# Relativity and impossibility of superluminal motion*

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I prove that not only particle motion but also wave propagation cannot be faster than light according to relativity.

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## 1. Introduction

The OPERA experiment[1,2] has revived the research of superluminal motion, though its result has become negative recently. It is well known that a classical particle cannot move faster than light. The relativistic mass–velocity relation shows that the mass of a classical particle with a velocity $v$ larger than the light velocity $c$ will be imaginary, and therefore is nonphysical.[3,4] Suppose a particle initially has a velocity $v < c$, and then is accelerated by certain mechanism. When the velocity of the particle approaches light speed $c$, the mass of the particle will approach infinity. This in turn prevents the velocity from reaching and exceeding the limit value $c$. However, micro particles are not classical but quanta of certain wave fields. In the microscopic world, the position and velocity of a particle could not be defined simultaneously at any given time. It makes the above arguments based on the mass–velocity relation invalid. According to quantum theory,[5] particles do not move along trajectories. Moreover, one is even not able to trace a single particle, and to identify a particle at time $t$ and position $x$ being the one which was originally at time $t_0$ and position $x_0$. The space distribution of micro particles at a given time is statistically described by wave functions, and their time evolution is governed by wave equations. The possibility of superluminal motion of micro particles is therefore dependent on the possibility of the superluminal propagation of corresponding waves. Whether the relativity also forbids the superluminal propagation of waves or not is an independent problem that is still not solved. In the following, by use of the contour integral technics, we prove that relativity indeed forbids superluminal wave propagation.

In Section 2, we solve the relativistic wave equation for free particles. In Section 3, we prove the impossibility of superluminal wave propagation on the basis of the solution. Section 4 contains discussions.

## 2. Solution of the relativistic wave equation

Consider some identical micro particles, like neutrinos in the OPERA experiment, moving freely from position $A$ to position $B$. This process is described by a wave packet moving freely from $A$ to $B$. Suppose the dimension of the wave packet is negligibly small compared with the distance between $A$ and $B$, one may consider it as a point whose position and velocity are well defined at any time. The velocity of the wave packet as a whole is its group velocity $\boldsymbol{v}_{\rm g} = \boldsymbol{k}c^2/\omega = \boldsymbol{p}c^2/E$, where $\boldsymbol{k}$ and $\omega$ are the wave vector and the circular frequency of the wave, and $\boldsymbol{p}$ and $E$ are the momentum and energy of the particle described by the wave, respectively. There are relativistic relations $\boldsymbol{p} = \hbar\boldsymbol{k}$, $E = \hbar\omega$, $\omega = \sqrt{k^2 + \kappa^2}c$, and $E = \sqrt{p^2 + m^2c^2}c$, with $m$ and $\kappa \equiv mc/\hbar$ being the mass and the reciprocal of the Compton wavelength of the particle, respectively. The dimension of the wave packet, on the other hand, is assumed to be quite large compared with its wavelength so that its wave vector and thus the circular frequency are quite well defined. Momentum $\boldsymbol{p}$ and energy $E$ are good quantum

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numbers of the particle. From the above relations we see that the group velocity of the wave packet exactly equals the classical velocity of the particle described by the wave, and does not exceed the light speed $c$. It seems that nothing is superluminal. However, one thing has to be considered further. Waves associated with massive particles are dispersive. The wave packets not only move but also diffuse. It means that different points in the wave packet move with different velocities. Especially, the wave front may move with a velocity higher than the group velocity. Could it be superluminal?

The wave function $\Psi$ describing spin-1/2 massive particles, like neutrinos, is governed by the Dirac equation

$$(\gamma_\mu \partial_\mu + \kappa)\Psi = 0\,, \tag{1}$$

where $\gamma_\mu$ ($\mu = 1, 2, 3, 4$) are covariant Dirac matrices satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}\,. \tag{2}$$

The wave function satisfying Dirac equation (1) also satisfies the Klein–Gordon equation

$$\partial_\mu \partial_\mu \Psi - \kappa^2 \Psi = 0\,. \tag{3}$$

From the wave function $\Psi_0$ with four components satisfying Eq. (3), one may construct a solution of the Dirac equation. Take $\Psi_1 \equiv (\gamma_\mu \partial_\mu - \kappa)\Psi_0$. If $\Psi_1 \neq 0$, $\Psi \equiv \Psi_1$ is a solution of the Dirac equation (1) because of the relation (2). If $\Psi_1 = 0$, one may construct $\Psi \equiv \gamma_5 \Psi_0$ with $\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4$, which is a solution to Eq. (1) because $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$ for $\mu = 1, 2, 3, 4$. This means, if one is only interested in the space distribution of particles, one can ignore the complexity introduced by the spinor property of the wave function, and consider a one-component wave function $\Phi$ satisfying the Klein–Gordon equation instead. This is true not only for spin-1/2 particles but also for any other micro particles in general.

The motion of a wave packet is along the straight line from $A$ to $B$. Our problem is therefore one-dimensional along the $x$ direction. To consider the wave packet diffusion, let us use a coordinate system moving with the packet, in which the center of the wave packet does not move. This is the wave-packet coordinate system $\mathrm{S}_0$. Suppose the wave is initially nonzero in the interval $-a/2 \leq x \leq a/2$ only, and $a > 0$ is the wave-packet length. Its most general form is

$$\Phi(x,0) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \left[ b_n \cos\left(\frac{2n+1}{a}\pi x\right) + b'_n \sin\left(\frac{2n}{a}\pi x\right)\right], & \text{if } -a/2 \leq x \leq a/2, \\ 0, & \text{otherwise}\,, \end{cases} \tag{4}$$

according to the theory of Fourier expansion. Here $b_n$ and $b'_n$ are free constant parameters characterizing the wave packet. The statistical interpretation of wave function is somewhat complicated in relativistic quantum theory, but the qualitative aspect is clear. Anyway, the probability of finding a particle in a region with zero wave function is zero. Contrarily, in the region with nonzero wave function, the probability of finding a particle is nonzero. Therefore, Eq. (4) shows that at the time $t = 0$ one cannot find any particle outside the region $-a/2 \leq x \leq a/2$.

The Klein–Gordon equation (3) is exactly solved by Fourier transformation. A solution satisfying the initial condition (4) is

$$\begin{aligned} \Phi(x,t) = \sum_0^{\infty} (-1)^n \Bigg[ & b_n \int_{-\infty}^{\infty} \frac{(2n+1)a\cos(ka/2)}{(2n+1)^2\pi^2 - k^2a^2} \\ & \times \cos(kx - \sqrt{k^2+\kappa^2}ct)\mathrm{d}k \\ & - b'_n \int_{-\infty}^{\infty} \frac{2na\sin(ka/2)}{(2n)^2\pi^2 - k^2a^2} \\ & \times \sin(kx - \sqrt{k^2+\kappa^2}ct)\mathrm{d}k \Bigg]\,, \end{aligned} \tag{5}$$

which gives the wave function at any time $t$. The denominator of each integrand has two zero points at $k = \pm(2n+1)\pi/a$ or $k = \pm 2n\pi/a$. Since the numerator is also zero at these points, we may arbitrarily shift the zero points upward or downward from the real axis by an infinitesimal positive amount $\epsilon$, because of the well-known formula[5]

$$\frac{1}{\xi \pm \mathrm{i}\epsilon} = \frac{1}{\xi} \mp \mathrm{i}\pi\delta(\xi). \tag{6}$$

To make it clear, we write solution (5) in the following form:

$$\Phi(x,t)$$

$$
\begin{aligned}
&= \sum_0^\infty \int_{-\infty}^{\infty} B_n(k)\Big\{ \cos\left[k\left(x+\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right] \\
&\quad + \cos\left[k\left(x-\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\Big\}\mathrm{d}k \\
&\quad + \sum_0^\infty \int_{-\infty}^{\infty} B'_n(k)\Big\{ \cos\left[k\left(x+\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right] \\
&\quad - \cos\left[k\left(x-\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\Big\}\mathrm{d}k, \qquad (7)
\end{aligned}
$$

with

$$
\begin{aligned}
B_n(k) &= \frac{1}{2}(B_n(k,+) + B_n(k,-))\,, \\
B'_n(k) &= \frac{1}{2}(B'_n(k,+) + B'_n(k,-))\,, \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
B_n(k,\pm) &= \frac{(-1)^n b_n (n+1/2)a}{(2n+1)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2}\,, \\
B'_n(k,\pm) &= \frac{(-1)^n b'_n na}{(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2}\,. \qquad (9)
\end{aligned}
$$

## 3. Impossibility of superluminal wave propagation

From the initial condition (4), we know that, for $t>0$, there are two superluminal regions of the wave (5). They are

$$
x > \frac{a}{2} + ct, \quad x < -\frac{a}{2} - ct\,. \qquad (10)
$$

In this section, we will prove that the wave function (5) is exactly zero in these regions. Therefore, the superluminal wave propagation is impossible.

**Theorem** The value of wave function (5) is zero in the superluminal regions (10).

**Proof** Consider the integral

$$
\begin{aligned}
I(\pm) &= \int_{-\infty}^{\infty} \frac{\cos[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]}{(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2}\mathrm{d}k \\
&= I_1(\pm) + I_2(\pm), \qquad (11)
\end{aligned}
$$

with

$$
I_1(\pm) \equiv \int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k, \qquad (12)
$$

and

$$
\begin{aligned}
I_2(\pm) &\equiv \int_{-\infty}^{\infty} \frac{\exp\{-\mathrm{i}[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k \\
&= \int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}[k(x+a/2) + \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\mp \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k\,. \qquad (13)
\end{aligned}
$$

The substitution $k\to -k$ made the last step. Instead of $I_1(\pm)$ and $I_2(\pm)$, let us consider the contour integrals

$$
J_1(\pm) = \int_\Gamma \frac{\exp\{\mathrm{i}[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k, \qquad (14)
$$

and

$$
J_2(\pm) = \int_\Gamma \frac{\exp\{\mathrm{i}[k(x+a/2) + \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\mp \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k, \qquad (15)
$$

where $\Gamma$ is a close contour on the complex $k$ plane composed of six parts: 1) from $+0$ to $+\infty$ along the positive real axis; 2) along a half circle of infinite radius on the upper half complex $k$-plane anticlockwise to $-\infty$ on negative real axis; 3) from $-\infty$ to $-0$ along the negative real axis; 4) from $-0$ to $-0+\mathrm{i}\kappa$, in parallel with the positive imaginary axis; 5) clockwise from $-0+\mathrm{i}\kappa$ to $+0+\mathrm{i}\kappa$ along a half circle of infinitesimal radius; 6) from $+0+\mathrm{i}\kappa$ to $+0$, antiparallel relative to the positive imaginary axis.

For $t>0$, in the superluminal region $x > a/2+ct$, we have $x+a/2-ct>0$ and $x+a/2+ct>0$. It makes the parts of integrals $J_1(\pm)$ and $J_2(\pm)$ on part 2 of the contour $\Gamma$, which is on the half circle of infinite radius on the upper half complex $k$-plane, be zeros. Since the integrands are continuous and finite near the branch point $\mathrm{i}\kappa$, the parts of integrals $J_1(\pm)$ and $J_2(\pm)$ on part 5 of the contour $\Gamma$, which is on the infinitesimal half circle centered at $\mathrm{i}\kappa$, are zeros too. Taking the influence of the cut from $-\mathrm{i}\kappa$ to $\mathrm{i}\kappa$ into account, we find that the remaining parts of the contour $\Gamma$ give the integrals

$$
\begin{aligned}
J_1(\pm) &= \int_0^\infty \frac{\exp\{\mathrm{i}[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k \\
&\quad + \int_{-\infty}^0 \frac{\exp\{\mathrm{i}[k(x+a/2) + \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\pm \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k \\
&\quad + \int_0^\kappa \mathrm{i}\frac{\exp\{-[\chi(x+a/2) - \mathrm{i}\sqrt{\kappa^2-\chi^2}ct]\}}{2[(2n)^2\pi^2 + \chi^2 a^2]}\mathrm{d}\chi \\
&\quad + \int_\kappa^0 \mathrm{i}\frac{\exp\{-[\chi(x+a/2) + \mathrm{i}\sqrt{\kappa^2-\chi^2}ct]\}}{2[(2n)^2\pi^2 + \chi^2 a^2]}\mathrm{d}\chi,
\end{aligned}
$$

and

$$
\begin{aligned}
J_2(\pm) &= \int_0^\infty \frac{\exp\{\mathrm{i}[k(x+a/2) + \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\mp \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k \\
&\quad + \int_{-\infty}^0 \frac{\exp\{\mathrm{i}[k(x+a/2) - \sqrt{k^2+\kappa^2}ct]\}}{2[(2n)^2\pi^2 - (k\mp \mathrm{i}\epsilon)^2 a^2]}\mathrm{d}k \\
&\quad + \int_0^\kappa \mathrm{i}\frac{\exp\{-[\chi(x+a/2) + \mathrm{i}\sqrt{\kappa^2-\chi^2}ct]\}}{2[(2n)^2\pi^2 + \chi^2 a^2]}\mathrm{d}\chi \\
&\quad + \int_\kappa^0 \mathrm{i}\frac{\exp\{-[\chi(x+a/2) - \mathrm{i}\sqrt{\kappa^2-\chi^2}ct]\}}{2[(2n)^2\pi^2 + \chi^2 a^2]}\mathrm{d}\chi.
\end{aligned}
$$

Therefore,

$$\begin{aligned} & J_1(+) + J_2(+) + J_1(-) + J_2(-) \\ = \; & I(+) + I(-) \equiv I\,. \end{aligned} \tag{16}$$

On the other hand, integrands of $J_1(+)$ and $J_2(-)$ are analytic inside and on the contour $\Gamma$. Integrands of $J_1(-)$ and $J_2(+)$ are analytic inside and on the contour $\Gamma$ except at two poles $\pm 2n\pi/a + \mathrm{i}\epsilon$, with residuals

$$\begin{aligned} & -\frac{\exp\{\mathrm{i}[2n\pi(x+a/2)/a - \sqrt{(2n\pi/a)^2+\kappa^2}ct]\}}{8n\pi a}, \\ & \frac{\exp\{-\mathrm{i}[2n\pi(x+a/2)/a + \sqrt{(2n\pi/a)^2+\kappa^2}ct]\}}{8n\pi a}, \\ & -\frac{\exp\{\mathrm{i}[2n\pi(x+a/2)/a + \sqrt{(2n\pi/a)^2+\kappa^2}ct]\}}{8n\pi a}, \\ & \frac{\exp\{-\mathrm{i}[2n\pi(x+a/2)/a - \sqrt{(2n\pi/a)^2+\kappa^2}ct]\}}{8n\pi a}. \end{aligned}$$

According to Cauchy theorem, we see from Eq. (16) that

$$\begin{aligned} I \; &= \; \frac{1}{na}\sin\left(\frac{2n\pi x}{a} + n\pi\right)\cos\left(\sqrt{\left(\frac{2n\pi}{a}\right)^2 + \kappa^2}ct\right) \\ &= (-1)^n \frac{1}{na}\sin\left(\frac{2n\pi x}{a}\right) \\ &\quad \times \cos\left(\sqrt{\left(\frac{2n\pi}{a}\right)^2 + \kappa^2}ct\right). \end{aligned} \tag{17}$$

From Eqs. (7)–(9), (11) and (17), we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} B_n'(k)\cos\left[k\left(x+\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\mathrm{d}k \\ = \; & \frac{b_n'}{2}\sin\left(\frac{2n\pi x}{a}\right)\cos\left(\sqrt{\left(\frac{2n\pi}{a}\right)^2 + \kappa^2}ct\right). \end{aligned} \tag{18}$$

For $t > 0$ and $x > a/2 + ct$, we have $x - a/2 - ct > 0$ and $x - a/2 + ct > 0$. Therefore, we may repeat the derivation from Eq. (11) to Eq. (17), but with $+a/2$ in the numerator of integrand in Eq. (11) being substituted by $-a/2$. Then we have

$$\begin{aligned} & \int_{-\infty}^{\infty} B_n'(k)\cos\left[k\left(x-\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\mathrm{d}k \\ = \; & \frac{b_n'}{2}\sin\left(\frac{2n\pi x}{a}\right)\cos\left(\sqrt{\left(\frac{2n\pi}{a}\right)^2 + \kappa^2}ct\right). \end{aligned} \tag{19}$$

Because this substitution only changes $\sin(2n\pi x/a + n\pi)$ in Eq. (17) into

$$\begin{aligned} \sin\left(\frac{2n\pi x}{a} - n\pi\right) &= (-1)^n \sin\left(\frac{2n\pi x}{a}\right) \\ &= \sin\left(\frac{2n\pi x}{a} + n\pi\right), \end{aligned} \tag{20}$$

it actually changes nothing. Under the same condition, the substitutions of $a/2$ by $\pm a/2$ in the numerator and $n$ by $n + 1/2$ in the denominator in the integrand of Eq. (11) change $\sin(2n\pi x/a + n\pi)$ in Eq. (17) into

$$\begin{aligned} & \sin\left[\frac{(2n+1)\pi x}{a} \pm (n+\frac{1}{2})\pi\right] \\ = \; & \pm(-1)^n \cos\frac{(2n+1)\pi x}{a}. \end{aligned} \tag{21}$$

This leads to

$$\begin{aligned} & \int_{-\infty}^{\infty} B_n(k)\cos\left[k\left(x \pm \frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\mathrm{d}k \\ = \; & \pm\frac{b_n}{2}\cos\left(\frac{2n+1}{a}\pi x\right) \\ & \times \cos\left(\sqrt{\left(\frac{2n+1}{a}\pi\right)^2 + \kappa^2}ct\right). \end{aligned} \tag{22}$$

From Eqs. (18), (19), and (22), we obtain that, for $t > 0$ and in the superluminal region $x > a/2 + ct$,

$$\begin{aligned} & \int_{-\infty}^{\infty} B_n(k)\Big\{\cos\left[k\left(x+\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right] \\ & + \cos\left[k\left(x-\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\Big\}\mathrm{d}k = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} B_n'(k)\Big\{\cos\left[k\left(x+\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right] \\ & - \cos\left[k\left(x-\frac{a}{2}\right) - \sqrt{k^2+\kappa^2}ct\right]\Big\}\mathrm{d}k = 0. \end{aligned} \tag{24}$$

A substitution $k \to -k$ shows the left-hand sides of Eqs. (23) and (24) are even and odd functions of $x$, respectively. It indicates that Eqs. (23) and (24) are also valid in the superluminal region $x < -a/2 - ct$. The wave function (7) is zero in the superluminal regions (10). The statement of the theorem is therefore proved. The velocity $v_0$ of wave packet expansion does not exceed the light velocity $c$.

Let us return to the laboratory system S, in which laboratories at positions $A$ and $B$ are at rest, but the wave packet with the coordinate system $\mathrm{S}_0$ moves along the $x$ direction at its group velocity $v_\mathrm{g}$. The velocity $v$ of the wave propagation seen from the system S is related to $v_0$ according to the relativistic theorem for velocity addition. It is

$$v = \frac{v_0 + v_\mathrm{g}}{1 + v_0 v_\mathrm{g}/c^2}\,. \tag{25}$$

For $v_0 \le c$ and $v_\mathrm{g} < c$, this formula results in $v \le c$. The wave propagation seen in any coordinate system is

therefore not superluminal. Micro particles described by relativistic wave functions cannot move faster than light either.

An often referred statement in the relativistic quantum field theory is that the value of a propagator between two space-like space–time points is zero. The above argument offers a reasoning for it.

## 4. Discussion

The impossibility of superluminal motion for classical bodies has been well known. The impossibility of superluminal wave propagation and the impossibility of superluminal motion of micro particles are proved here, based on the relativistic equations of motion. However, a general proof for the impossibility of superluminal motion based on the principle of relativity only is still lacking. The question of whether relativity forbids any superluminal motion is therefore still open.

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