Effects of error feedback on a nonlinear bistable system with stochastic resonance*

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In this paper, we discuss the effects of error feedback on the output of a nonlinear bistable system with stochastic resonance. The bit error rate is employed to quantify the performance of the system. The theoretical analysis and the numerical simulation are presented. By investigating the performances of the nonlinear systems with different strengths of error feedback, we argue that the presented system may provide guidance for practical nonlinear signal processing.

Keywords: stochastic resonance, nonlinear bistable system, error feedback, bit error rate

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1. Introduction

Nonlinear systems present rich potential for signal processing. In certain nonlinear systems, however, the noise can in fact improve rather than degrade the performance of the systems. This interesting phenomenon is called stochastic resonance (SR).[1,2] Since the concept of SR was originally put forward in 1981,[3–5] it has attracted the attention of many researchers in various fields including biology,[6,7] signal processing,[8–10] chemistry,[11] and other scientific fields.[1,2,12] Particularly, in the field of signal processing, some promising results have been obtained,[13–20] which compensate the deficiency of linear systems. The SR phenomenon has become one interesting research field of nonlinear systems.

One of the earliest systems with the SR is the nonlinear bistable system, and most work has been done based on this system. It is well known that the error feedback plays a significant role in dynamical systems. Systems with error feedback modules have high control accuracies and run stably. However, in existing research work, the SR with error feedback driven by aperiodic signals has not been discussed yet.

In this paper, we investigate a nonlinear bistable dynamical system with error feedback and an aperiodic input signal. We derive the corresponding error feedback Fokker–Planck equation (FPE) by using the dynamical probability density function (PDF). The bit error rate (BER) is employed to measure the performance of the nonlinear bistable system with the bipolar binary signal. The effect of error feedback strength on the BER of the SR system is investigated in detail.

2. System model and its performance measurement

Consider a nonlinear bistable system with the error feedback. With the input signal and the noise, it can be described as

$$\dot{x}(t) = ax(t) - bx^3(t) + c(x(t) - s(t)) + s(t) + \Gamma(t), \tag{1}$$

where a and b are system parameters, $s(t)$ is an aperiodic input signal, $x(t) - s(t)$ is the error of the system output, c is the strength of the feedback, and $\Gamma(t)$ denotes the additive Gaussian white noise having the correlation function

$$\langle \Gamma(t)\Gamma(0)\rangle = 2D\delta(t), \tag{2}$$

with D being the noise intensity.

Assume that the input signal $s(t)$ is a pseudorandom bipolar binary signal, positive and negative levels are denoted by $s_0(t) = +A$ and $s_1(t) = -A$, respectively, $s_0(t)$ and $s_1(t)$ are randomly transmitted in a

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bit duration $T_{\rm b}$, and A is the signal level. With the simplification, Eq. (1) becomes

$$\dot{x}(t) = (a+c)x(t) - bx^3(t) + (1-c)s(t) + \Gamma(t). \quad (3)$$

The corresponding potential function is

$$U(x) = -\frac{a+c}{2}x^2 + \frac{b}{4}x^4 \mp (1-c)Ax. \quad (4)$$

From Eq. (4), we find that the error feedback strength c affects the system parameters and the effective signal amplitude. The variation of the strength c has a great effect on the input signal. With the increase of parameter c (when $0 < c < 1$), the effective signal amplitude is reduced, and this effect is equivalent to the increase of the noise intensity.

Equation (1) is a bistable system with error feedback. Its statistically equivalent description for the corresponding dynamical probability density $\rho(x,t|\pm A)$ is governed by the FPE[21]

$$\frac{\partial \rho(x,t|\pm A)}{\partial t} = -\frac{\partial}{\partial x}[U'(x)\rho(x,t|\pm A)] + D\frac{\partial^2}{\partial x^2}[\rho(x,t|\pm A)], \quad (5)$$

where the dynamical PDF $\rho(x,t|\pm A)$ can be approximately calculated from the FPE, and the approach of calculating $\rho(x,t|\pm A)$ without the error feedback has been discussed in Ref. [22]. The derivation of the dynamical PDF $\rho(x,t|\pm A)$ from Eq. (5) is given in Appendix A.

Since the input signal $s(t)$ is a bipolar binary one, we employ the BER to measure the performance of the nonlinear system in Eq. (1). Assume that the signal levels $s_0(t) = +A$ and $s_1(t) = -A$ are statistically independent and transmitted equiprobably ($P_0 = P_1 = \frac{1}{2}$). The BER can be expressed as

$$\begin{aligned} \text{BER} &= P_0 P(A|-A) + P_1 P(-A|A) \\ &= \frac{1}{2}[P(A|-A) + P(-A|A)], \end{aligned} \quad (6)$$

where $P(A|-A)$ is the error probability when the decision is A but the transmitted is $-A$, and $P(-A|A)$ is the error probability when the decision is $-A$ but the transmitted is A. According to the maximum likelihood ratio decision rule,[23] the optimal decision threshold for the bipolar binary signal is 0. Assume the current symbol is $+A$, if the number of the successive amplitudes with the same symbol is K (K = 0,1,...) and the amplitude of the (K+1)-st symbol (after $(K+1)T_{\rm b}$ time interval) turns $-A$, the probability of error detection for the i-th ($0 < i \leq K+1$) symbol is

$$P_i(-A|A) = \int_{-\infty}^{0} \rho(x, iT_{\rm b}|A){\rm d}x. \quad (7)$$

Since the dynamical PDF is symmetric, we have $P_i(A|-A) = P_i(-A|A)$, and Eq. (6) can be expressed as

$$\begin{aligned} \text{BER} &= \frac{1}{2}\int_{-\infty}^{0} \rho(x, T_{\rm b}|A){\rm d}x \\ &\quad + \frac{1}{4}\int_{-\infty}^{0} \rho(x, 2T_{\rm b}|A){\rm d}x + \cdots \\ &= \sum_{i=1}^{+\infty} \frac{1}{2^i}\int_{-\infty}^{0} \rho(x, i\cdot T_{\rm b}|A){\rm d}x, \end{aligned} \quad (8)$$

where $1/2^i$ is the probability of occurrence of the successive amplitudes with the same symbol. In the simulation, we take $i = 1, 2, 3$, because the prior probability and the probability of error detection with $i > 3$ are small and the corresponding terms can be neglected.

3. Effects of error feedback on a nonlinear bistable system with SR

In this section, we will discuss the effect of error feedback strength c on the BER of the SR system with theoretical calculation and simulation analysis. First, we discuss the situation with $c > 0$ (when $c = 1$, the effect is equivalent to no input signal).

Without loss of generality, assume $a = 10$, $b = 100$, $A = 1$, and $T_{\rm b} = 1$. Figure 1 shows the BER versus the noise intensity D with different c. Lines are theoretical calculation results, where black solid, blue dot-dashed, red dashed, and green dotted lines correspond to $c = 0.1$, $c = 0.3$, $c = 0.5$, and $c = 0.7$, respectively. From Fig. 1, we can see the phenomenon of SR becomes more obvious with the decrease of c, and the BER at the corresponding noise resonance level decreases. Moreover, we find that the noise resonance levels are 0.15, 0.175, 0.2, and 0.225 corresponding to $c = 0.1, 0.3, 0.5, 0.7$, respectively, which shows that the noise resonance level increases with the increase of c, the reason may be found in Fig. 2, where the corresponding potential functions are plotted with the input signal. As shown in Fig. 2, the left potential

barrier of $U(x)$ increases with the increase of c. As the error feedback strength c increases, the left potential barrier becomes deeper, the noise intensity to produce the SR increases. Consequently, the noise resonance level increases with the increasing c.

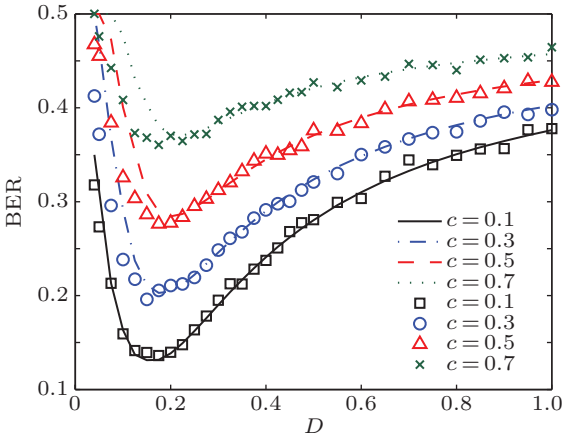


Fig. 1. (colour online) The BER versus noise intensity D with $a = 10$, $b = 100$, $A = 1$, $T_b = 1$, and different values of c ($0 < c < 1$). Lines are the theoretical results, and numerical results are presented with data points.

The simulation is performed to validate the theoretical calculation. Figure 3 describes the block schematic diagram of the numerical simulation for calculating the BER, where $f(x) = -x^3$. The simulation can be implemented by using MATLAB software .

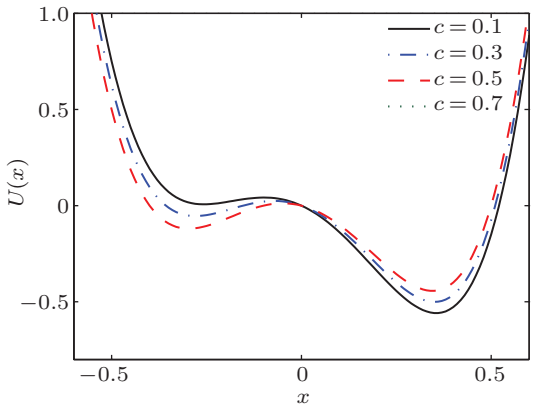


Fig. 2. (colour online) The potential function $U(x)$ with $a = 10$, $b = 100$, $A = 1$, $T_b = 1$, and different values of c ($0 < c < 1$).

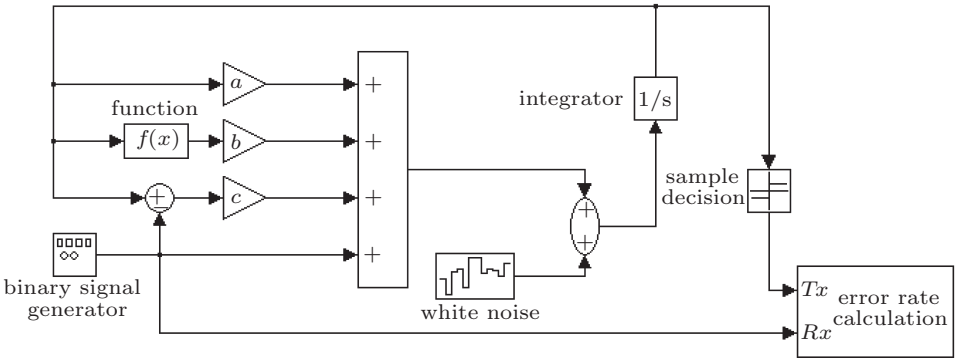


Fig. 3. Block schematic diagram for calculating the BER.

In the simulation, the input signal is a bipolar binary one with randomly transmitted levels $+A$ and $-A$. All the parameters are the same as those in the theoretical calculation. The input noise is modeled as the Gaussian white noise with noise rms amplitude σ. The relationship between noise intensity D and σ is[23]

$$2D = \sigma^2 \Delta T, \tag{9}$$

where ΔT is the sampling time. In the numerical simulation, ΔT is taken to be 1 ms, which satisfies $\Delta T \ll T_b$. Figure 1 also plots BERs obtained from the numerical simulation. It is seen from Fig. 1 that the theoretical and the numerical results are well consistent, except those with small D. The reason is that the system respond speed is very small with small D, which leads to large errors in the calculation of the PDFs and the theoretical BERs due to the finite terms being kept.

The above discussion is mainly about the situation with $0 < c < 1$. In the following, we investigate the effect of the strength of error feedback on the performance of the system when $c > 1$. From Eq. (4), we can see that the polarity of the input signal is opposite when $c > 1$. Figure 4 shows the relation between the BER and the noise intensity. Lines denote the theoretical calculation results, and the numerical simulation results are shown with the data points.

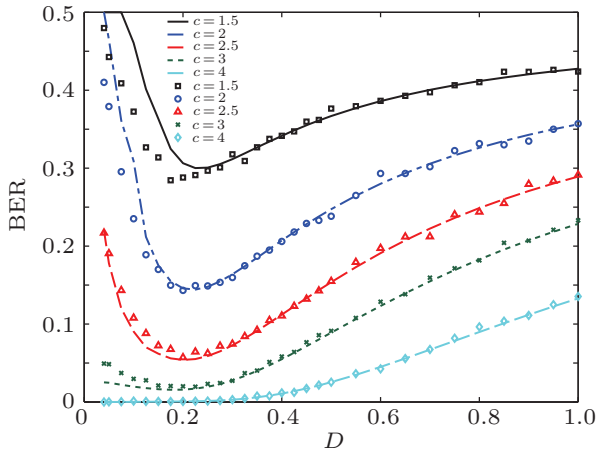


Fig. 4. (colour online) The BER versus noise intensity D with $a = 10$, $b = 100$, $A = 1$, $T_b = 1$, and different values of c ($c > 1$). Lines are the theoretical results, and numerical results are presented with data points.

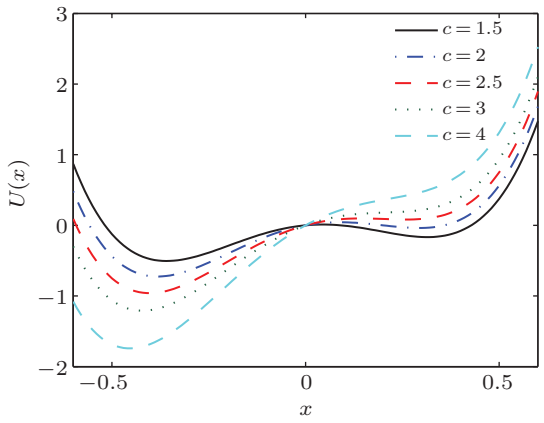


Fig. 5. (colour online) The potential function $U(x)$ with $a = 10$, $b = 100$, A=1, $T_b = 1$, and different values of c ($c > 1$).

As presented in Fig. 4, the BER at the corresponding noise resonance level decreases with the increasing c. The reason is that the intensity of the input signal increases with the increasing c although the polarity of the input signal is opposite. Moreover, we find that the noise resonance levels are 0.225, 0.225, 0.2, and 0.175 corresponding to $c = 1.5$, 2, 2.5, 3, respectively, which is opposite to the situation for $0 < c < 1$. From Fig. 5, we can see that the right potential barrier of $U(x)$ decreases with the increase of c, so the noise intensity to produce the SR decreases with the increase of c. We also find that the phenomenon of SR gradually disappears with the increase of c in Fig. 5. Especially when $c = 4$, the phenomenon of SR totally disappears. The reason is that when

$$|(1-c)A| \geq \sqrt{\frac{4(a+c)^3}{27b}}, \tag{10}$$

the system bistability is destroyed, and system (1) is monostable (the potential function has only one well). In this case, the input signal is suprathreshold,[24] and the noise does not play a constructive role. As c increases, the right potential barrier becomes shallow. When condition (10) is satisfied, the right potential barrier disappears, and the system becomes monostable.

In the remaining part of this section, we discuss the case of $c < 0$. The input signal, the noise, and the parameters are the same as those in the situation with $c > 0$. Figure 6 shows the BER versus the noise intensity D with different c. Lines denote the theoretical calculation results, and the numerical simulation results are shown with the data points.

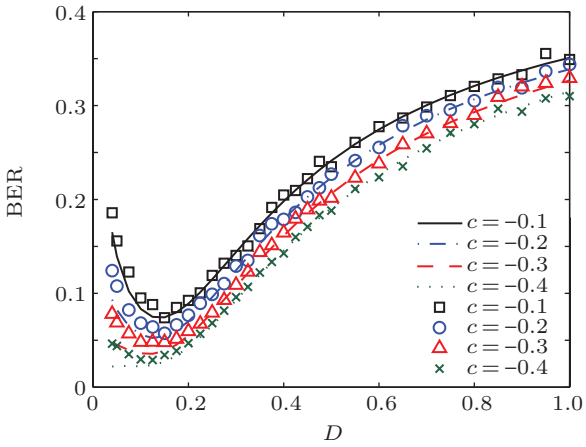


Fig. 6. (colour online) The BER versus noise intensity D with $a = 10$, $b = 100$, $A = 1$, $T_b = 1$, and different values of c ($c < 0$). Lines are the theoretical results, and numerical results are presented with data points.

From Fig. 6, we find that the BER at the corresponding noise resonance level decreases with the increasing $|c|$, the reason is that the effective signal amplitude is amplified with the increase of $|c|$. Moreover, it is seen from Fig. 6 that the phenomenon of SR gradually disappears. When c reaches -0.4, the nonlinear system does not show the phenomenon of SR. The reason is that system parameters a, b and error feedback strength c satisfy condition (10), the left potential barrier totally disappears. Figure 7 shows us the corresponding potential functions. We can see that the left potential barrier disappears in the situations of $c = -0.2$, $c = -0.3$, and $c = -0.4$, but when $c = -0.2$ and $c = -0.3$, the residual SR phenomenon still present,[25] it only occurs $|(1-c)A|$ little greater than $\sqrt{4(a+c)^3/27b}$.

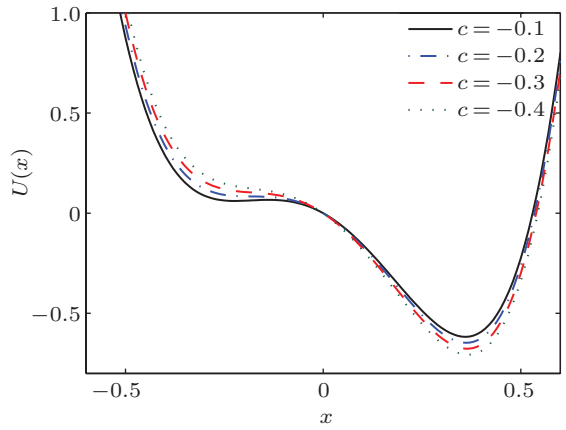


Fig. 7. (colour online) The potential function $U(x)$ with $a = 10$, $b = 100$, A=1, $T_b = 1$, and different values of c ($c < 0$).

4. Conclusion

In this paper, we discuss the effects of the error feedback strength on the SR in detail. For an input bipolar binary signal, the BER is employed to measure the performance of the nonlinear system. We have theoretically and numerically evaluated the results of the BER, and the figures show that the theoretical and the numerical results are matched quite well, except for those with small noise intensities.

For a nonlinear bistable system with error feedback, the performance of the system output and the noise resonance level varies with error feedback strength c. In the case of $c > 0$, with the increasing c, the performance of the nonlinear system decreases, and the noise resonance level increases. When the strength c is greater than the critical point $c = 1$, the polarity of the input signal turns opposite, the BER and the noise resonance level both decrease with the increase of c. In the case of $c < 0$, the BER of the system and the noise resonance level both decrease with the increasing $|c|$.

The variation of c may break the bistability of the nonlinear system, and the system becomes monostable. As a result, the system does not produce the phenomenon of SR. Tuning the system parameters may be a good choice to realize SR.

Appendix A: Derivation of the dynamical probability density function

Assume a solution of Eq. (5) as

$$\rho(x,t|\pm A) = \psi(x,t)\exp\left[\frac{1}{2D}U(x)\right]. \tag{A1}$$

Substituting the above solution into Eq. (5) gives

$$\begin{aligned}&\frac{1}{D}\frac{\partial\psi(x,t)}{\partial t}\\ &= \frac{\partial^2\psi(x,t)}{\partial x^2} - \left[\frac{1}{2D}U''(x) + \frac{1}{4D^2}U'^2(x)\right]\psi(x,t).\end{aligned} \tag{A2}$$

Define $\psi(x,t)$ as

$$\psi(x,t) = \phi(x)\exp(\lambda t). \tag{A3}$$

Putting $\psi(x,t)$ into Eq. (A2) yields

$$\begin{aligned}&-\lambda\frac{\phi(x)}{D}\\ &= \phi''(x) - \left[\frac{1}{2D}U''(x) + \frac{1}{4D^2}U'^2(x)\right]\phi(x).\end{aligned} \tag{A4}$$

Equation (A4) gives an eigenvalue problem. The right-hand side of Eq. (A4) can be viewed as a self-conjugate differential operator acting on $\psi(x,t)$, so the eigenvalues of Eq. (A4) are real, and the eigenfunctions must be orthogonal to each other. By multiplying both sides of Eq. (A4) by $\phi(x)$ and integrating it from minus infinity to infinity with respect to $\lim\limits_{x\to\pm\infty}\phi(x) = \lim\limits_{x\to\pm\infty}\phi'(x) = 0$, the equation can be transformed into the Rayleigh quotient form, i.e.,

$$\lambda = \underset{\phi\neq 0}{\text{st.}}\ \frac{\int_{-\infty}^{+\infty}\{[\frac{1}{2D}U''(x) + \frac{1}{4D^2}U'^2(x)]\phi^2(x) + \phi'^2(x)\}\mathrm{d}x}{\int_{-\infty}^{+\infty}\phi^2(x)/D\ \mathrm{d}x}, \tag{A5}$$

where st. denotes the stationary value. The evaluation of λ was introduced in Ref. [26]. The minimum eigenvalue is $\lambda_0 = 0$, and the corresponding eigenfunction is $\phi_0(x) = \exp[U(x)/(2D)]$, so $\rho_0(x,t|\pm A) = \exp[U(x)/D]$ is the steady state solution. The minimum nonzero eigenvalue λ_1 is the system response speed,[27] and the inverse of λ_1 is the characteristic time of the system, which dominates the time of the system tending to the steady state. The solution corresponding to eigenvalue λ_j for $j = 1, 2, \ldots$ is

$$\begin{aligned}\rho_j(x,t|\pm A) &= \phi_j(x)\exp[U(x)/(2D)]\exp(-\lambda_j t)\\ &= \varphi_j^{\pm A}(x)\exp(-\lambda_j t).\end{aligned} \tag{A6}$$

Therefore, the complete solution of Eq. (5) can be expressed as

$$\rho(x,t|\pm A)=\sum_{j=0}^{\infty}c_j\varphi_j^{\pm A}(x)\exp(-\lambda_j t), \tag{A7}$$

where $\varphi_0^{\pm A}(x)=\exp[U(x)/D]$, c_0 can be determined by the normalization condition of the steady state probability density, and c_j's ($j=1,2,\ldots$) are constant coefficients and can be obtained by the initial condition and the orthogonality of the eigenfunctions. If the eigenfunctions are normalized, we have the orthonormality relation

$$\sum_{-\infty}^{+\infty}\frac{\phi_i(x)\phi_j(x)}{D}\mathrm{d}x=\delta_{ij}. \tag{A8}$$

Assume that a bit symbol takes level $-A$, and after time $t=iT_{\mathrm{b}}$, level $+A$ is transmitted. Using Eq. (A7), we have

$$\begin{aligned}\rho(x,t|\pm A)\simeq&\sum_{j=0}^{n}c_j\varphi_j^{\pm A}(x)\exp(-\lambda_j t)\\&+Q(x)\exp(-\lambda_{n+1}t)\\&+\mathrm{O}(\mathrm{e}^{-\lambda_{n+2}t}),\end{aligned} \tag{A9}$$

where $Q(x)\exp(-\lambda_{n+1}t)$ can be truncated. Here, we assume $\lambda_1<\lambda_2<\cdots<\lambda_{n+2}$. The above equation must satisfy the initial condition

$$\rho(x,0|+A)=\rho_0(x|-A), \tag{A10}$$

where $\rho_0(x|-A)$ is the steady state solution with the input level $-A$. Hence, by omitting the trace quantity $\mathrm{O}(e^{-\lambda_{n+2}t})$, from Eqs. (A9) and (A10), $Q(x)$ can be expressed as

$$Q(x)=\rho_0(x|-A)-\sum_{j=0}^{n}c_j\varphi_j^{+A}(x). \tag{A11}$$

Substituting Eq. (A11) into Eq. (A9), we can obtain the dynamical PDF as

$$\begin{aligned}\rho(x,t|\pm A)\simeq&\sum_{j=0}^{n}c_j\varphi_j^{\pm A}(x)\exp(-\lambda_j t)+\Big[\rho_0(x|-A)\\&-\sum_{j=0}^{n}c_j\varphi_j^{+A}(x)\Big]\exp(-\lambda_{n+1}t).\end{aligned} \tag{A12}$$

Obviously, with the system approaching the steady state (i.e., $t\to+\infty$), the above equation is still satisfied. Substituting Eq. (A7) into Eq. (A10), we have

$$\rho_0(x|-A)=\sum_{j=0}^{\infty}c_j\varphi_j^{+A}(x). \tag{A13}$$

Multiplying both sides of Eq. (A13) by $\phi_j(x)/\exp[U(x)/(2D)]$ and integrating from minus infinity to infinity, according to the orthonormality (A8), we find

$$\begin{aligned}c_j=&\frac{\int_{-\infty}^{+\infty}\phi_j(x)\rho_0(x|-A)/\exp[U(x)/(2D)]\mathrm{d}x}{\int_{-\infty}^{+\infty}\phi_j^2(x)\mathrm{d}x},\\&j\geq 1,\end{aligned} \tag{A14}$$

where the signal level in $U(x)$ and $\phi_j(x)$ is taken to be $+A$. Similarly, $\rho(x,t|-A)$ can be evaluated.

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