambda_2,-1)) + \delta(\lambda_2,0)(\delta(\lambda_1,1)+\delta(\lambda_1,-1))], \quad (10)$$

where g is a constant related to the intensity of interaction.

3. Model of tunnelling

Let us now consider the tunnelling problem by dealing with the plane of our array with two connected semi-infinite leads orthogonal to the plane (see Fig. 1). To construct the model, we use ideas similar to those in Refs. [11] and [12]. The Hilbert space for the above-described array will be marked as $\mathcal{H}_\mathrm{d}$ corresponding to the "internal" states. The state spaces of the channels are $\mathcal{H}_+ = \mathcal{L}^\in(\mathcal{R}_+)$, and $\mathcal{H}_- = \mathcal{L}^\in(\mathcal{R}_-)$. Thus, the whole state space for the system $\mathcal{H}$ is given as follows:

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_\mathrm{d} \oplus \mathcal{H}_+. \quad (11)$$

We start from the Hamiltonian H^0 which corresponds to the system having no interaction between the channels and the array. It is the orthogonal sum of the corresponding operators

$$H^0 = H_- \oplus H_\mathrm{d} \oplus H_+, \quad (12)$$

where $H_\pm$ is the operator $-\mathrm{d}^2/\mathrm{d}x^2$ in $\mathrm{L}^2(\boldsymbol{R}_\pm)$ under the Neumann condition (absence of transmission through the contact point) at the edge (point 0), and H_d is the Hamiltonian of the charged particle in the quantum-dot array in a homogeneous magnetic field described in the last section.

The well-known Green functions $G_\pm$ read

$$G_\pm(x,y;E) = \mathrm{i}(2k)^{-1}[\exp(\mathrm{i}k|x-y|) + \exp(\pm\mathrm{i}k(x+y))], \quad (13)$$

where $E = k^2$ $(k \geq 0)$. Thus,

$$G_\pm(x,0;E) = \mathrm{i}k^{-1}\exp(\pm\mathrm{i}k(x+y)), \quad \pm x > 0. \quad (14)$$

To switch on the interaction between the channels and the array, we use the operator extension theory again (the mathematical construction is analogous to that in Ref. [11] for a quantum dot with two leads). That is, one has the second stage extension (the first stage is the construction of the array Hamiltonian H_d). Namely, we restrict our operator H onto the set of smooth functions vanishing at the points of lead–plane contacts, and consider its closure. It is a symmetric non-self-adjoint operator. The extension of this operator gives us the model Hamiltonian. Considering the scattering problem for this Hamiltonian (having the incoming and reflected wave in one lead and the outgoing wave in the other lead), one obtains the following formula for the transmission coefficient $T(E)$:

$$T(E) = \frac{|B_1|^2}{E|\det[\boldsymbol{Q}_\mathrm{d}(E)+A]|^2}, \quad (15)$$

where B_1 is the second-stage extension parameter which describes the quality of the lead–plane contact, and $\boldsymbol{Q}_\mathrm{d}(E)$ is the Krein function (2×2 matrix) for the second-stage extension. Its diagonal terms are the regular parts of the Green function G for the array at the lead–plane contact points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, correspondingly. The non-diagonal terms are

$$Q_\mathrm{d}^{ij}(E) = G(\boldsymbol{r}_i,\boldsymbol{r}_j;E), \quad i \neq j. \quad (16)$$

We calculate the transmission coefficient as a function of electron energy E for different values of the magnetic field $\boldsymbol{B}$. It has an irregular form with many sharp peaks close to each other. It seems difficult to observe the peaks in real experiments, due to the temperature effect in real systems (the phenomenon is observable if the distances between peaks are greater than the product of the Boltzmann constant and the temperature value). However, our calculations show that there exist some wide segments on the energy axis for which the transmission coefficient is zero (zero transmission band). The position of this band depends on the magnetic field. This dependence is shown by Fig. 2. This phenomenon is physically measurable. It is interesting because one obtains the possibility to control the transmission by the magnetic field.

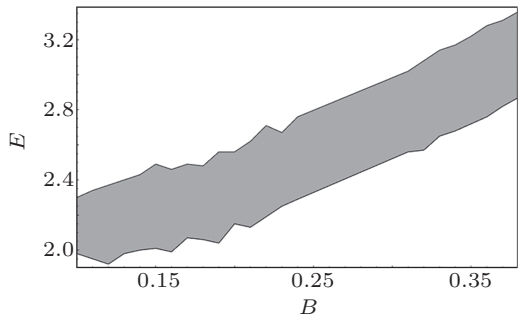


Fig. 2. The position and width of zero transmission band of the electron energy E versus the magnetic field B.

4. Conclusion

Solvable model of tunneling through two-dimensional periodic array of quantum dots in a magnetic field is constructed. The mathematical background is the theory of self-adjoint extensions of symmetric operators. The construction procedure consists of two stages: creation of the array model and construction of the model of tunneling. For the array we use zero-range potentials model. We look for the tunneling for different values of the magnetic field. An interesting phenomenon is observed: there is rather wide range of energies (depending on the magnetic field) for which one has zero transmission.

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