Effects of external fields on a two-dimensional Klein–Gordon particle under pseudo-harmonic oscillator interaction

Sameer M. Ikhdair[a)] and Majid Hamzavi[b)†]

[a)] *Physics Department, Near East University, Nicosia 922022, Turkey*

[b)] *Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran*

(Received 5 April 2012; revised manuscript received 2 May 2012)

We study the effects of the perpendicular magnetic and Aharonov–Bohm (AB) flux fields on the energy levels of a two-dimensional (2D) Klein–Gordon (KG) particle subjected to an equal scalar and vector pseudo-harmonic oscillator (PHO). We calculate the exact energy eigenvalues and normalized wave functions in terms of chemical potential parameter, magnetic field strength, AB flux field, and magnetic quantum number by means of the Nikiforov–Uvarov (NU) method. The non-relativistic limit, PHO, and harmonic oscillator solutions in the existence and absence of external fields are also obtained.

Keywords: Klein–Gordon equation, two-dimensional pseudo-harmonic oscillator (PHO) potential, magnetic and Aharonov–Bohm (AB) flux fields, Nikiforov–Uvarov method

PACS: 03.65.−w, 03.65.Pm, 03.65.Ge **DOI:** 10.1088/1674-1056/21/11/110302

1. Introduction

The pseudo-harmonic oscillator (PHO) potential is extensively used to describe the bound state of the interaction systems, and has been applied for both classical and modern physics. It plays a basic role in chemical and molecular physics since it can be used to calculate the molecular vibration–rotation energy spectrum of linear and non-linear systems.[1−3] This potential is considered as an intermediate between harmonic oscillator (HO) and Morse-type potentials which are more realistic anharmonic potentials. In the non-relativistic quantum mechanics, the PHO is one of the exactly solvable potentials in the frame of the Schrödinger equation and has also been studied in one-dimensional (1D), two-dimensional (2D), three-dimensional (3D), and even higher-dimensional space.

Therefore, this problem has attracted a great deal of interest in solving the Schrödinger and Klein–Gordon (KG) equations with the PHO interaction. The discrete (bound) and continuous (scattering) energy spectra of the PHO have been investigated by the SU(1,1) spectrum generating algebra.[4] The PHO potential has been obtained depending on the dimension and angular momentum, and the zero-point energy in 2D space was found to be minimum. The exact polynomial solution of the Schrödinger equation for the PHO has been obtained in 3D space.[5] Recently, solutions to the Schrödinger and KG equations with the PHO potential have been investigated.[6−10] The realization of the creation and annihilation (ladder) operators for the solution to the Schrödinger equation with a PHO in 2D space was studied in Refs. [11] and [12]. The operators satisfy the commutation relations of an SU(1,1) group. The exact solution to the Schrödinger equation with a PHO in an arbitrary D-dimensional space was presented in Ref. [13]. The exact analytical solutions to the Schrödinger equation of the D-dimensional space for the PHO potential have been presented by means of the ansatz method, and the energy eigenvalues were calculated from the eigenfunction ansatz.[14] The exact bound-state solutions of the KG and the Dirac equations with equal scalar and vector PHO potentials have been obtained using the supersymmetric quantum mechanics, shape invariance, and other alternative methods.[15] The bound-state solutions of the Dirac equation with PHO have been obtained in the presence of spin and pseudo-spin symmetries.[16] In relativistic quantum mechanics, the treatment of this problem is relatively inadequate.

Recently, the solutions of non-relativistic and rel-

†Corresponding author. E-mail: majid.hamzavi@gmail.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

ativistic equations with different potentials for their bound and continuum states were investigated. The approximate analytical solutions of the D-dimensional Schrödinger equation with modified Pöschl–Teller, Eckart, modified Morse, and generalized Hulthén potentials were found for any angular momentum state using new approximation scheme to centrifugal term.[17−20] It was shown that the energy levels of the continuum states of modified Morse potential are reduced to those of bound states at the poles of the scattering amplitude.[19] Using the supersymmetry and shape-invariance, exact bound-state solutions to the Schrödinger equation with Pö schl–Teller double-ring-shaped Coulomb potential were presented.[21] The exact solution to the 1D KG equation with mixed scalar and vector linear potentials was studied in the context of deformed quantum mechanics characterized by a finite minimal uncertainty in position using the momentum space representation.[22] A new ring-shaped harmonic oscillator for spin-1/2 particles was studied, and corresponding eigenfunctions and eigenvalues were obtained by using the Dirac equation with an equal mixture of scalar and vector potentials.[23] Furthermore, the approximate analytical solution to the Dirac equation with the Pöschl–Teller potential has been presented for arbitrary spin–orbit quantum number in view of spin-symmetry.[24]

Recently, the spectral properties of a 2D charged particle (electron or hole) confined by a PHO potential in the presence of external strong uniform magnetic field $\boldsymbol{B}$ along the z direction and Aharonov–Bohm (AB) flux field created by a solenoid have been studied. The Schrödinger equation is solved exactly for its bound states (energy spectrum and wave functions).[25,26] Hence, it is natural that the relativistic effects for a charged particle under the effect of this potential could become important, especially for a strong coupling.

The aim of the present work is to study the exact analytical bound state energy eigenvalues and normalized wave functions of the spinless relativistic equation with equal scalar and vector PHO interaction under the effect of external uniform magnetic field and AB flux field in the framework of the Nikiforov–Uvarov (NU) method.[27,28] The non-relativistic limit of our solution is obtained by making an appropriate mapping of parameters. In addition, the KG–PHO and KG–HO special cases are also considered.

The structure of this paper is given as follows. We study the effect of external uniform magnetic and AB flux fields on a relativistic spinless particle (anti-particle) under equal scalar and vector PHO interaction in Section 2. We discuss some special cases in Section 3. Finally, we give our concluding remarks in Section 4.

2. KG equation with PHO interaction under external fields

The KG equation is a wave equation most used in describing particle dynamics in relativistic quantum mechanics. Nonetheless, this equation physically describes a scalar particle (spin 0). Moreover, this wave equation, for free particles, is constructed using two objects: the four-vector linear momentum operator $P_\mu = \mathrm{i}\hbar\partial_\mu$ and the scalar rest mass M, which allows one to introduce naturally two types of potential couplings. One is the gauge-invariant coupling to the four-vector potential $\{A_\mu(\boldsymbol{r})\}_{\mu=0}^{3}$ which is introduced via the minimal substitution $P_\mu \to P_\mu - gA_\mu$, where g is a real coupling parameter. The other is an additional coupling to the space–time scalar potential $S_{\rm conf}(\boldsymbol{r})$, which is introduced by the substitution $M \to M + S_{\rm conf}(\boldsymbol{r})$. The terms "four-vector" and "scalar " refer to the corresponding unitary irreducible representation of the Poincaré space–time symmetry group (the group of rotations and translations in $(3+1)$-dimensional Minkowski space time). Gauge invariance of the vector coupling allows for the freedom to fix the gauge (eliminating the non-physical gauge modes) without altering the physical content of the problem. Many people choose to simplify the solution of the problem by neglecting the space component of the vector potential (i.e., $\boldsymbol{A}$). One may write the time-component of the four-vector potential as $gA_0 = V_{\rm conf}(\boldsymbol{r})$, then it ends with two independent potential functions in the KG equation. These are the "vector" $V_{\rm conf}(\boldsymbol{r})$ and the "scalar" $S_{\rm conf}(\boldsymbol{r})$ potentials.[29,30]

The free KG equation is written as

$$(\partial^\mu\partial_\mu + M^2c^4)\psi_{\rm KG}(t,\boldsymbol{r}) = 0. \tag{1}$$

Moreover, the vector and scalar couplings mentioned above introduce potential interactions by mapping the free KG equation for a 2D single charged electron as

$$\Big[c^2\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\Big)^2 - (E - V_{\rm conf}(\boldsymbol{r}))^2 + (Mc^2 + S_{\rm conf}(\boldsymbol{r}))^2\Big]\psi(\boldsymbol{r},\phi) = 0, \tag{2}$$

where we used the transformation $\boldsymbol{p} \to \boldsymbol{p} + (e/c)\boldsymbol{A}$. Furthermore, the 2D cylindrical wave function $\psi(\boldsymbol{r}, \phi)$ is defined as

$$\psi(\boldsymbol{r}, \phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\phi} g(r), \ \ (m = 0, \pm 1, \ldots), \qquad (3)$$

where m is the magnetic quantum number. This type of coupling has attracted much attention due to the resulting simplification in the solution of the relativistic problem. The scalar-like potential coupling is added to the scalar mass so that when $S_{\mathrm{conf}}(\boldsymbol{r}) = \pm V_{\mathrm{conf}}(\boldsymbol{r})$, the KG equation could always be reduced to a Schrödinger-type second-order differential equation as follows:

$$\Big[c^2\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\Big)^2 + 2(E \pm Mc^2)V_{\mathrm{conf}}(\boldsymbol{r}) + M^2c^4 - E^2\Big]\psi(\boldsymbol{r}, \phi) = 0. \qquad (4)$$

The potential $V_{\mathrm{conf}}(\boldsymbol{r})$ is taken as the following repulsive PHO potential:[1−3,11]

$$V_{\mathrm{conf}}(\boldsymbol{r}) = V_0\Big(\frac{r}{r_0} - \frac{r_0}{r}\Big)^2, \qquad (5)$$

where r_0 and V_0 are the zero point (effective radius) and the chemical potential, respectively. The vector potential $\boldsymbol{A}$ may be represented as a sum of two terms: $\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2$, so that $\boldsymbol{\nabla} \times \boldsymbol{A}_1 = \boldsymbol{B}$ and $\boldsymbol{\nabla} \times \boldsymbol{A}_2 = 0$, where $\boldsymbol{B} = B\boldsymbol{z}$ is the applied magnetic field, and $\boldsymbol{A}_2$ describes the additional AB flux field Φ_{AB} created by a solenoid in cylindrical coordinates.[31] Hence, the vector potentials have the following azimuthal components:[32]

$$\boldsymbol{A}_1 = \frac{Br}{2}\boldsymbol{\phi}, \ \ \boldsymbol{A}_2 = \frac{\Phi_{\mathrm{AB}}}{2\pi r}\boldsymbol{\phi}, \ \ \boldsymbol{A} = \Big(\frac{Br}{2} + \frac{\Phi_{\mathrm{AB}}}{2\pi r}\Big)\boldsymbol{\phi}. \qquad (6)$$

In the following, the bound-state solutions of the two cases in Eq. (4) are to be treated separately.

2.1. Positive-energy bound states

The positive-energy states (corresponding to $S_{\mathrm{conf}}(\boldsymbol{r}) = +V_{\mathrm{conf}}(\boldsymbol{r})$) in the non-relativistic limit ($E - Mc^2 \to E$ and $(E + Mc^2) \to 2\mu c^2$, where M is an effective mass and $|E| \ll Mc^2$) are solutions of the following equation:

$$\Big[\frac{1}{2\mu}\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\Big)^2 + 2V_{\mathrm{conf}}(\boldsymbol{r}) - E\Big]\psi(\boldsymbol{r}, \phi) = 0, \qquad (7)$$

where $\psi(\boldsymbol{r}, \phi)$ stands for either $\psi^{(+)}(\boldsymbol{r}, \phi)$ or $\psi^{(\mathrm{KG})}(\boldsymbol{r}, \phi)$. This is the Schrödinger equation for the potential $2V_{\mathrm{conf}}(\boldsymbol{r})$. Thus, the choice $S_{\mathrm{conf}}(\boldsymbol{r}) = +V_{\mathrm{conf}}(\boldsymbol{r})$ produces a nontrivial non-relativistic limit with a potential function $2V_{\mathrm{conf}}(\boldsymbol{r})$ instead of $V_{\mathrm{conf}}(\boldsymbol{r})$. Accordingly, it would be natural to scale the potential term in Eqs. (4) and (7) so that in the non-relativistic limit the interaction potential becomes V_{conf} rather than $2V_{\mathrm{conf}}$. Thus, we need to recast Eq. (4) for the $S(\boldsymbol{r}) = V(\boldsymbol{r})$ as[29,33]

$$\Big[c^2\Big(-\mathrm{i}\hbar\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{A}\Big)^2 + 2(E + Mc^2)V_{\mathrm{conf}}(\boldsymbol{r})\Big]\psi(\boldsymbol{r}, \phi) = (E^2 - M^2c^4)\psi(\boldsymbol{r}, \phi). \qquad (8)$$

We introduce new parameters $\lambda_1 = E + Mc^2$ and $\lambda_2 = E - Mc^2$ so that equation (8) can be reduced to the following form:

$$\Big[c^2\Big(-\mathrm{i}\hbar\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{A}\Big)^2 - \lambda_1(\lambda_2 - V_{\mathrm{conf}}(\boldsymbol{r}))\Big]\psi(\boldsymbol{r}, \phi) = 0. \qquad (9)$$

Now, substituting Eqs. (3), (5), and (6) into the KG equation (9), and further introducing a variable substitution $s = r^2$ that maps $r \in (0, \infty)$ to s $\in (0, \infty)$, we obtain the second-order differential equation satisfying the radial wave function $g(s)$ as

$$g''(s) + \frac{1}{s}g'(s) + \frac{1}{s^2}\Big(-\frac{\gamma^2}{4}s^2 + \frac{\nu^2}{4}s - \frac{\beta^2}{4}\Big)g(s) = 0, \ \ (g(0) = 0, \ g(\infty) = 0), \qquad (10)$$

where we used the following abbreviation symbols:

$$\nu^2 = \frac{1}{\hbar^2c^2}\big[\lambda_1(\lambda_2 + 2V_0) - Mc^2\omega_{\mathrm{c}}\hbar m'\big], \qquad (11a)$$

$$\beta^2 = m'^2 + \frac{1}{\hbar^2c^2}r_0^2V_0\lambda_1, \qquad (11b)$$

$$\gamma^2 = \Big(\frac{M\omega_{\mathrm{c}}}{2\hbar}\Big)^2 + \frac{1}{\hbar^2c^2}\frac{V_0\lambda_1}{r_0^2}, \qquad (11c)$$

where $m' = m + \xi$ ($m' = 1, 2, \ldots$) is a new quantum number, $\xi = \Phi_{\mathrm{AB}}/\Phi_0$ is an integer with the flux quantum $\Phi_0 = hc/e$, and $\omega_{\mathrm{c}} = eB/Mc$ is the cyclotron frequency. Now, we use the basic ideas of the NU method[27] and the parametric NU derived in Ref. [28] to obtain energy-spectrum equation

$$\nu^2 = 2(2n + 1 + \beta)\gamma, \ \ (n = 0, 1, \ldots), \qquad (12)$$

where the constant parameters used in our calculations are displayed in Table 1. Using Eqs. (11a)–(11c), we finally arrive at the following transcendental energy formula,

$$2\Big(1 + 2n + \sqrt{m'^2 + \frac{V_0r_0^2\lambda_1}{\hbar^2c^2}}\Big)\sqrt{\Big(\frac{M\omega_{\mathrm{c}}}{2\hbar}\Big)^2 + \frac{V_0\lambda_1}{\hbar^2c^2r_0^2}} = \frac{1}{\hbar^2c^2}\big[\lambda_1(\lambda_2 + 2V_0) - Mc^2\hbar\omega_{\mathrm{c}}m'\big], \qquad (13)$$

We may find the solution to the above transcendental equation to be $E = E_{\mathrm{KG}}^{(+)}$. In the non-relativistic limit $\lambda_1 \to 2Mc^2$ and $\lambda_2 \to E$, the above energy equation

has a solution given by Eq. (39) in Ref. [25].

Table 1. Specific values of the constants in the solution of Eq. (20) for the case of $\alpha_3 = 0$.

Parameter	Value.	Parameter	Value.
$\xi_1 = \alpha_6$	$\gamma^2/4$	α_9	$\gamma^2/4$
$\xi_2 = \alpha_7$	$\nu^2/4$	α_{10}	$\beta + 1$
$\xi_3 = \alpha_8$	$\beta^2/4$	α_{11}	γ
α_1	1	α_{12}	$\beta/2$
$\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5$	0	α_{13}	$-\gamma/2$

Using Eq. (38) in Ref. [28] and Table 1, we obtain the corresponding radial wave function $g(r)$ as

$$g(r) = C_{n,m} r^{|\beta|} \mathrm{e}^{-\gamma r^2/2} \mathrm{F}(-n, |\beta| + 1; \gamma r^2), \quad (14)$$

and hence the total 2D KG wave function (3) takes the explicit form as follows:

$$\psi^{(+)}_{n,m}(\boldsymbol{r}, \phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\phi} \sqrt{\frac{\gamma^{|\beta|+1} n!}{\pi(n + |\beta|)!}} r^{|\beta|} \mathrm{e}^{-\gamma r^2/2} \times \mathrm{L}^{(\beta)}_n(\gamma r^2), \quad (15)$$

where $\mathrm{L}^{(b)}_a(x) = \dfrac{(a+b)!}{a!b!} \mathrm{F}(-a, b+1; x)$ is the associated Laguerre polynomial, and $\mathrm{F}(-a, b; x)$ is the confluent hypergeometric function. Note that the wave function (15) is finite and satisfies the standard asymptotic analysis for $r = 0$ and $r \to \infty$. An alternative short solution to Eq. (9) can be easily found using the asymptotic analysis in Appendix A.

2.2. Negative-energy bound states

When $S_{\mathrm{conf}}(\boldsymbol{r}) = -V_{\mathrm{conf}}(\boldsymbol{r})$, we need to follow same procedure of deriving solution in Subsection 2.1 and consider the solution given by Eq. (12) with the substitutions

$$\nu^2 \to \widetilde{\nu}^2 = \frac{1}{\hbar^2 c^2} \big[\lambda_2(\lambda_1 + 2V_0) - Mc^2 \omega_{\mathrm{c}} \hbar m'\big], \quad (16a)$$

$$\beta^2 \to \widetilde{\beta}^2 = m'^2 + \frac{1}{\hbar^2 c^2} r_0^2 V_0 \lambda_2, \quad (16b)$$

$$\gamma^2 \to \widetilde{\gamma}^2 = \Big(\frac{M\omega_{\mathrm{c}}}{2\hbar}\Big)^2 + \frac{1}{\hbar^2 c^2} \frac{V_0 \lambda_2}{r_0^2}. \quad (16c)$$

Hence, the negative-energy solution for an anti-particle can be easily given by

$$2\left(2n + 1 + \sqrt{m'^2 + \frac{1}{\hbar^2 c^2} r_0^2 V_0 \lambda_2}\right) \sqrt{\Big(\frac{M\omega_{\mathrm{c}}}{2\hbar}\Big)^2 + \frac{1}{\hbar^2 c^2} \frac{V_0 \lambda_2}{r_0^2}} = \frac{1}{\hbar^2 c^2} [\lambda_2(\lambda_1 + 2V_0) - Mc^2 \hbar \omega_{\mathrm{c}} m'], \quad (m' = 1, 2, \ldots), \quad (17)$$

and the 2D wave function is

$$\psi^{(-)}_{n,m}(\boldsymbol{r}, \phi) = \sqrt{\frac{\widetilde{\gamma}^{|\widetilde{\beta}|+1} n!}{\pi(n + |\widetilde{\beta}|)!}} r^{|\widetilde{\beta}|} \mathrm{e}^{-\widetilde{\gamma} r^2/2} \times \mathrm{L}^{(\widetilde{\beta})}_n(\widetilde{\gamma} r^2) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\phi}. \quad (18)$$

It should be noted that the negative-energy states are free fields since under these conditions, equation (4) can be rewritten as

$$\Big[-\frac{1}{2\mu}\Big(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\Big)^2 + E\Big] \psi_{n,m}(\boldsymbol{r}, \phi) = 0, \quad (19)$$

which is a simple free-interaction mode. Then, the set of parameters given by Eqs. (16a)–(16c) becomes

$$\widetilde{\nu} = \sqrt{\frac{2ME}{\hbar^2} - \frac{M\omega_{\mathrm{c}}}{\hbar} m'}, \ \widetilde{\beta} = m', \ \widetilde{\gamma} = \frac{M\omega_{\mathrm{c}}}{2\hbar}. \quad (20)$$

Thus, equation (17) leads to the following energy formula:

$$E^{(-)}_{nm'} = \Big(n + m' + \frac{1}{2}\Big) \hbar \omega_{\mathrm{c}}, \quad (21)$$

and hence the wave function reads

$$\psi^{(-)}_{nm'}(\boldsymbol{r}, \phi) = \sqrt{\frac{\Big(\frac{M\omega_{\mathrm{c}}}{2\hbar}\Big)^{m'+1} n!}{\pi(n + m')!}} r^{m'} \mathrm{e}^{-[(M\omega_{\mathrm{c}})/(4\hbar c)] r^2} \times \mathrm{L}^{(m')}_n\Big(\frac{M\omega_{\mathrm{c}}}{2\hbar} r^2\Big) \mathrm{e}^{\mathrm{i}m\phi}. \quad (22)$$

3. Discussion

In this section, we briefly study some special cases and relationship between our results and those of some others.

3.1. Schrödinger–PHO problem under the effect of magnetic and AB flux fields

In the non-relativistic limit ($\lambda_1 \to 2M$ and $\lambda_2 \to E$), equation (13) can be easily reduced into the fol-

lowing form:

$$E_{nm}(\xi,\beta)=\hbar\Omega\Big(n+\frac{|\widetilde{m}|+1}{2}\Big)+\frac{1}{2}\hbar\omega_{\rm c}m'-2V_0, \quad (23a)$$

$$\Omega=\sqrt{\omega_{\rm c}^2+4\omega_D^2},\ \omega_D=\sqrt{2V_0/Mr_0^2},$$
$$|\widetilde{m}|=\sqrt{m'^2+a^2}, \quad (23b)$$

where $a=k_{\rm F}r_0$ with $k_{\rm F}=\sqrt{2MV_0/\hbar^2}$ being the Fermi wave vector of the electron.[25] Then, the wave function becomes

$$\psi_{n,m}^{(+)}(\boldsymbol{r},\phi)=\frac{1}{\sqrt{2\pi}}{\rm e}^{{\rm i}m\phi}\sqrt{\frac{c^{b+1}n!}{\pi(n+b)!}}r^b \times{\rm e}^{-cr^2/2}{\rm L}_n^{(b)}(cr^2), \quad (24)$$

where

$$b=\sqrt{m'^2+\frac{2MV_0r_0^2}{\hbar^2}},\ c=\sqrt{\Big(\frac{M\omega_{\rm c}}{2\hbar}\Big)^2+\frac{2MV_0}{\hbar^2r_0^2}}.$$

3.2. KG—PHO problem

The energy spectrum of a relativistic spinless particle in the absence of magnetic and AB flux fields has the following form:

$$\frac{2\hbar}{r_0}\sqrt{V_0}\Big(1+2n+\sqrt{m^2+\frac{V_0r_0^2}{\hbar^2}\lambda_1}\Big) =(\lambda_2+2V_0)\sqrt{\lambda_1}, \quad (25)$$

and is reduced to its non-relativistic limit:

$$E_{nm}=-2V_0+\Big(1+2n+\sqrt{m^2+\frac{2MV_0r_0^2}{\hbar^2}}\Big) \times\sqrt{\frac{2V_0\hbar^2}{Mr_0^2}}, \quad (26)$$

which is completely identical to Eq. (27) in Ref. [7]. The wave function can be expressed as

$$\psi_{n,m}^{(+)}(\boldsymbol{r},\phi)=\frac{1}{\sqrt{2\pi}}{\rm e}^{{\rm i}m\phi}\sqrt{\frac{C^{B+1}n!}{\pi(n+B)!}}r^B \times{\rm e}^{-Cr^2/2}{\rm L}_n^{(B)}(Cr^2), \quad (27)$$

where

$$B=\sqrt{m^2+\frac{r_0^2V_0}{\hbar^2}\lambda_1},\quad C=\frac{1}{\hbar r_0}\sqrt{V_0\lambda_1}.$$

3.3. KG—HO problem

The energy equation of the relativistic spinless particle subjected to the HO field reads

$$n'\hbar c\sqrt{2{\rm k}}-\sqrt{\lambda_1}\lambda_2=0,\ (n'=1,2,\ldots), \quad (28)$$

where $n'=1+|m|+2n$ $(n=0,1,\ldots)$, which is completely identical to Eqs. (11) and (26) in Ref. [34] when one uses the notation $k=2V_0/r_0^2=M\omega_{\rm D}'^2$. Following Ref. [35], equation (28) has three solutions, and the only real solution giving energy reads

$$E_{nm}=\frac{1}{3}(Mc^2+M^2c^4T^{-1/3}+T^{1/3}), \quad (29)$$

with

$$T=27kn'^2\hbar^2c^2-8M^3c^6 +3n'\hbar c\sqrt{3k(27kn'^2\hbar^2c^2-16M^3c^6)}. \quad (30)$$

The wave function takes the following form:

$$\psi_{n,m}^{(+)}(\boldsymbol{r},\phi)=\sqrt{\frac{D^{|m|+1}n!}{\pi(n+|m|)!}}r^{|m|}{\rm e}^{-Dr^2/2}{\rm L}_n^{(|m|)}(Dr^2) \times{\rm e}^{{\rm i}m\phi},\ D=\frac{1}{\hbar r_0}\sqrt{V_0\lambda_1}. \quad (31)$$

If one expands Eq. (28) into a series of λ_2, it becomes

$$n'\hbar=\sqrt{\frac{M}{k}}\Big[\lambda_2+\frac{1}{4Mc^2}\lambda_2^2-\frac{1}{32M^2c^4}\lambda_2^3+O(\lambda_2^4)\Big]. \quad (32)$$

Taking the first order of λ_2 by neglecting the higher-order relativistic corrections, we finally arrive at the non-relativistic solution

$$E'_{nm}=E_{nm}-Mc^2=\hbar\sqrt{\frac{k}{M}}(1+2n+|m|) =(1+|m|+2n)\hbar\omega'_{\rm D},\ (n=0,1,\ldots). \quad (33)$$

The wave function resembles that given by Eq. (31) with $D=(\hbar r_0)^{-1}\sqrt{2MV_0}$.

4. Concluding remarks

In this work, we have obtained bound-state energies and wave functions of the relativistic spinless particle subjected to a PHO interaction and expressed in terms of external uniform magnetic field and AB flux field. We explored the solution of both positive (particle) and negative (anti-particle) KG energy states. The Schrödinger bound-state solution is found to be the non-relativistic limit of the present model. It is noticed that the solution with an equal mixture of scalar and vector potentials can be easily reduced into the well-known Schrödinger solution for a particle with an interaction potential field and a free field, respectively. We have also studied the bound-state solutions for some special cases including the non-relativistic limits (Schrödinger equation for PHO and HO under external magnetic and AB flux fields) and the KG equation for HO and PHO interactions. The results show that the splitting is not constant and dependent on the strength of the external magnetic field and AB flux field.

Acknowledgments

S. M. Ikhdair acknowledges the support of the Scientific and Technological Research Council of Turkey.

Appendix A

Here, we seek to find a short solution to Eq. (9). By substituting Eqs. (3), (5), and (6) into Eq. (9), we obtain a 2D Schrödinger-type equation for the anharmonic oscillator $V(r) = \gamma^2 r^2$ satisfying the radial wave function $R(r)$

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{(\beta^2 - 1/4)}{r^2} + \nu^2 - \gamma^2 r^2\Big) R(r) = 0, \quad \text{(A1)}$$

where $g(r) = r^{-1/2} R(r)$. For brevity, the solution of the above equation can be found based on Eq. (17) in Ref. [5] by simply making the changes: $2\mu E'_{nL}/\hbar^2 \to \nu^2$, $2\mu B^2/\hbar^2 \to \gamma^2$, and $2L + 1 \to 2\beta$ in Eq. (19) in Ref. [5]. Finally, we obtain Eqs. (12) and (13). By first inspecting the asymptotic behavior of Eq. (A1), we find out that when r approaches 0, the radial wave function $R(r) \sim r^q$ ($q = \beta + 1/2 > 0$), and when $r \to \infty$, $R(r) \sim \exp(-\gamma r^2/2)$. Both solutions satisfy the finiteness of the radial wave function $g(r = 0) = 0$ and $g(r \to \infty) \to 0$. Therefore, for the entire range $r \in (0, \infty)$, we consider the general solution $g(r) = r^\beta \exp(-\gamma r^2/2) h(r)$ ($\beta > 0$), where $h(r)$ is the associated Laguerre polynomials, i.e., $h(r) = \mathrm{L}_n^{(a)}(br^2)$. With these behaviors, the 3D wave function (20) in Ref. [5] results in the 2D wave function given by Eq. (15) when one uses the relation

$$\begin{aligned} g(r) &= N r^{-1/2} R(r) \\ &= N r^{L+1/2} \exp\Big(-\sqrt{\frac{\mu}{2\hbar^2}} B r^2\Big) \\ &\quad \times \mathrm{L}_n^{(L+1/2)}\Big(\sqrt{\frac{2\mu}{\hbar^2}} B r^2\Big) \\ &= N r^\beta \exp\Big(-\frac{1}{2}\gamma r^2\Big) \mathrm{L}_n^{(\beta)}(\gamma r^2), \end{aligned} \quad \text{(A2)}$$

where N is the normalization constant.

References

[1] Goldman I I, Krivchenkov V D, Kogan V I and Golitskii V M 1961 *Problems in Quantum Mechanics* (London: Pergamon Press)
[2] Sage M and Goodisman J 1985 *Am. J. Phys.* **53** 350
[3] Maitland G C, Rigby M, Smith E B and Wakeham W A 1987 *Intermolecular Forces* (London: Oxford University Press)
[4] Büyükkılıç F, Demirhan D and Özeren S F 1992 *Chem. Phys. Lett.* **194** 9
[5] Ikhdair S M and Sever R 2007 *J. Mol. Struct.: Theochem.* **806** 155
[6] Sever R, Tezcan C, Aktaş M and Yeşiltaş Ö 2007 *J. Math. Chem.* **43** 845
[7] Ikhdair S M and Sever R 2007 *Cent. Eur. J. Phys.* **5** 516
[8] Ikhdair S M and Sever R 2008 *Int. J. Mod. Phys.* C **20** 361
[9] Ikhdair S M and Sever R 2008 *Int. J. Mod. Phys.* C **19** 1425
[10] Ikhdair S M and Sever R 2008 *Cent. Eur. J. Phys.* **6** 685
[11] Dong S H 2003 *Appl. Math. Lett.* **16** 199
[12] Dong S H and Ma Z Q 2002 *Int. J. Mod. Phys.* E **11** 155
[13] Wang L Y, Gu X Y, Ma Z Q and Dong S H 2002 *Found. Phys. Lett.* **15** 569
[14] Oyewumi K J, Akinpelu F O and Agboola A D 2008 *Int. J. Theor. Phys.* **47** 1039
[15] Chen G, Chen Z D and Lou Z M 2004 *Chin. Phys.* **13** 279
[16] Aydoğdu O and Sever R 2009 *Phys. Scr.* **80** 015001
[17] Chen C Y, Lu F L and You Y 2012 *Chin. Phys.* B **21** 030302
[18] Taşkin F and Koçak G 2010 *Chin. Phys.* B **19** 090314
[19] Wei G F and Chen W L 2010 *Chin. Phys.* B **19** 090308
[20] Chen W L, Wang H Y and Li Y Y 2009 *Chin. Phys.* B **18** 451
[21] Lu F L and Chen C Y 2010 *Chin. Phys.* B **19** 100309
[22] Chargui Y, Chetouani L and Trabelsi A 2010 *Chin. Phys.* B **19** 020305
[23] Zhou Y and Guo J Y 2008 *Chin. Phys.* B **17** 380
[24] Taşkın F and Koçak G 2011 *Chin. Phys.* B **20** 070302
[25] Ikhdair S M 2012 arXiv: 1111.2442v1 [cond-mat.mes-hall]
[26] Khordad R 2010 *Solid State Sciences* **12** 1253
[27] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Berlin: Birkhauser)
[28] Tezcan C and Sever R 2009 *Int. J. Theor. Phys.* **48** 337
[29] Greiner W 2000 *Relativistic Quantum Mechanics: Wave Equations* (Berlin: Springer-Verlag)
[30] Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett.* A **349** 87
[31] Bogachek E N and Landman U 1995 *Phys. Rev.* B **52** 14067
[32] Çetin A 2008 *Phys. Lett.* A **372** 3852
[33] Xu Y, He S and Jia C S 2010 *Phys. Scr.* **81** 045001
[34] Qiang W C 2004 *Chin. Phys.* **13** 283
[35] Ikhdair S M 2012 *J. Mod. Phys.* **3** 170