Cluster projective synchronization of complex networks with nonidentical dynamical nodes*

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We investigate a new cluster projective synchronization (CPS) scheme in time-varying delay coupled complex dynamical networks with nonidentical nodes. Based on the community structure of the networks, the controllers are designed differently for the nodes in one community, which have direct connections to the nodes in the other communities and the nodes without direct connections to the nodes in the other communities. Some sufficient criteria are derived to ensure the nodes in the same group projectively synchronize and there is also projective synchronization between nodes in different groups. Particularly, the weight configuration matrix is not assumed to be symmetric or irreducible. The numerical simulations are performed to verify the effectiveness of the theoretical results.

Keywords: cluster projective synchronization, complex network, time-varying delay adaptive controller

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1. Introduction

In recent years, the dynamics of complex networks have been extensively investigated. As a typical kind of dynamics, the synchronization in complex networks has received a great deal of attention.[1−6] There are different types of definitions for synchronization, such as complete synchronization,[7] phase synchronization,[8] lag synchronization,[9,10] generalized synchronization,[11] cluster synchronization,[12] and projective synchronization.[13−16] It is well known that many effective control methods, including adaptive control,[17] pinning control,[18,19] impulsive control,[20,21] hybrid control,[22] intermittent control,[23] and so on, have been proposed to realize the synchronized dynamics for complex networks.

Cluster synchronization is significant in biological science[24] and communication engineering.[25] Recently, some works for the cluster synchronization in complex networks have been published. Belykh *et al.*[26] studied the effect of cluster partitioning in the networks of coupled oscillators, and the conditions for the existences of the unconditional and the conditional clusters were given. Wang *et al.*[12] investigated the cluster synchronization problem for time-varying delay coupling networks with nonidentical delayed dynamical systems by using the pinning control method. Liu *et al.*[27] investigated the cluster synchronization problem for linearly coupled networks by adding some simple intermittent pinning controls and some sufficient conditions to guarantee the global cluster synchronization were presented. Wu *et al.*[28] investigated the cluster synchronization in adaptive complex dynamical networks with nonidentical nodes by using a local control method and a novel adaptive strategy for the coupling strengths of the networks. Wu and Fu[29] investigated the cluster mixed synchronization of complex networks with non-delayed coupling. By using some linear pinning control schemes and the adaptive coupling strength method, only the nodes in one community which have direct connections to the nodes in the other communities were controlled to achieve the cluster mixed synchronization. However, there are few theoretical results on the cluster projective synchronization (CPS) of the linearly coupled complex networks with time-varying delay couplings

Motivated by the above discussion, we investigate the cluster projective synchronization in time-varying

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delay coupled complex dynamical networks. The local dynamics in each community are identical, but those of different communities are not identical. To achieve the synchronization, based on the community structure of the networks, the controllers are designed differently for the nodes in one community which have direct connections to the nodes in the other communities and the nodes without direct connections to the nodes in the other communities. Some sufficient criteria are derived to ensure that the nodes in the same group are projectively synchronized and there is also projective synchronization between nodes in different groups. According to the Lyapunov stability theory, the sufficient conditions for achieving the cluster projective synchronization are obtained analytically.

The paper is organized as follows. The network model is introduced followed by some definitions, lemmas, and hypotheses in Section 2. The cluster projective synchronization of the complex coupled networks is discussed in Section 3. Simulation results are given in Section 4. Finally, in Section 5, various conclusions are discussed.

2. Model and preliminaries

The network with non-delayed and time-varying delay couplings can be described by

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t))+c\sum_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &+c\sum_{j=1}^{N}b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t)),\\ &i=1,2,\cdots,N,\end{aligned}\tag{1}$$

where $\boldsymbol{x}_i(t)=(x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^{\mathrm{T}}\in\mathbf{R}^n$ is the state vector of node i; $c>0$ is the coupling strength, and $\boldsymbol{f}_{\phi_i}:\mathbf{R}^n\to\mathbf{R}^n$ describes the local dynamics of nodes in the ϕ_i-th community. For any pair of nodes i and j, if $\phi_i\neq\phi_j$, i.e., nodes i and j belong to different communities, then $\boldsymbol{f}_{\phi_i}\neq\boldsymbol{f}_{\phi_j}$. The $\eta_{\phi_i}(t)$ is the time-varying delay. The $\boldsymbol{\Gamma}_1\in\mathbf{R}^{n\times n}$ and $\boldsymbol{\Gamma}_2\in\mathbf{R}^{n\times n}$ are the inner-coupling matrices, for simplicity, we assume that $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are diagonal matrices with positive diagonal elements, i.e., $\boldsymbol{\Gamma}_1=\mathrm{diag}(\rho_1,\rho_2,\ldots,\rho_n)$ with $\rho_i\geq 0$, $\boldsymbol{\Gamma}_2=\mathrm{diag}(\theta_1,\theta_2,\ldots,\theta_n)$ with $\theta_i\geq 0$. The $\boldsymbol{A}=(a_{ij})_{N\times N}$ and $\boldsymbol{B}=(b_{ij})_{N\times N}$ are the weight configuration matrices. If there is a connection from node i to node j ($j\neq i$), then $a_{ij}>0$, $b_{ij}>0$, otherwise $a_{ij}=a_{ji}=0$, $b_{ij}=b_{ji}=0$. The diagonal elements of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as

$$\begin{aligned}a_{ii}&=-\sum_{j=1,j\neq i}^{N}a_{ji},\\ b_{ii}&=-\sum_{j=1,j\neq i}^{N}b_{ji},\quad i=1,2,\ldots,N.\end{aligned}\tag{2}$$

Particularly, the weight configuration matrix is not assumed to be symmetric or irreducible.

When the control inputs $\boldsymbol{v}_i(t)$ and $\boldsymbol{u}_i(t)\in\mathbf{R}^n$ ($i=1,2,\ldots,N$) are introduced, the controlled dynamical network with respect to network (1) can be written as

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t))+c\sum_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &+c\sum_{j=1}^{N}b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t))+\boldsymbol{u}_i(t),\\ &\phi_i(t)\in\bar{J}_{\phi_i},\\ \dot{\boldsymbol{x}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t))+c\sum_{j=1}^{N}a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &+c\sum_{j=1}^{N}b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t))+\boldsymbol{v}_i(t),\\ &\phi_i(t)\in J_{\phi_i}-\bar{J}_{\phi_i},\end{aligned}\tag{3}$$

where J_{ϕ_i} denotes all the nodes in the ϕ_i-th community, and $\bar{J}_{\phi_i}$ represents the nodes in the ϕ_i-th community that have direct links with nodes in the other communities.

In this paper, $\{C_1,C_2,\cdots,C_m\}$ denotes m ($2\leq m\leq N$) communities of the networks, and $\cup_{i=1}^{m}C_i=\{1,2,\cdots N\}$. If node i belongs to the j-th community, then we denote $\phi_i=j$. We employ $\boldsymbol{f}_i(\cdot)$ to represent the local dynamics of all nodes in the i-th community, the local dynamics of individual nodes in different communities are assumed to be nonidentical, i.e., if $\phi_i\neq\phi_j$, then $\boldsymbol{f}_{\phi_i}\neq\boldsymbol{f}_{\phi_j}$

Definition 1 Assume that $\boldsymbol{s}(t)\in\mathbf{R}^n$ is any smooth dynamics. The controlled complex community network (1) is said to achieve the cluster projective synchronization with respect to the scaling factor p_{ϕ_i} (p_{ϕ_i} are constants), if $\lim_{t\to\infty}\|\boldsymbol{x}_i(t)-p_{\phi_i}\boldsymbol{s}(t)\|=0$, $i=1,2,\ldots,N$ for any initial conditions.

Definition 2[28] The error variables are $\boldsymbol{e}_i(t)=\boldsymbol{x}_i(t)-p_{\phi_i}\boldsymbol{s}(t)$, where $\boldsymbol{s}(t)$ satisfies $\boldsymbol{F}_{\phi_i}(t)=p_{\phi_i}\dot{\boldsymbol{s}}(t)-p_{\phi_i}\boldsymbol{f}_{\phi_i}(t,\boldsymbol{s}(t))$.

Lemma 1 For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, a matrix $\boldsymbol{Q}>0$ with compatible dimensions, one has $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}\leq\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{y}$.

Assumption 1 The $\eta_{\phi_i}(t)$ is a differential function with $0 \leq \dot{\eta}_{\phi_i}(t) \leq \varepsilon \leq 1$. Clearly, this assumption is certainly ensured if the delay $\eta_{\phi_i}(t)$ is a constant.

3. Main results

In this section, a control scheme is developed to synchronize a delayed complex network with non-identical nodes to any smooth dynamics $\boldsymbol{s}(t)$. Let synchronization errors be $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - p_{\phi_i}\boldsymbol{s}(t)$ for $i = 1, 2, \cdots, N$, according to system (1), the error dynamical system can be derived as

$$\begin{aligned}
\dot{\boldsymbol{e}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t, \boldsymbol{x}_i(t)) - p_{\phi_i}\dot{\boldsymbol{s}}(t) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t) \\
& + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\eta_{\phi_i}(t)) \\
& + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1 p_{\phi_i}\boldsymbol{s}(t) \\
& + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2 p_{\phi_i}\boldsymbol{s}(t-\eta_{\phi_i}(t)) + \boldsymbol{u}_i(t), \\
& \phi_i(t) \in \bar{J}_{\phi_i}, \\
\dot{\boldsymbol{e}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t, \boldsymbol{x}_i(t)) - p_{\phi_i}\dot{\boldsymbol{s}}(t) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t) \\
& + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\eta_{\phi_i}(t)) + \boldsymbol{v}_i(t), \\
& \phi_i(t) \in J_{\phi_i} - \bar{J}_{\phi_i}. \qquad (4)
\end{aligned}$$

According to the diffusive coupling condition (2) of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we have

$$\begin{aligned}
& c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1 p_{\phi_i}\boldsymbol{s}(t) + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2 p_{\phi_i}\boldsymbol{s}(t-\eta_{\phi_i}(t)) = 0, \\
& \quad i \in J_{\phi_i} - \bar{J}_{\phi_i}. \qquad (5)
\end{aligned}$$

It is easy to see that the synchronization of the controlled complex network (1) is achieved if the zero solution of the error system (4) is globally asymptotically stable, which is ensured by the following theorem.

Let $\tilde{\boldsymbol{e}}_j(t) = (e_{1j}(t), e_{2j}(t), \ldots, e_{Nj}(t))^{\mathrm{T}}$, then we have the following result.

Theorem 1 Suppose Assumption 1 hold. On the basis of property (5), for achieving the cluster projective synchronization, we design the controllers as follows:

$$\begin{aligned}
\boldsymbol{u}_i(t) =\ & -c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1 p_{\phi_i}\boldsymbol{s}(t) \\
& - c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2 p_{\phi_i}\boldsymbol{s}(t-\eta_{\phi_i}(t)) + \boldsymbol{v}_i(t), \\
& i \in \bar{J}_{\phi_i}, \\
\boldsymbol{v}_i(t) =\ & -\boldsymbol{f}_{\phi_i}(t, \boldsymbol{x}_i(t)) + p_{\phi_i}\boldsymbol{f}_{\phi_i}(t, \boldsymbol{s}(t)) \\
& + \boldsymbol{F}_{\phi_i}(t) - d_i\boldsymbol{e}_i(t), \quad i \in J_{\phi_i} - \bar{J}_{\phi_i}. \qquad (6)
\end{aligned}$$

By $\boldsymbol{F}_{\phi_i}(t) = p_{\phi_i}\dot{\boldsymbol{s}}(t) - p_{\phi_i}\boldsymbol{f}_{\phi_i}(t, \boldsymbol{s}(t))$, we have

$$\begin{aligned}
\boldsymbol{u}_i(t) =\ & -c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1 p_{\phi_i}\boldsymbol{s}(t) \\
& - c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2 p_{\phi_i}\boldsymbol{s}(t-\eta_{\phi_i}(t)) \\
& + \boldsymbol{v}_i(t), \quad i \in \bar{J}_{\phi_i}, \\
\boldsymbol{v}_i(t) =\ & -\boldsymbol{f}_{\phi_i}(t, \boldsymbol{x}_i(t)) + p_{\phi_i}\dot{\boldsymbol{s}}(t) - d_i\boldsymbol{e}_i(t), \\
& i \in J_{\phi_i} - \bar{J}_{\phi_i}, \qquad (7)
\end{aligned}$$

$$\dot{d}_i = k_i\boldsymbol{e}_i^{\mathrm{T}}(t)\,\boldsymbol{e}_i(t), \qquad (8)$$

where d_i are the feedback strengths, and k_i are arbitrary positive constants. Then system (3) can realize the cluster projective synchronization.

Proof Construct the following Lyapunov functional:

$$\begin{aligned}
V(t) =\ & \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\
& + \frac{c}{2(1-\varepsilon)}\int_{t-\eta_{\phi_i}(t)}^{\mathrm{t}} \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(\xi)\boldsymbol{e}_i(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{2}\sum_{i=1}^{N} \frac{(d_i - d^*)^2}{k_i},
\end{aligned}$$

where d^* is a sufficiently large positive constant to be determined, Calculating the derivative of $V(t)$, we have

$$\begin{aligned}
\dot{V}(t) =\ & \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\left[c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\right. \\
& \left. + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\eta_{\phi_i}(t)) - d_i\boldsymbol{e}_i(t)\right] \\
& + \frac{c}{2(1-\varepsilon)}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\
& - c\frac{1-\dot{\eta}_{\phi_i}(t)}{2(1-\varepsilon)}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t-\eta_{\phi_i}(t))\boldsymbol{e}_i(t-\eta_{\phi_i}(t)) \\
& + \sum_{i=1}^{N} (d_i - d^*)\boldsymbol{e}_i^{\mathrm{T}}(t)\,\boldsymbol{e}_i(t).
\end{aligned}$$

By Assumption 1, we have

$$\begin{aligned}\dot{V}(t) \le\ & c\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\\ &+ c\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\eta_{\phi_i}(t))\\ &+ \frac{c}{2(1-\varepsilon)}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\\ &- \frac{c}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\eta_{\phi_i}(t))\boldsymbol{e}_i(t-\eta_{\phi_i}(t))\\ &+ \sum_{i=1}^{N} d^*\boldsymbol{e}_i^{\mathrm{T}}(t)\,\boldsymbol{e}_i(t)\\ \le\ & c\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{A}\otimes\boldsymbol{\Gamma}_1)\boldsymbol{e} + c\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{B}\otimes\boldsymbol{\Gamma}_2)\boldsymbol{e}(t-\eta_{\phi_i}(t))\\ &+ \frac{c}{2(1-\varepsilon)}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\\ &- \frac{c}{2}\boldsymbol{e}^{\mathrm{T}}(t-\eta_{\phi_i}(t))\boldsymbol{e}(t-\eta_{\phi_i}(t)) - d^*\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t).\end{aligned}$$

Let $\boldsymbol{e}(t)=(\boldsymbol{e}_1^{\mathrm{T}}(t),\boldsymbol{e}_2^{\mathrm{T}}(t),...,\boldsymbol{e}_N^{\mathrm{T}}(t))^{\mathrm{T}}\in\mathbf{R}^{nN}$, $\boldsymbol{Q}=(\boldsymbol{A}\otimes\boldsymbol{\Gamma}_1)$, $\boldsymbol{P}=(\boldsymbol{B}\otimes\boldsymbol{\Gamma}_2)$, where $\otimes$ represents the Kronecker product. Then

$$\begin{aligned}\dot{V}(t) \le\ & c\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}(t) + c\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t-\eta_{\phi_i}(t))\\ &+ \frac{c}{2(1-\varepsilon)}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\\ &- \frac{c}{2}\boldsymbol{e}^{\mathrm{T}}(t-\eta_{\phi_i}(t))\boldsymbol{e}(t-\eta_{\phi_i}(t))\\ &- d^*\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t).\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}\dot{V}(t) \le\ & c\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}(t) + \frac{c}{2}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}\boldsymbol{e}(t)\\ &+ \frac{c}{2(1-\varepsilon)}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t) - d^*\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\\ \le\ & \left(c\lambda_{\max}(\boldsymbol{Q}) + \frac{1}{2}c\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}}) + \frac{c}{2(1-\varepsilon)} - d^*\right)\\ &\times \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t).\end{aligned}$$

Choose $d^* \ge c\lambda_{\max}(\boldsymbol{Q}) + c\lambda_{\max}(\boldsymbol{P}\boldsymbol{P}^{\mathrm{T}})/2 + c/[2(1-\varepsilon)]+1$. Thus, we obtain $\dot{V}(t)\le -\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\le 0$. It is obvious that the largest invariant set contained in set $\boldsymbol{E}=\{\dot{V}(t)=0\}=\{\boldsymbol{e}_i(t)=0, i=1,2,\ldots,N\}$ is $\boldsymbol{Q}=\{\boldsymbol{e}_i(t)=0, d_i(t)=d^*, i=1,2,\ldots,N\}$. Based on the LaSalle invariance principle,[30] starting with any initial values of system (4), the trajectory asymptotically converges to the largest invariant $\boldsymbol{Q}$, which implies that $\lim_{t\to\infty}\|\boldsymbol{e}_i(t)\|=0$ for $i=1,2,\ldots,N$. Therefore, the cluster projective synchronization in network (3) is achieved. This completes the proof.

Corollary 1 When $\boldsymbol{A}=0$, network (1) is translated into

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t)),\quad i=1,2,\ldots,N. \tag{9}$$

We design the controllers as follows:

$$\begin{aligned}\boldsymbol{u}_i(t) &= -c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2 p_{\phi_i}\boldsymbol{s}(t-\eta_{\phi_i}(t)) + \boldsymbol{v}_i(t),\\ &\quad i\in\bar{J}_{\phi_i},\\ \boldsymbol{v}_i(t) &= -\boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + p_{\phi_i}\dot{\boldsymbol{s}}(t) - d_i\boldsymbol{e}_i(t),\\ &\quad i\in J_{\phi_i}-\bar{J}_{\phi_i},\end{aligned} \tag{10}$$

then the complex networks can also achieve the synchronization.

Corollary 2 When $\boldsymbol{B}=0$, network (1) is translated into

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t),\quad i=1,2,\ldots,N. \tag{11}$$

We design the controllers as follows:

$$\begin{aligned}\boldsymbol{u}_i(t) &= -c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1 p_{\phi_i}s(t) + \boldsymbol{v}_i(t),\\ &\quad i\in\bar{J}_{\phi_i},\\ \boldsymbol{v}_i(t) &= -\boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + p_{\phi_i}\dot{s}(t) - d_i\boldsymbol{e}_i(t),\\ &\quad i\in J_{\phi_i}-\bar{J}_{\phi_i},\end{aligned} \tag{12}$$

then the complex networks can also achieve the synchronization.

4. Illustrative example

In this section, a numerical example will be given to demonstrate the validity of the synchronization criteria obtained in the previous sections. The node dynamics are studied in the following well-known modified Chua's circuit[31] with different system parameters. Consider the following network:

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) =\ & \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &+ c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t)) + \boldsymbol{u}_i(t),\\ &\phi_i(t)\in\bar{J}_{\phi_i},\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{f}_{\phi_i}(t,\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\\ &\quad + c\sum_{j=1}^{N} b_{ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\eta_{\phi_i}(t)) + \boldsymbol{v}_i(t),\\ &\quad \phi_i(t)\in J_{\phi_i} - \bar{J}_{\phi_i},\end{aligned}$$

where $i = 1, 2, \ldots, N$, $\boldsymbol{x}_i(t) = (x_{1i}(t), x_{2i}(t), x_{3i}(t))^{\mathrm{T}}$, $\boldsymbol{f}_1(t,\boldsymbol{x}_i(t)) = \boldsymbol{D}_1\boldsymbol{x}_i(t) + \boldsymbol{h}_1(\boldsymbol{x}_i(t))$, $\boldsymbol{f}_2(t,\boldsymbol{x}_i(t)) = \boldsymbol{D}_2\boldsymbol{x}_i(t) + \boldsymbol{h}_2(\boldsymbol{x}_i(t))$, $\boldsymbol{f}_3(t,\boldsymbol{x}_i(t)) = \boldsymbol{D}_3\boldsymbol{x}_i(t) + \boldsymbol{h}_3(\boldsymbol{x}_i(t))$, $\boldsymbol{h}_1(\boldsymbol{x}_i) = \boldsymbol{h}_2(\boldsymbol{x}_i) = \boldsymbol{h}_3(\boldsymbol{x}_i) = (-20\boldsymbol{x}_{i1}^3/7, 0, 0)^{\mathrm{T}}$, $\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2 = \mathrm{Diag}\{1,1,1\}$, $c = 0.02$, $\eta_1(t) = e^t/(1+e^t)$, $\eta_2(t) = 2\,e^t/(1+e^t)$, $\eta_2(t) = 1.1\,e^t/(1+e^t)$, $\boldsymbol{D}_1 = \begin{bmatrix} -10/7 & 10 & 0\\ 1 & -1 & 1\\ 0 & -12 & 0\end{bmatrix}$, $\boldsymbol{D}_2 = \begin{bmatrix} -10/7 & 10 & 0\\ 1 & -1 & 1\\ 0 & -13 & 0\end{bmatrix}$, $\boldsymbol{D}_3 = \begin{bmatrix} -10/7 & 10 & 0\\ 1 & -1 & 1\\ 0 & -14 & 0\end{bmatrix}$, and $d = 15$. For simplicity, we analyze the networks with 16 nodes, namely $N = 16$. We choose the coupling matrices as

$$\boldsymbol{A} = \boldsymbol{B} = \begin{bmatrix}
-7 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & -5 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & -6 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 1 & 1 & -8 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 0 & -5 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 0 & -6 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 1 & 1 & 1 & 0 & -9 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0\\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -5 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -6 & 0 & 1 & 1 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & -7 & 0 & 0 & 1 & 1 & 0\\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & -8 & 0 & 1 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & -5 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 0 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -6 & 0\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & -9
\end{bmatrix}_{16\times 16}.$$

The following quantities are utilized to measure the process of cluster projective synchronization:

$$\begin{aligned}E(t) &= \|\boldsymbol{x}_u(t) - p_1\boldsymbol{s}(t)\| + \|\boldsymbol{x}_v(t) - p_2\boldsymbol{s}(t)\|\\ &\quad + \|\boldsymbol{x}_w(t) - p_3\boldsymbol{s}(t)\|, u\in C_1, v\in C_2, w\in C_3,\\ E_{12}(t) &= \left\|\boldsymbol{x}_u(t) - \frac{p_1}{p_2}\boldsymbol{x}_v(t)\right\|, u\in C_1, v\in C_2,\\ E_{13}(t) &= \left\|\boldsymbol{x}_u(t) - \frac{p_1}{p_3}\boldsymbol{x}_w(t)\right\|, u\in C_1, w\in C_3,\\ E_{23}(t) &= \left\|\boldsymbol{x}_v(t) - \frac{p_2}{p_3}\boldsymbol{x}_w(t)\right\|, v\in C_2, w\in C_3, \quad (13)\end{aligned}$$

where $\{C_1, C_2, C_3\}$ denote three communities of the networks, $\cup_{i=1}^{3} C_i = \{1, 2, \ldots N\} = 16$, $E(t)$ denotes the total error of the projective synchronization of the nodes in the three communities, $E_{12}(t)$, $E_{13}(t)$, and $E_{23}(t)$ denote that there are the projective synchronization between nodes in different communities. The cluster projective synchronization is achieved if the synchronization errors $E(t)$, $E_{12}(t)$, $E_{13}(t)$, and $E_{23}(t)$ converge to zero as $t\to\infty$. Let $p_1 = -2$, $p_2 = 3$, $p_3 = 4$. The simulation results are given in Figs. 1–4. Figure 1 shows the evolution of synchronization total errors of the nodes in the three communities. Figures 2–4 show the evolution of synchronization errors between the nodes in different communities.

Remark 1 We investigate a new cluster projective synchronization scheme in the time-varying delay coupled complex dynamical networks with nonidentical nodes. Based on the community structure of the networks, the controllers are designed to ensure that the nodes in the same group projective synchronize with each other and there is also projective synchronization between nodes in different groups.

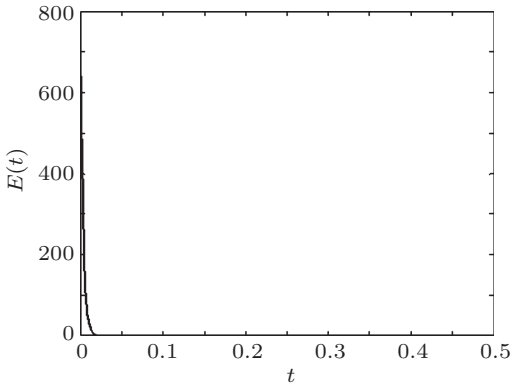


Fig. 1. Time evolution of synchronization error $E(t)$.

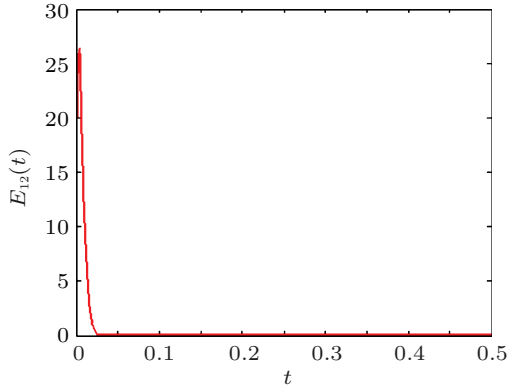


Fig. 2. (colour online) Time evolution of synchronization error $E_{12}(t)$.

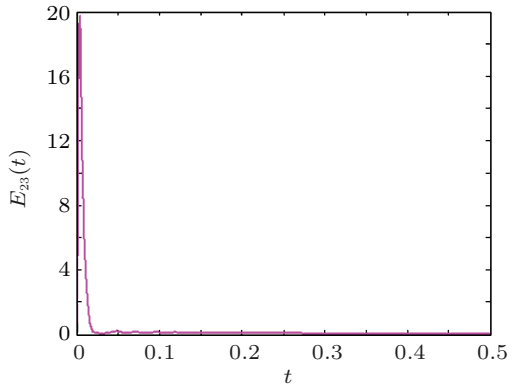


Fig. 3. (colour online) Time evolution of synchronization error $E_{23}(t)$.

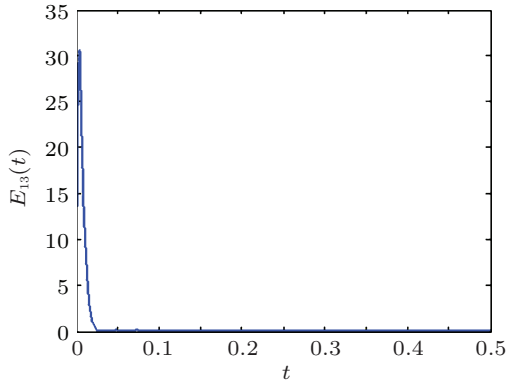


Fig. 4. (colour online) Time evolution of synchronization error $E_{13}(t)$.

Remark 2 In this paper, the weight configuration matrix is not assumed to be symmetric or irreducible, and the synchronization criteria are independent of the time delay.

5. Conclusion

The cluster projective synchronization in community networks has been studied in this paper. Based on the community structure of the networks, some simple and useful criteria are derived to ensure that the nodes in the same group projective synchronize and there is also projective synchronization between nodes in different groups. Particularly, the weight configuration matrix is not assumed to be symmetric or irreducible, and the synchronization criteria are independent of the time delay. Finally, the developed techniques are applied to three complex community networks. The numerical simulations are performed to verify the effectiveness of the theoretical results.

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