Enhancement of density divergence in an insect outbreak model driven by colored noise*

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The steady states and the transient properties of an insect outbreak model driven by Gaussian colored noise are studied in this paper. According to the Fokker–Planck equation in the unified colored-noise approximation, we analyse the stationary probability distribution and the mean first-passage time of this model. By numerical analysis, the effects of the self-correlation time of insect birth rate and predation rate respectively reveal a manifest population divergence on the insect density. The decrease of the mean first-passage time indicates an enhancement dynamic on the density divergency with colored noise of a large self-correlation time based on the insect outbreak model.

Keywords: colored noise, insect outbreak model, stationary probability distribution function, mean first-passage time

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1. Introduction

In the past decades, nonlinear systems subjected to noisy disturbance have been extensively studied in both theory and experiment. Noise plays an important role in many nonlinear system phenomena, such as noise-induced transition[1,2] and stochastic resonance,[3,4] which are widely present in a large variety of systems, such as biological systems,[1−8] laser systems,[9−12] bistable kinetic systems,[13−19] and so on. Especially, noise-induced stochastic resonance can explain many complicated biological phenomena.[4] In recent years, the logistic biological growth equation was used to investigate different aspects of the noisy effect and has become a hot topic in biological dynamic systems. Ai *et al.*[2] introduced noise into an insect outbreak model and found that multiplicative noise can make insects go to extinction. They only studied the steady state of the insect growth but did not consider the system evolution with time. The analysis on the evolution of the system can more comprehensively show the effect of the noise. The mean first-passage time (MFPT) is a useful physical quantity to reveal the dynamic properties of the system,[16,17,20−22,24] so the study of MFPT by using the insect outbreak model driven by Gaussian colored noise is naturally an important aspect for the transient behaviors. However, in the insect outbreak model, the noises considered in all the previous studies are ideal Gaussian white noises or partial Gaussian colored noises, while, in a real biological system, the memory should be particularly considered, and a full colored noise is more significant. In this paper, we study the steady state and the transient properties of the insect outbreak model driven by Gaussian colored noises in the insect birth rate and the predation rate.

2. Insect outbreak model

The deterministic equation of insect growth can be described by the following differential equation, named the insect outbreak model:[2]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h_0(x) = rx\left(1 - \frac{x}{q}\right) - \frac{\beta x^2}{1+x^2}. \qquad (1)$$

Here, x is the insect density, r is the insect birth rate, q is the environmental capacity, and β is the predation rate. The corresponding potential function to Eq. (1)

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is

$$U(x) = \frac{1}{3}\frac{rx^3}{q} - \frac{1}{2}rx^2 + \beta x - \beta \arctan(x). \quad (2)$$

Under the condition $x > 0$, Eq. (2) has two stable states and one unstable state, which are shown in Fig. 1. Without any fluctuation, the density distribution of the ideal model can be expressed as

$$P(x) \propto \mathrm{e}^{-U(x)}, \quad (3)$$

which has two peaks near $x \approx 0.7283$ and $x \approx 7.2939$ directly corresponding to the two minima of the potential function for static behavior.

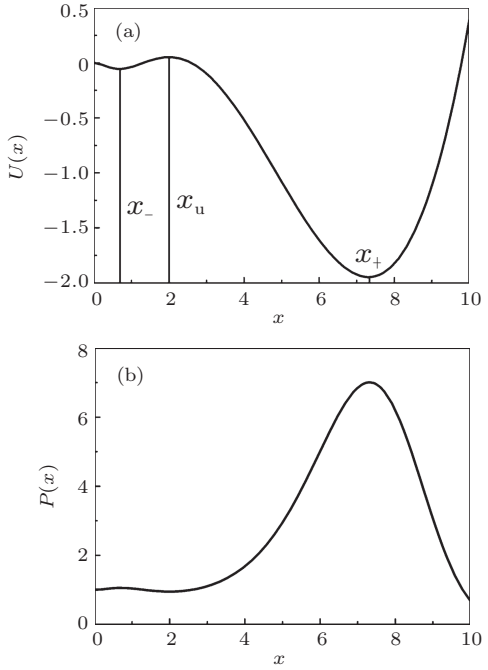


Fig. 1. The bistable potential $U(x)$ (a) and the corresponding un-normalized density distribution $P(x)$ (b). The parameters are $r = 1.0$, $\beta = 2.0$, and $q = 10.0$. The two stable states are $x_- \approx 0.7283$ and $x_+ \approx 7.2939$, and the unstable state is $x_\mathrm{u} \approx 2.0937$.

Equation (1) is an ideal case without considering any disturbance of fluctuation, but an actual environment may change the growth of the insect, such as temperature, climate, predation of birds, and so on, all of which influence the population of insects and bring noise. When fluctuations of birth rate and predation rate are considered, Eq. (1) should be written into a stochastic differential equation as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = [r + \Gamma(t)]\, x\left(1 - \frac{x}{q}\right) - [\beta + \xi(t)]\,\frac{x^2}{1+x^2}. \quad (4)$$

Here $\Gamma(t)$ and $\xi(t)$ are Gaussian colored noise with the following statistical properties:

$$\langle \Gamma(t) \rangle = \langle \xi(t) \rangle = 0, \quad (5)$$

$$\langle \Gamma(t)\Gamma(t') \rangle = \frac{D}{\tau_1}\exp\left(-\frac{1}{\tau_1}|t-t'|\right), \quad (6)$$

$$\langle \xi(t)\xi(t') \rangle = \frac{\alpha}{\tau_2}\exp\left(-\frac{1}{\tau_2}|t-t'|\right), \quad (7)$$

where D and α are the strengths of noise for $\Gamma(t)$ and $\xi(t)$, t and t' denote two different moments, τ_1 and τ_2 are the self-correlation time of $\Gamma(t)$ and $\xi(t)$, respectively. According to the unified colored-noise approximation method,[14] we can obtain an approximate Fokker–Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}A(x)P(x,t) + \frac{\partial^2}{\partial x^2}B(x)P(x,t), \quad (8)$$

where $P(x,t)$ is the probability distribution function and

$$A(x) = h(x) + G'(x)G(x), \quad (9)$$

$$B(x) = [G(x)]^2, \quad (10)$$

$$h(x) = \frac{h_0(x)}{C(\tau,x)}, \quad (11)$$

$$G(x) = \frac{g(x)}{C(\tau,x)}, \quad (12)$$

$$C(\tau,x) = 1 - \tau_1\left[h_0'(x) - \frac{g_1'(x)}{g_1(x)}h_0(x)\right] - \tau_2\left[h_0'(x) - \frac{g_2'(x)}{g_2(x)}h_0(x)\right], \quad (13)$$

$$g(x) = \{D[g_1(x)]^2 + \alpha[g_2(x)]^2\}^{1/2}, \quad (14)$$

$$h_0(x) = rx\left(1 - \frac{x}{q}\right) - \beta\frac{x^2}{1+x^2}, \quad (15)$$

$$g_1(x) = x\left(1 - \frac{x}{q}\right), \quad (16)$$

$$g_2(x) = -\frac{x^2}{1+x^2}. \quad (17)$$

Then the steady-state probability distribution function (SPDF) of Eq. (8) reads

$$P_\mathrm{st}(x) = \frac{N}{B(x)}\exp\left[\int \frac{A(x)}{B(x)}\mathrm{d}x\right], \quad (18)$$

where N is the normalization constant. As $\Gamma(t)$ and $\xi(t)$ are two uncorrelated noises, we can consider the fluctuations of insect birth rate and predation rate separately.

3. Fluctuation of insect birth rate in the model

When the fluctuation of the insect birth rate is considered only, we set $\alpha = 0$ and obtain

$$A(x) = \left[rx\left(1 - \frac{x}{q}\right) - \frac{\beta x^2}{1+x^2}\right]/C(\tau_1, x)$$

$$+\frac{\sqrt{D}x(1-x/q)}{C(\tau_1,x)}\left[\frac{\sqrt{D}x(1-x/q)}{C(\tau_1,x)}\right]', \quad (19)$$

$$B(x) = Dx^2(1-x/q)^2/C^2(\tau_1,x). \quad (20)$$

From Eq. (18), we have the following expression of the SPDF:

$$P_{\rm st}(x) = \frac{N}{B(x)}\exp\left[-\frac{\Phi_1(x)}{D}\right]. \quad (21)$$

In Eq. (21), $\Phi_1(x)$ is the generalized potential function determined by

$$\begin{aligned}\Phi_1(x) =& \frac{(Fx+M)x^2+(F-P)x+M+R}{(1+x^2)^2}\\ &+\frac{2K(x-q)-P(q^2+1)}{2(x-q)^2}+J\ln(x-q)\\ &+N\ln(1+x^2)-(D+r)\ln(x)\\ &+D\ln(Q)+E\arctan(x), \end{aligned}\quad (22)$$

and the specific functions defined in the above equations are

$$\begin{aligned}C(\tau_1,x) =& 1-\tau_1\left\{r-\frac{2rx}{q}-\frac{2\beta x}{1+x^2}\left(\frac{x}{1+x^2}-1\right)\right.\\ &\left.-\frac{q-2x}{q-x}\left[rx\left(1-\frac{x}{q}\right)-\frac{\beta x^2}{1+x^2}\right]\right\},\\ E =& \beta q^2[q^3(q+3/2\beta\tau_1)-1]/(1+q^2)^3,\\ F =& \beta q^2\tau_1[\beta(5q^2-1)-r(1+q^2)^2]/(1+q^2)^3,\\ J =& [(1+q^2)^2(r-2D)-2\beta q^3]/(1+q^2)^2,\\ K =& \beta q^2[(1+q^2)^2(r\tau_1-1)\\ &+\beta q\tau_1(1-q^2)]/(1+q^2)^3,\\ M =& \beta q\tau_1[\beta q^3(q^2-3)+2r(1+q^2)^2]/2(1+q^2)^3,\\ N =& 2-\beta q^3/(1+q^2)^2, \quad P=\beta^2q^4\tau_1/(1+q^2)^2,\\ Q =& (1+x^2)^2(x-q)+\beta q\tau_1 x(x^2-1-2x^3/q),\\ R =& \beta^2q^2\tau_1(1-q^2)/2(1+q^2)^2. \end{aligned}$$

The above analytical results modified by colored noise seem to be complicated, so we choose $D=0.3$, $\beta=2.0$, $r=1.0$, and $q=10.0$ to numerically analyse the steady-state probability distribution (SPD). As shown in Fig. 2, when the noise intensity of the birth rate is fixed, the SPD is sensitive to the self-correlation time τ_1. The two peaks are located at the points of $x_-\approx 0.7283$ and $x_+\approx 7.2939$, but obviously their heights grow with the increase of τ_1. This shows that the increase of memory on the fluctuation of birth rate can reinforce the state of the system and enhance the divergence of the SPD in two distinct population states: small density state and large density state. The population trapping of the insects is clearly improved by the self-correlation time as the widths of both peaks are narrowed.

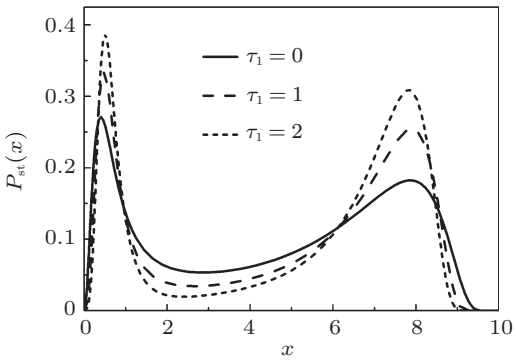


Fig. 2. Plots of $P_{\rm st}(x)$ versus x for different self-correlation time, where $D=0.3$, $r=1.0$, $\beta=2.0$, $q=10.0$.

In order to quantitatively investigate the shift of the steady population, we investigate the mean value of the insect density by $\langle x\rangle=\int_0^\infty xP_{\rm st}(x){\rm d}x$ shown in Fig. 3. When τ_1 is fixed, the insect density decreases with the increase of the noise strength D, which leads to a normal noise-suppress effect. When the noise strength D is fixed, the total insect density grows with τ_1, indicating a significant effect of "memory" on the population scale. Comparing Fig. 3 with Fig. 2, we will find that the mean density of the insects grows with τ_1 in a small insect state and a large insect state. These results imply some new aberrance appearing in the insects with the increase of τ_1 and the number of insects can recover to a certain extent through the "memory" on birth rate.

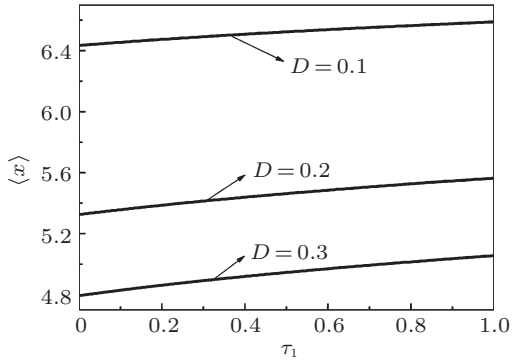


Fig. 3. Plots of $\langle x\rangle$ versus τ_1 for different noise strengths, where $r=1.0$, $\beta=2.0$, $q=10.0$.

For a stochastic dynamic system, we also need to consider the transient properties beyond its steady states by investigating the mean first-passage time (MFPT). The MFPT is the time passing through a potential barrier from one state to the other. The MFPT of a process from the unstable insect population state $x_{\rm u}$ to the large population state x_+ can be given by[16,21,22]

$$T(x_+\to x_{\rm u}) = \int_{x_+}^{x_{\rm u}}\frac{{\rm d}x}{B(x)P_{\rm st}(x)}\int_0^x P_{\rm st}(y){\rm d}y. \quad (23)$$

When the noise intensity is small enough compared with the energy barrier height $\Delta U(x) = U(x_{\rm u}) - U(x_+)$, we can use the steepest-descent approximation in Eq. (23) to simplify T by[23,25]

$$T(x_+ \to x_{\rm u}) \approx \frac{2\pi}{\sqrt{|U''(x_+)U''(x_{\rm u})|}} \exp\left[\frac{\Phi_1(x_{\rm u}) - \Phi_1(x_+)}{D}\right], \quad (24)$$

where the potential $U(x)$ is given by Eq. (2).

Based on Eq. (24), the numerical calculations are given in Fig. 4. When D is constant, T decreases monotonically with τ_1 increasing, which verifies the population divergent process. When τ_1 is fixed, T also decreases with the increase of D. That means the escape rates are speeded up in both cases above. At this point, we can conclude that both D and τ_1 accelerate the divergent process from an unstable population state $x_{\rm u}$ to a large or small insect population state.

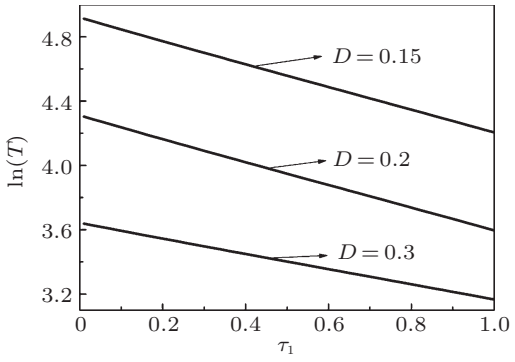


Fig. 4. Plots of the MFPT versus τ_1 for different noise strengths, where $r = 1.0$, $\beta = 2.0$, $q = 10.0$.

4. Fluctuation of predation rate in the model

Similarly, if the fluctuation of predation rate is solely considered, we can set $D = 0$, and obtain

$$A(x) = \left[rx\left(1 - \frac{x}{q}\right) - \frac{\beta x^2}{1+x^2}\right]/C(\tau_2, x) + \frac{\sqrt{\alpha}x^2/(1+x^2)}{C(\tau_2, x)}\left[\frac{\sqrt{\alpha}x^2/(1+x^2)}{C(\tau_2, x)}\right]', \quad (25)$$

$$B(x) = \alpha x^4/(1+x^2)^2/C^2(\tau_2, x). \quad (26)$$

From Eq. (18), we reach the following expression of the SPDF:

$$P_{\rm st}(x) = \frac{N}{B(x)} \exp\left[-\frac{\Phi_2(x)}{\alpha}\right]. \quad (27)$$

In Eq. (27), N is the normalization constant, and $\Phi_2(x)$ is the generalized potential and expressed as

$$\Phi_2(x) = E'x^4 + F'x^3 + J'x^2 + K'x - M'\ln(x) + (N' - P'x)/x^2 + \alpha\ln(Q'). \quad (28)$$

In this case, the auxiliary functions are

$$C(\tau_2, x) = [r\tau_2 x^2(q - 2x) - q(x^2+1) - rq\tau_2]/q(x^2+1),$$
$$E' = r^2\tau_2/2q, \quad F' = r\left(\frac{1}{3} - r\tau_2\right)/q,$$
$$J' = r\left[\frac{1}{2}(r\tau_2 - 1) + \tau_2(\beta + r/q)/q\right],$$
$$K' = (1 - r\tau_2)(2r/q + \beta),$$
$$M' = 2(r + \alpha), \quad N' = r(1 + r\tau_2)/2,$$
$$P' = (r\tau_2 + 1)(r/q + \beta),$$
$$Q' = r\tau_2 x^2(2x - q) + q(1 + x^2) + rq\tau_2.$$

When $\alpha = 0.9$, $\beta = 2.0$, $r = 1.0$, and $q = 10.0$, we investigate the SPD changing with the self-correlation time τ_2. As shown in Fig. 5, when the fluctuating strength in the predation rate is fixed, both the insect population and its corresponding SPD vary with τ_2. The two peaks appearing at $x_- \approx 0.7283$ and $x_+ \approx 7.2939$ still grow with the increase of τ_2. The calculations show that the increase of the self-correlation time in the predation rate also strengthens the state of the system and makes the SPD splitting of insect density become more obvious.

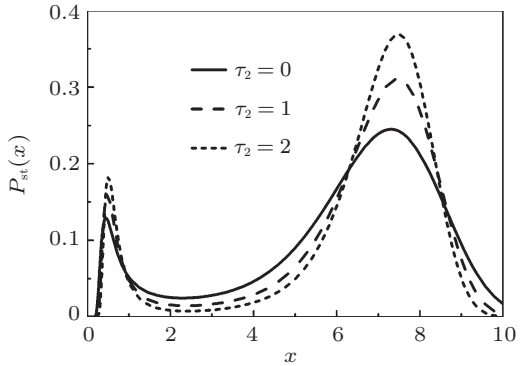


Fig. 5. Plots of $P_{\rm st}(x)$ versus x for different self-correlation times, where $\alpha = 0.9$, $r = 1.0$, $\beta = 2.0$, $q = 10.0$.

The total scale of the insect population with mean density $\langle x\rangle = \int_0^\infty x P_{\rm st}(x){\rm d}x$ is again considered in this case. As shown in Fig. 6, when τ_2 is fixed, the density of the insects decreases with the increase of noise strength α. While when α is fixed, the density of the insects increases with τ_2 increasing. Comparing Fig. 6 with Fig. 5, we still find that the mean density of insects grows with τ_2 increasing in both small insect state and large insect state. Therefore, we can draw

a similar conclusion that a certain new aberrance is present among insects when τ_2 increases to an extent and the scale of the insects can be recovered by the "memory" of predation fluctuation.

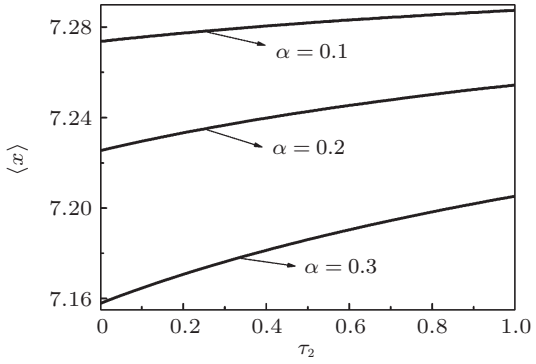


Fig. 6. Plots of $\langle x \rangle$ versus τ_2 for different noise strengths, where $r = 1.0$, $\beta = 2.0$, $q = 10.0$.

Similarly, we calculate the MFPT by

$$T(x_+ \to x_{\rm u}) \approx \frac{2\pi}{\sqrt{|U''(x_+)U''(x_{\rm u})|}} \times \exp\left[\frac{\Phi_2(x_{\rm u}) - \Phi_2(x_+)}{\alpha}\right], \quad (29)$$

and the result is shown in Fig. 7. When the noise strength α is fixed, T decreases monotonically with the increase of τ_2. When τ_2 is fixed, T also decreases with

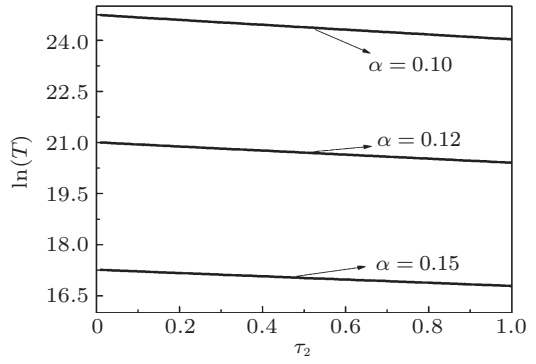


Fig. 7. Plots of the MFPT versus τ_1 for different noise strengths, where $r = 1.0$, $\beta = 2.0$, $q = 10.0$.

the increase of α, and the escape rates are speeded up in two ways. Similarly, we conclude that both α, and τ_2 can accelerate the transition process from unstable states to two different population trapping states in this case.

5. Conclusion

In this paper, we study the steady-state properties and the transient properties of an insect outbreak model driven by Gaussian colored noise. Applying the Fokker–Planck equation with the SPD, we analyse the changes of the SPD with the self-correlation time of the fluctuations of insect birth rate and predation rate at certain values. In the steepest-descent approximation, we obtain the expression of the MFPT, and analyse the influence of the self-correlation time on the MFPT. We conclude that under the influence of the colored noise on the insect birth rate and predation rate, the self-correlation time exhibits an enhancement effect of the population divergence on the insect outbreak model.

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