Deriving some quantum optical identities using the general ordering theorem*

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(Received 19 March 2012; revised manuscript received 19 April 2012)

Using the newly introduced general ordering theorem by Shähandeh and Bazrafkan, we derive and generalize some quantum optical identities and give their applications.

Keywords: s-ordered expansion of operators, general ordering theorem

PACS: 42.50.–p, 31.15.–p, 03.65.–w **DOI:** 10.1088/1674-1056/21/11/114202

1. Introduction

The s-ordering of operators has many applications in quantum optics, such as constructing the phase space s-parameterized quasiprobabilities.[1−3] Perhaps the most well-known method of s-ordering of operators is Fan's integration within s-ordered product of operators (IWSOP).[4,5] For a review of the method and some of its applications, see Refs. [6]–[8]. In a recent work, the authors introduced the most general theorem regarding the ordering of operators, which will be referred to as the general ordering theorem or just GOT.[9] In this regard, one may t-order any multiplicative sequence of s_j-ordered functions with $j \in \{2,3,\ldots,k\}$ as

$$\left\{\hat{F}\left(a^{\dagger},a\right)\right\}_{s_2}\left\{\hat{G}\left(a^{\dagger},a\right)\right\}_{s_3}\cdots\left\{\hat{H}\left(a^{\dagger},a\right)\right\}_{s_k} = \sum_{i\text{-pair }(u_l,\,t)\text{-contractions}} \left\{\left(\hat{F}\hat{G}\cdots\hat{H}\right)_i^{\boldsymbol{u}}\left(a^{\dagger},a\right)\right\}_t, \quad (1)$$

where $l \in \{1,2,\ldots,i\}$, $\boldsymbol{u} \equiv (u_1,u_2,\ldots,u_i)$ with $u_l \in \{-1,1,s_j\}$, and $\left(\hat{F}\hat{G}\cdots\hat{H}\right)_i^{\boldsymbol{u}}$ is the i-pair (u_l,t)-contracted form of the multiplicative sequence $\hat{F}\hat{G}\cdots\hat{H}$.

This statement has an obvious meaning by virtue of the definition of the relative order of operators. Strictly speaking, for a multiplicative sequence of s-ordered monomials, each pair of operators $a^{\dagger}$ and a have a relative order depending on their relative positions. 1) If $a^{\dagger}$ stands on the left of a, and they lie in two different s-ordering symbols, they are relatively normally ordered, specified by the parameter $s=1$. 2) If a stands on the left of $a^{\dagger}$, and they lie in two different s-ordering symbols, they are relatively antinormally ordered, specified by the parameter $s=-1$. 3) If $a^{\dagger}$ and a both lie in the same $\{\cdots\}_s$ symbol, they are relatively s-ordered. In this regard, for two operators $a^{\dagger}$ and a with $\left[a,a^{\dagger}\right]=1$, a (s,t)-contraction is defined to be the replacement of a pair of relatively s-ordered operators

$$\left\{a^{\dagger}a\right\}_s - \left\{a^{\dagger}a\right\}_t = \frac{t-s}{2}, \quad (2)$$

where the right-hand side is a direct result of the s-ordering definition.

Relation (1) provides the combinatorial structure underlying the ordering procedure. In fact, this is a generalization of the Wick's idea of normal ordering. In this way, one may s-order any given function granted that he could count the number of different contractions.

In the present paper, we derive some useful quantum optical identities by using the above theorem.

2. Factorial identities

We may simply derive the Glauber's formula as[10]

$$\lambda^{a^{\dagger}a} =: \mathrm{e}^{(\lambda-1)a^{\dagger}a} : . \quad (3)$$

To this end, we write

$$\left(a^{\dagger}a\right) : \left(a^{\dagger}a\right)^{n-1} : = \sum_{i=0}^{\min\{1,n-1\}} \binom{1}{i}\binom{n-1}{i} i!(1)^i : \left(a^{\dagger}a\right)^{n-i} :$$

*Project supported by the Imam Khomeini International University, I. R. Iran (Grant Nos. 751075-91 and 383042-91).

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$$= :\left(a^\dagger a\right)^n: + (n-1):\left(a^\dagger a\right)^{n-1}:,$$

where the coefficient $\binom{1}{i}\binom{n}{i}i!$ is just the number of possible contractions between the left a with the inside $a^\dagger$s. This gives

$$:\left(a^\dagger a\right)^n: := \left(a^\dagger a - n + 1\right):\left(a^\dagger a\right)^{n-1}:,$$

and then

$$:\left(a^\dagger a\right)^n: := \frac{\left(a^\dagger a\right)!}{\left(a^\dagger a - n\right)!}.$$

Now, we have

$$:\mathrm{e}^{(\lambda-1)a^\dagger a}: = \sum_{n=0}^{\infty}\binom{a^\dagger a}{n}(\lambda-1)^n = \lambda^{a^\dagger a}, \quad (4)$$

in which we have used the binomial formula. In the same manner, we may write

$$\begin{aligned}&\left(aa^\dagger\right)\vdots\left(a^\dagger a\right)^{n-1}\vdots\\ &= \sum_{i=0}^{n}\binom{1}{i}\binom{n-1}{i}i!(-1)^i\vdots\left(a^\dagger a\right)^{n-i}\vdots\\ &= \vdots\left(a^\dagger a\right)^n\vdots - (n-1)\vdots\left(a^\dagger a\right)^{n-1}\vdots,\end{aligned}$$

which leads to

$$\vdots\left(a^\dagger a\right)^n\vdots = \frac{\left(a^\dagger a + n\right)!}{\left(a^\dagger a\right)!}. \quad (5)$$

Thus, we may use the negative-binomial relation

$$\sum_{n=0}^{\infty}\binom{m+n}{n}(-x)^n = (x+1)^{-(m+1)},$$

together with Eq. (5) to write

$$\lambda\vdots\mathrm{e}^{(1-\lambda)a^\dagger a}\vdots = \lambda^{-a^\dagger a}. \quad (6)$$

3. Generalized derivative lemmas

In this section, we use the GOT to generalize some derivative formulas. We begin with the product $a^n\hat{F}\left(a^\dagger, a\right)$. Consider the case for which we know the s-ordered form of $\hat{F}\left(a^\dagger, a\right)$ to be $\hat{F}^{(s)}\left(a^\dagger, a\right)$. Then,

$$\begin{aligned}&a^n\hat{F}^{(s)}\left(a^\dagger, a\right)\\ &= a^n\sum_{k,l}f_{kl}^s\left\{a^{\dagger k}a^l\right\}_s\\ &= \left\{\sum_{k,l}f_{kl}^s\sum_{i=0}^{\min\{k,n\}}\binom{k}{i}\binom{n}{i}\right.\\ &\quad\left.\times i!\left(\frac{s+1}{2}\right)^i a^{\dagger k-i}a^{n+l-i}\right\}_s.\end{aligned}$$

Now, we may substitute the coefficient $\binom{k}{i}i!$ by the formal differentiation $\partial_{a^\dagger}^i$ to have

$$\begin{aligned}&a^n\hat{F}^{(s)}\left(a^\dagger, a\right)\\ &= \left\{\left[a + \left(\frac{s+1}{2}\right)\frac{\partial}{\partial a^\dagger}\right]^n F^{(s)}\left(a^\dagger, a\right)\right\}_s. \quad (7)\end{aligned}$$

This is a generalization of the well-known derivative lemma (s=1)[11]

$$a^n\hat{F}^{(\mathrm{N})}\left(a^\dagger, a\right) =: \left(a + \frac{\partial}{\partial a^\dagger}\right)^n F^{(\mathrm{N})}\left(a^\dagger, a\right):, \quad (8)$$

with the trivial case of $s = -1$ as

$$a^n\hat{F}^{(\mathrm{A})}\left(a^\dagger, a\right) = \vdots a^n F^{(\mathrm{A})}\left(a^\dagger, a\right)\vdots. \quad (9)$$

In a similar way, we may write

$$\begin{aligned}&\hat{F}^{(s)}\left(a^\dagger, a\right)a^n\\ &= \sum_{k,l}f_{kl}^s\left\{a^k a^{\dagger l}\right\}_s a^n\\ &= \left\{\sum_{k,l}f_{kl}^s\sum_{i=0}^{\min\{l,n\}}\binom{n}{i}\binom{l}{i}\right.\\ &\quad\left.\times i!\left(\frac{s-1}{2}\right)^i a^{\dagger l-i}a^{n+k-i}\right\}_s\\ &= \left\{\left[a + \left(\frac{s-1}{2}\right)\frac{\partial}{\partial a^\dagger}\right]^n F^{(s)}\left(a^\dagger, a\right)\right\}_s, \quad (10)\end{aligned}$$

which in the special case of $s = -1$ gives the formula

$$\hat{F}^{(\mathrm{A})}\left(a^\dagger, a\right)a^n = \vdots\left(a - \frac{\partial}{\partial a^\dagger}\right)^n F^{(\mathrm{A})}\left(a^\dagger, a\right)\vdots. \quad (11)$$

Alternatively, we may examine the multiplication by $a^{\dagger n}$. This leads to

$$\begin{aligned}&a^{\dagger n}\hat{F}^{(s)}\left(a^\dagger, a\right)\\ &= \left\{\left[a^\dagger + \left(\frac{s-1}{2}\right)\frac{\partial}{\partial a}\right]^n \hat{F}^{(s)}\left(a^\dagger, a\right)\right\}_s, \quad (12)\\ &\hat{F}^{(s)}\left(a^\dagger, a\right)a^{\dagger n}\\ &= \left\{\left[a^\dagger + \left(\frac{s+1}{2}\right)\frac{\partial}{\partial a}\right]^n \hat{F}^{(s)}\left(a^\dagger, a\right)\right\}_s, \quad (13)\end{aligned}$$

with the special cases

$$a^{\dagger n}\hat{F}^{(\mathrm{A})}\left(a^\dagger, a\right) = \vdots\left(a^\dagger - \frac{\partial}{\partial a}\right)^n \hat{F}^{(\mathrm{A})}\left(a^\dagger, a\right)\vdots, \quad (14)$$

$$\hat{F}^{(\mathrm{N})}\left(a^\dagger, a\right)a^{\dagger n} = :\left(a^\dagger + \frac{\partial}{\partial a}\right)^n \hat{F}^{(\mathrm{N})}\left(a^\dagger, a\right):. \quad (15)$$

These are solely new identities regarding the transformations of ordered operators. For example, we may use Eqs. (7) and (12) to obtain

$$\mathrm{e}^{\lambda a}\hat{F}^{(s)}\left(a^{\dagger},a\right)=\left\{\mathrm{e}^{\lambda a}\hat{F}^{(s)}\left(a^{\dagger}+\tau_{+}\lambda,a\right)\right\}_{s},$$
$$\tau_{+}\equiv\left(\frac{s+1}{2}\right), \tag{16}$$
$$\mathrm{e}^{\lambda a^{\dagger}}\hat{F}^{(s)}\left(a^{\dagger},a\right)=\left\{\mathrm{e}^{\lambda a^{\dagger}}\hat{F}^{(s)}\left(a^{\dagger}+\tau_{-}\lambda,a\right)\right\}_{s},$$
$$\tau_{-}\equiv\left(\frac{s-1}{2}\right). \tag{17}$$

4. Fan's Hermite polynomials

We may use the GOT to simply write down Fan's ordering formulas regarding the Hermite polynomials as well.[7] In this regard, we have

$$a^{\dagger n}a^{m}=\left\{\sum_{i=0}^{\min\{n,m\}}\binom{n}{i}\binom{m}{i}\times i!\left(\frac{s-1}{2}\right)^{i}a^{\dagger n-i}a^{m-i}\right\}_{s}.$$

Using the definition of two-variable Hermite polynomials[12]

$$\mathrm{H}_{n,m}(x,y)=\sum_{i=0}^{\min\{n,m\}}\binom{n}{i}\binom{m}{i}i!(-1)^{i}x^{n-i}y^{m-i} \tag{18}$$

gives

$$a^{\dagger n}a^{m}=\left\{\left(\frac{1-s}{2}\right)^{\frac{m+n}{2}}\times\mathrm{H}_{n,m}\left(\sqrt{\frac{2}{1-s}}a^{\dagger},\sqrt{\frac{2}{1-s}}a\right)\right\}_{s}. \tag{19}$$

In a similar way, we may write the general transformation formula

$$\left\{a^{\dagger n}a^{m}\right\}_{s}=\left\{\sum_{i=0}^{\min\{n,m\}}\binom{n}{i}\binom{m}{i}\times i!\left(\frac{t-s}{2}\right)^{i}a^{\dagger n-i}a^{m-i}\right\}_{t}$$
$$=\left\{\left(\frac{s-t}{2}\right)^{\frac{m+n}{2}}\times\mathrm{H}_{m,n}\left(\sqrt{\frac{2}{s-t}}a,\sqrt{\frac{2}{s-t}}a^{\dagger}\right)\right\}_{t}. \tag{20}$$

We can use the incomplete two-variable Hermite polynomials[13]

$$\mathrm{h}_{m,n}(x,y|\tau)=\sum_{i=0}^{\min\{n,m\}}\binom{m}{i}\binom{n}{i}i!\tau^{i}x^{m-i}y^{n-i}, \tag{21}$$

with the generating function

$$\sum_{m,n=0}^{\infty}\frac{\lambda^{m}\mu^{n}}{m!n!}\mathrm{h}_{m,n}(x,y|\tau)=\mathrm{e}^{\lambda x+\mu y+\tau\lambda\mu}, \tag{22}$$

to write Eqs. (19) and (20) as

$$a^{\dagger n}a^{m}=\left\{\mathrm{h}_{n,m}\left(a^{\dagger},a|\tau_{-}\right)\right\}_{s}, \tag{23}$$
$$\left\{a^{\dagger n}a^{m}\right\}_{s}=\left\{\mathrm{h}_{n,m}\left(a^{\dagger},a|\tau_{st}\right)\right\}_{t},\quad\tau_{st}\equiv\frac{t-s}{2}. \tag{24}$$

5. Application of the results

Using the definition of two-variable Hermite polynomial Eq. (18), we have

$$\mathrm{h}_{m,n}(x,y|\kappa)=\left(-\mathrm{i}\sqrt{\kappa}\right)^{m+n}\mathrm{H}_{m,n}\left(\mathrm{i}\frac{x}{\sqrt{\kappa}},\mathrm{i}\frac{y}{\sqrt{\kappa}}\right). \tag{25}$$

As a special case, we have

$$\mathrm{h}_{m,n}(x,y|-1)=\mathrm{H}_{m,n}(x,y). \tag{26}$$

The relation between the two-variable Hermite polynomials and the generalized Laguerre polynomials $\mathrm{L}_{n}^{m}(x)$[12]

$$\mathrm{H}_{m,n}(x,y)=(-1)^{n}n!x^{m-n}\mathrm{L}_{n}^{m-n}(xy)=(-1)^{m}m!y^{n-m}\mathrm{L}_{m}^{n-m}(xy) \tag{27}$$

implies

$$\mathrm{h}_{m,n}(x,y|\kappa)=\kappa^{n}n!x^{m-n}\mathrm{L}_{n}^{m-n}\left(-\frac{xy}{\kappa}\right)=\kappa^{m}m!y^{n-m}\mathrm{L}_{n}^{n-m}\left(-\frac{xy}{\kappa}\right). \tag{28}$$

This gives the special case of $m=n$ as

$$\mathrm{h}_{n,n}(x,y|\kappa)=\kappa^{n}n!\mathrm{L}_{n}\left(-\frac{xy}{\kappa}\right), \tag{29}$$

in which $\mathrm{L}_{n}(x)$ is just the usual Laguerre polynomial.

Using Eqs. (24) and (29) and the generating function of the usual Laguerre polynomials,[14] we may write

$$\left\{\mathrm{e}^{\lambda a^{\dagger}a}\right\}_{s}=\left\{\sum_{n=0}^{\infty}\frac{\lambda^{n}}{n!}\mathrm{h}_{n,n}\left(a^{\dagger},a|\tau_{st}\right)\right\}_{t}=\frac{1}{1-\lambda\tau_{st}}\left\{\mathrm{e}^{\frac{\lambda}{1-\lambda\tau_{st}}a^{\dagger}a}\right\}_{t}. \tag{30}$$

In addition, we can multiply both sides of Eq. (30) by a^{n} on the left and employ the relation (7) to get

$$a^{n}\left\{\mathrm{e}^{\lambda a^{\dagger}a}\right\}_{s}=\left[\frac{1-\lambda\left(\tau_{st}+\tau_{+}^{\prime}\right)}{\left(1-\lambda\tau_{st}\right)^{2}}\right]^{n}\left\{a^{n}\,\mathrm{e}^{\frac{\lambda}{1-\lambda\tau_{st}}a^{\dagger}a}\right\}_{t},$$

$$\tau'_{+} \equiv \frac{t+1}{2}. \tag{31}$$

Similarly, multiplying both sides of Eq. (31) by $a^{\dagger m}$ on the right and after using Eq. (13), we have

$$a^n \left\{ \mathrm{e}^{\lambda a^\dagger a} \right\}_s a^{\dagger m} = \left[\frac{1-\lambda\left(\tau_{st}+\tau'_{+}\right)}{\left(1-\lambda\tau_{st}\right)^2} \right]^n \times \left\{ \left(a^\dagger + \tau'_{+}\partial_a\right)^m a^n \, \mathrm{e}^{\frac{\lambda}{1-\lambda\tau_{st}} a^\dagger a} \right\}_t,$$

which by the substitution of $a^n \to \left(\frac{1-\lambda\tau_{st}}{\lambda}\right)^n \partial^n_{a^\dagger}$ and using the general Leibniz rule of differentiation gives

$$a^n \left\{ \mathrm{e}^{\lambda a^\dagger a} \right\}_s a^{\dagger m} = \left[\frac{1-\lambda\left(\tau_{st}+\tau'_{+}\right)}{\lambda\left(1-\lambda\tau_{st}\right)} \right]^{n+m} \times \lambda^m \left\{ \mathrm{h}_{m,n}\left(a^\dagger, \frac{\lambda}{1-\lambda\tau_{st}} a^\dagger \middle| 1 \right) \mathrm{e}^{\frac{\lambda}{1-\lambda\tau_{st}} a^\dagger a} \right\}_t. \tag{32}$$

This is the most general ordered product rule of this kind.

6. Conclusion

In summary, we have obtained a simple method for the s-ordering of operators. This is especially suitable for evaluating the product of ordered operators, as well as ordering transformations. Indeed, we have re-derived and generalized some elementary quantum optical identities in a simple fashion. We have shown that some of the previous ordering identities have simple combinatorial meanings. We have also represented the extent of our method through a simple example.

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