Robust H_∞ control for uncertain systems with heterogeneous time-varying delays via static output feedback*

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This paper is concerned with the problem of robust H_∞ control for a novel class of uncertain linear continuous-time systems with heterogeneous time-varying state/input delays and norm-bounded parameter uncertainties. The objective is to design a static output feedback controller such that the closed-loop system is asymptotically stable while satisfying a prescribed H_∞ performance level for all admissible uncertainties. By constructing an appropriate Lyapunov–Krasovskii functional, a delay-dependent stability criterion of the closed-loop system is presented with the help of the Jensen integral inequality. From the derived criterion, the solutions to the problem are formulated in terms of linear matrix inequalities and hence are tractable numerically. A simulation example is given to illustrate the effectiveness of the proposed design method.

Keywords: robust H_∞ control, Lyapunov stability, Jensen integral inequality, linear matrix inequality

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1. Introduction

In practical systems, unavoidable uncertainties and exogenous disturbances due to inaccurate modeling, exterior conditions, and measurement errors exist widely and can badly affect system responses. For preciseness, controller design and performance analysis must be conducted to accommodate these uncertainties and disturbances. Therefore, in the past few decades, the robust control problem has grown as one of the leading topics in modern control theory, and several elegant robust control techniques addressing the disturbance rejection have been proposed, including H_∞ control,[1,2] H_2 control,[3] and $L_2 - L_\infty$ control,[4] to name just a few.

As an important tool solving the robust control problems, the robust H_∞ control theory has received particular attention of many researchers in the recent years. The goal of the robust H_∞ control is to design a controller to stabilize an uncertain system while satisfying an H_∞ norm bound constraint on the disturbance attenuation for all admissible uncertainties. Some attempts have been made to design robust H_∞ controllers for uncertain systems with various parameter uncertainties.[5−10] It is noted that a common assumption made in those designs is that the system states are all available online. However, in many real control systems, the state vector is not always accessible, and only partial information is available via measured outputs. When this happens, the designed robust H_∞ controllers based on the state feedbacks are no longer feasible. This has motivated the study of the robust H_∞ control for both continuous-time and discrete-time systems via output feedbacks (including static-output feedback (SOF) and dynamic-output feedback (DOF)) (see Refs. [11]–[15] and the references therein). However, time delays are not considered in the aforementioned works.

It is well known that time delays frequently occur in physical and engineering systems and are generally determined by the distances and the speeds of information processing.[16] The presence of time delays in practical systems may severely degrade system performances, derail the stability of the relevant systems,

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and even cause sustained oscillations.[17−20] During the past few years, much effort has been devoted to the H_∞ control of linear or nonlinear dynamical systems with time delays.[21−25] For example, by introducing a new type of Lyapunov–Krasovskii functional, a non-fragile H_∞ dynamic output feedback controller was designed for a class of linear systems with time-varying state delays.[22] For Takagi–Sugeno fuzzy systems with interval time-varying state delays, Peng *et al.*[23] presented some delay-dependent stability conditions for the robust H_∞ control. In Ref. [25], a new delay fraction technique was proposed to investigate the robust H_∞ control of uncertain systems with time delays in state or input. These relevant studies mainly concentrated on uncertain systems with either purely state time delays or purely input time delays.

On the other hand, the robust H_∞ control of uncertain systems with both state and input time delays has also begun to gain some attention. Su and Chu designed a robust H_∞ controller for a class of linear time-varying uncertain systems with delayed states and inputs when the full states could not be measured.[26] But the derivative bounds of the time-varying delays in Ref. [26] were required to be less than one. The delay-dependent robust H_∞ control for uncertain stochastic linear systems[27] and uncertain Takagi–Sugeno fuzzy systems[28] with time-varying state and input delays was addressed, where the state delay was assumed to equal to the input delay. Nevertheless, in many practical control problems, the state and the input delays are generally unknown differentiable functions and non-identical, often with large derivative bounds. In this case, the control design becomes more complicated and thus remains challenging.[29] To the best of our knowledge, the robust H_∞ control for such systems via the SOF is still open and remains unsolved.

Motivated by the above discussion, we deal with the robust H_∞ SOF control problem for uncertain linear systems. Specifically, we first propose a novel class of systems with heterogeneous time delays, parameter uncertainties, and external disturbances. The uncertainties are assumed to be time varying, unknown, but bounded, which appear additively in the matrices of system model. The time-varying delays appeared in both state and control input can be any non-negative continuous and bounded functions, and do not require that their derivatives are less than one. By constructing an appropriate Lyapunov–Krasovskii functional, a delay-dependent stability criterion of the closed-loop system is presented with the help of the Jensen integral inequality. From the derived stability criterion, the sufficient conditions for designing a robust H_∞ SOF controller are formulated within the framework of linear matrix inequalities (LMIs), which can be efficiently solved by using the interior-point algorithm.

2. Problem statement

Throughout this paper, the notation * is used as an ellipsis for terms induced by symmetry, $L_2[0,\infty)$ denotes the space of square integrable vector functions on $[0,\infty)$, and $\|\cdot\|_2$ is the L_2-norm. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

We consider the following uncertain linear model with heterogenous state and input delays:

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)+(\boldsymbol{A}_d+\Delta\boldsymbol{A}_d(t))\boldsymbol{x}(t-\tau(t))\\
&\quad+(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\boldsymbol{u}(t)\\
&\quad+(\boldsymbol{B}_d+\Delta\boldsymbol{B}_d(t))\boldsymbol{u}(t-\sigma(t))\\
&\quad+(\boldsymbol{B}_{\boldsymbol{\omega}}+\Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t))\boldsymbol{\omega}(t),\\
\boldsymbol{z}(t) &= \boldsymbol{C}_1\boldsymbol{x}(t)+\boldsymbol{C}_{1d}\boldsymbol{x}(t-\tau(t))+\boldsymbol{D}_1\boldsymbol{u}(t)\\
&\quad+\boldsymbol{D}_{1d}\boldsymbol{u}(t-\sigma(t)),\\
\boldsymbol{y}(t) &= \boldsymbol{C}_2\boldsymbol{x}(t),\\
\boldsymbol{x}(t) &= \boldsymbol{\phi}(t),\ \forall t\in[-d,0]\,,
\end{aligned}\tag{1}$$

where as usual, $\boldsymbol{x}(t)\in\mathbb{R}^n$ is the state vector, $\boldsymbol{u}(t)\in\mathbb{R}^m$ is the control input, $\boldsymbol{z}(t)\in\mathbb{R}^q$ is the controlled output, $\boldsymbol{y}(t)\in\mathbb{R}^l$ is the measured output, and $\boldsymbol{\omega}(t)\in\mathbb{R}^p$ is the exogenous disturbance input, which belongs to $L_2[0,\infty)$. The functions $\tau(t)$ and $\sigma(t)$ denote time-varying state and input delays, respectively, satisfying

$$\begin{aligned}
&0\le\tau(t)\le\tau_1,\ \dot{\tau}(t)\le\tau_2,\\
&0\le\sigma(t)\le\sigma_1,\ \dot{\sigma}(t)\le\sigma_2,
\end{aligned}\tag{2}$$

where τ_1, τ_2, σ_1, σ_2 are known constants, and $d=\max\{\tau_1,\sigma_1\}$. The $\boldsymbol{A}$, $\boldsymbol{A}_d$, $\boldsymbol{B}$, $\boldsymbol{B}_d$, $\boldsymbol{B}_{\boldsymbol{\omega}}$, $\boldsymbol{C}_1$ $\boldsymbol{C}_{1d}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_{1d}$, and $\boldsymbol{C}_2$ are known constant matrices, and $\boldsymbol{C}_2$ has the full row rank. The $\Delta\boldsymbol{A}(t)$, $\Delta\boldsymbol{A}_d(t)$, $\Delta\boldsymbol{B}(t)$, $\Delta\boldsymbol{B}_d(t)$, and $\Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t)$ are unknown real-valued matrices representing the time-varying parameter uncertainties, which are assumed to be of the form

$$\begin{aligned}
&\left[\,\Delta\boldsymbol{A}(t)\ \Delta\boldsymbol{A}_d(t)\ \Delta\boldsymbol{B}(t)\ \Delta\boldsymbol{B}_d(t)\ \Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t)\,\right]\\
&=\boldsymbol{M}\boldsymbol{F}(t)\left[\,\boldsymbol{E}_1\ \boldsymbol{E}_2\ \boldsymbol{E}_3\ \boldsymbol{E}_4\ \boldsymbol{E}_5\,\right],
\end{aligned}\tag{3}$$

where $\boldsymbol{M}$, $\boldsymbol{E}_i$ $(i = 1, 2, \ldots, 5)$ are known real-constant matrices of appropriate dimensions, and $\boldsymbol{F}(t)$ is an unknown time-varying matrix satisfying the inequality

$$\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t) \leq \boldsymbol{I}, \ \forall t. \tag{4}$$

The parameter uncertainties are said to be admissible if both relations (3) and (4) hold.

Remark 1 In modeling physical systems, it is almost impossible to get exact mathematical models because of environmental noise, system complexity, the difficulty of measuring various parameters, *etc.* Hence, the linear plant model of the system to be controlled always contains some types of uncertainties. The plant uncertainty can be broadly classified into the structured uncertainty (including norm-bounded uncertainty, polytopic uncertainty, affine uncertainty, interval uncertainty, *etc*) and the unstructured uncertainty (including additive uncertainty, multiplicative uncertainty, nonlinear uncertainty, *etc*). The uncertainty satisfying relations (3) and (4) is of the norm-bounded type, which serves as a general description of the parameter uncertainties of many physical systems and has been widely studied in the robust H_∞ control problems.[26−31] It should be noted that parameter uncertainties for matrices $\boldsymbol{C}_1$, $\boldsymbol{C}_{1d}$, $\boldsymbol{D}_1$, $\boldsymbol{D}_{1d}$, and $\boldsymbol{C}_2$ can also be handled by the same line of treatments, and therefore are omitted here for brevity.

In this paper, we are concerned with designing the following SOF control law for the uncertain system (1):

$$\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{y}(\boldsymbol{t}), \tag{5}$$

where $\boldsymbol{K}$ is the constant controller gain matrix to be determined later.

Remark 2 It is worth pointing out that among various control schemes, the output feedback control is particularly useful and more realistic, since the state variables are not always fully measurable in practice. Generally, the output feedback control is classified into two fundamental categories: static-output-feedback (SOF) control and dynamics-output-feedback (DOF) control. Our interest in the SOF control is motivated by the following reasons. First, the SOF control can be easily implemented at low cost.[32] Second, the DOF stabilization controller can often be transformed into an SOF stabilization controller.[33]

By substituting the SOF controller (5) to system (1), we obtain the following closed-loop system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{A}}\boldsymbol{x}(t) + \bar{\boldsymbol{A}}_d\boldsymbol{x}(t-\tau(t)) \\ \qquad\quad + \boldsymbol{\mathcal{B}}_d\boldsymbol{x}(t-\sigma(t)) + \bar{\boldsymbol{B}}_{\boldsymbol{\omega}}\boldsymbol{\omega}(t), \\ \boldsymbol{z}(t) = (\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t) + \boldsymbol{C}_{1d}\boldsymbol{x}(t-\tau(t)) \\ \qquad\quad + (\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t-\sigma(t)), \end{cases} \tag{6}$$

where

$$\begin{aligned} &\boldsymbol{\mathcal{A}} = \bar{\boldsymbol{A}} + \bar{\boldsymbol{B}}\boldsymbol{K}\boldsymbol{C}_2, \ \boldsymbol{\mathcal{B}}_d = \bar{\boldsymbol{B}}_d\boldsymbol{K}\boldsymbol{C}_2, \ \ \bar{\boldsymbol{A}} = \boldsymbol{A} + \Delta\boldsymbol{A}, \\ &\bar{\boldsymbol{B}} = \boldsymbol{B} + \Delta\boldsymbol{B}, \ \ \bar{\boldsymbol{A}}_d = \boldsymbol{A}_d + \Delta\boldsymbol{A}_d, \\ &\bar{\boldsymbol{B}}_d = \boldsymbol{B}_d + \Delta\boldsymbol{B}_d, \ \ \bar{\boldsymbol{B}}_{\boldsymbol{\omega}} = \boldsymbol{B}_{\boldsymbol{\omega}} + \Delta\boldsymbol{B}_{\boldsymbol{\omega}}. \end{aligned} \tag{7}$$

So far, the problem to be addressed in this paper can be expressed as follows. Given system (1), a prescribed level of noise attention $\gamma > 0$, all admissible parametric uncertainties and time-varying delays satisfying relations (2)–(4), design a suitable SOF controller of the form (5) satisfying the following requirements: (i) the closed-loop system (6) is asymptotically stable when $\boldsymbol{\omega}(t) \equiv 0$; (ii) the closed-loop system (6) satisfies the H_∞ performance

$$\|\boldsymbol{z}(t)\|_2 \leq \gamma\|\boldsymbol{\omega}\|_2 \tag{8}$$

for any nonzero $\boldsymbol{\omega}(t) \in L_2[0, \infty)$ under the zero initial condition. In this case, we say that system (1) is robustly stabilizable with H_∞ disturbance attenuation γ.

3. The H_∞ performance analysis of the closed-loop system

In this section, we will derive the delay-dependent sufficient conditions for the H_∞ performance analysis of the closed-loop system (6). It is assumed that the feedback gain matrix $\boldsymbol{K}$ is known and the system parameter uncertainties $\Delta\boldsymbol{A}(t)$, $\Delta\boldsymbol{A}_d(t)$, $\Delta\boldsymbol{B}(t)$, $\Delta\boldsymbol{B}_d(t)$, and $\Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t)$ satisfy relations (3) and (4), which are common assumptions in the open literatures for stability analysis.

Theorem 1 For given scalars $\gamma > 0$, $\tau_1 > 0$, $\tau_2 > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, as well as a given matrix $\boldsymbol{K}$, the closed-loop system (6) is robustly asymptotically stable with a H_∞ disturbance attenuation level γ for all admissible additive parameter uncertainties $\Delta\boldsymbol{A}(t)$, $\Delta\boldsymbol{A}_d(t)$, $\Delta\boldsymbol{B}(t)$, $\Delta\boldsymbol{B}_d(t)$, and $\Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t)$ if there exist matrices $\hat{\boldsymbol{P}} > 0$, $\hat{\boldsymbol{Q}}_1 > 0$, $\hat{\boldsymbol{Q}}_2 > 0$, $\hat{\boldsymbol{Q}}_3 > 0$, $\hat{\boldsymbol{R}}_1 > 0$, $\hat{\boldsymbol{R}}_2 > 0$, $\hat{\boldsymbol{R}}_3 > 0$ such that the following inequality holds:

$$\boldsymbol{\Phi}=\begin{bmatrix}
\boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} & \hat{\boldsymbol{P}}\bar{\boldsymbol{B}}_{\boldsymbol{\omega}} & \tau_1\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2 & \sigma_1\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2 & -\hat{\boldsymbol{Q}}_2 & -\hat{\boldsymbol{R}}_2\\
* & \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & 0 & \tau_1\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2 & \sigma_1\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2 & 0 & 0\\
* & * & \boldsymbol{\Phi}_{33} & 0 & \tau_1\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2 & \sigma_1\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2 & 0 & 0\\
* & * & * & -\gamma^2\boldsymbol{I} & \tau_1\bar{\boldsymbol{B}}_{\boldsymbol{\omega}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2 & \sigma_1\bar{\boldsymbol{B}}_{\boldsymbol{\omega}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2 & 0 & 0\\
* & * & * & * & -\hat{\boldsymbol{Q}}_2 & 0 & 0 & 0\\
* & * & * & * & * & -\hat{\boldsymbol{R}}_2 & 0 & 0\\
* & * & * & * & * & * & -\hat{\boldsymbol{Q}}_1-\hat{\boldsymbol{Q}}_2 & 0\\
* & * & * & * & * & * & * & -\hat{\boldsymbol{R}}_1-\hat{\boldsymbol{R}}_2
\end{bmatrix}<0, \tag{9}$$

where

$$\begin{aligned}
\boldsymbol{\Phi}_{11} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{A}}+\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{P}}+\hat{\boldsymbol{Q}}_1+\hat{\boldsymbol{Q}}_3+\hat{\boldsymbol{R}}_1+\hat{\boldsymbol{R}}_3-\hat{\boldsymbol{Q}}_2\\
&\quad-\hat{\boldsymbol{R}}_2+(\boldsymbol{C}_1+\boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{C}_1+\boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2),\\
\boldsymbol{\Phi}_{12} &= \hat{\boldsymbol{P}}\bar{\boldsymbol{A}}_d+(\boldsymbol{C}_1+\boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}\boldsymbol{C}_{1d},\\
\boldsymbol{\Phi}_{13} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{B}}_d+(\boldsymbol{C}_1+\boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2),\\
\boldsymbol{\Phi}_{22} &= (\tau_2-1)\hat{\boldsymbol{Q}}_3+\boldsymbol{C}_{1d}^{\mathrm{T}}\boldsymbol{C}_{1d},\\
\boldsymbol{\Phi}_{23} &= \boldsymbol{C}_{1d}^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2),\\
\boldsymbol{\Phi}_{33} &= (\sigma_2-1)\hat{\boldsymbol{R}}_3+(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2).
\end{aligned}$$

Proof First, assuming that $\boldsymbol{\omega}(t)=0$, we will prove the asymptotic stability of the closed-loop system (6). Select a Lyapunov–Krasovskii functional candidate for the closed-loop system as

$$\begin{aligned}
V(t) &= V_1(t)+V_2(t)+V_3(t)+V_4(t)\\
&\quad+V_5(t)+V_6(t)+V_7(t),
\end{aligned}\tag{10}$$

with

$$\begin{aligned}
V_1(t) &= \boldsymbol{x}^{\mathrm{T}}\hat{\boldsymbol{P}}\boldsymbol{x},\quad V_2(t)=\int_{t-\tau_1}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(s)\mathrm{d}s,\\
V_3(t) &= \tau_1\int_{-\tau_1}^{0}\int_{t+s}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(v)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(v)\mathrm{d}v\mathrm{d}s,\\
V_4(t) &= \int_{t-\tau(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(s)\mathrm{d}s,\\
V_5(t) &= \int_{t-\sigma_1}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\hat{\boldsymbol{R}}_1\boldsymbol{x}(s)\mathrm{d}s,\\
V_6(t) &= \sigma_1\int_{-\sigma_1}^{0}\int_{t+s}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(v)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(v)\mathrm{d}v\mathrm{d}s,\\
V_7(t) &= \int_{t-\sigma(t)}^{t}\boldsymbol{x}^{\mathrm{T}}(s)\hat{\boldsymbol{R}}_3\boldsymbol{x}(s)\mathrm{d}s,
\end{aligned}$$

where matrices $\hat{\boldsymbol{P}}$, $\hat{\boldsymbol{Q}}_1$, $\hat{\boldsymbol{Q}}_2$, $\hat{\boldsymbol{Q}}_3$, $\hat{\boldsymbol{R}}_1$, $\hat{\boldsymbol{R}}_2$, $\hat{\boldsymbol{R}}_3$ are positive symmetric matrices to be determined. Then along the trajectory of system (6), the derivative of Eq. (10) is given by

$$\begin{aligned}
\dot{V}(t) &= 2\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{P}}\dot{\boldsymbol{x}}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(t)\\
&\quad-\boldsymbol{x}^{\mathrm{T}}(t-\tau_1)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(t-\tau_1)\\
&\quad+\tau_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(t)-\tau_1\int_{t-\tau_1}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\quad+\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{Q}}_3\boldsymbol{x}(t)-(1-\dot{\tau}(t))\\
&\quad\times\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\hat{\boldsymbol{Q}}_3\boldsymbol{x}(t-\tau(t))\\
&\quad+\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{R}}_1\boldsymbol{x}(t)-\boldsymbol{x}^{\mathrm{T}}(t-\sigma_1)\hat{\boldsymbol{R}}_1\boldsymbol{x}(t-\sigma_1)\\
&\quad+\sigma_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(t)-\sigma_1\int_{t-\sigma_1}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\quad+\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{R}}_3\boldsymbol{x}(t)-(1-\dot{\sigma}(t))\\
&\quad\times\boldsymbol{x}^{\mathrm{T}}(t-\sigma(t))\hat{\boldsymbol{R}}_3\boldsymbol{x}(t-\sigma(t))\\
&\le 2\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{P}}\dot{\boldsymbol{x}}(t)+\boldsymbol{x}^{\mathrm{T}}(t)(\hat{\boldsymbol{Q}}_1+\hat{\boldsymbol{Q}}_3+\hat{\boldsymbol{R}}_1+\hat{\boldsymbol{R}}_3)\boldsymbol{x}(t)\\
&\quad+\tau_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(t)+\sigma_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(t)\\
&\quad-(1-\tau_2)\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\hat{\boldsymbol{Q}}_3\boldsymbol{x}(t-\tau(t))\\
&\quad-(1-\sigma_2)\boldsymbol{x}^{\mathrm{T}}(t-\sigma(t))\hat{\boldsymbol{R}}_3\boldsymbol{x}(t-\sigma(t))\\
&\quad-\boldsymbol{x}^{\mathrm{T}}(t-\tau_1)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(t-\tau_1)\\
&\quad-\boldsymbol{x}^{\mathrm{T}}(t-\sigma_1)\hat{\boldsymbol{R}}_1\boldsymbol{x}(t-\sigma_1)\\
&\quad-\tau_1\int_{t-\tau_1}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\quad-\sigma_1\int_{t-\sigma_1}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(s)\mathrm{d}s.
\end{aligned}\tag{11}$$

By using the Jensen integral inequality,[17] we further have

$$\begin{aligned}
\dot{V}(t) &\le 2\boldsymbol{x}^{\mathrm{T}}(t)\hat{\boldsymbol{P}}(\boldsymbol{\mathcal{A}}\boldsymbol{x}(t)+\bar{\boldsymbol{A}}_d\boldsymbol{x}(t-\tau(t))\\
&\quad+\boldsymbol{\mathcal{B}}_d\boldsymbol{x}(t-\sigma(t)))\\
&\quad+\boldsymbol{x}^{\mathrm{T}}(t)\left(\hat{\boldsymbol{Q}}_1+\hat{\boldsymbol{Q}}_3+\hat{\boldsymbol{R}}_1+\hat{\boldsymbol{R}}_3\right)\boldsymbol{x}(t)\\
&\quad+\tau_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{Q}}_2\dot{\boldsymbol{x}}(t)+\sigma_1^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\hat{\boldsymbol{R}}_2\dot{\boldsymbol{x}}(t)\\
&\quad-(1-\tau_2)\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))\hat{\boldsymbol{Q}}_3\boldsymbol{x}(t-\tau(t))\\
&\quad-(1-\sigma_2)\boldsymbol{x}^{\mathrm{T}}(t-\sigma(t))\hat{\boldsymbol{R}}_3\boldsymbol{x}(t-\sigma(t))\\
&\quad-\boldsymbol{x}^{\mathrm{T}}(t-\tau_1)\hat{\boldsymbol{Q}}_1\boldsymbol{x}(t-\tau_1)\\
&\quad-\boldsymbol{x}^{\mathrm{T}}(t-\sigma_1)\hat{\boldsymbol{R}}_1\boldsymbol{x}(t-\sigma_1)\\
&\quad+\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-\tau_1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-\hat{\boldsymbol{Q}}_2 & \hat{\boldsymbol{Q}}_2\\ * & -\hat{\boldsymbol{Q}}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-\tau_1)\end{bmatrix}\\
&\quad+\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-\sigma_1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-\hat{\boldsymbol{R}}_2 & \hat{\boldsymbol{R}}_2\\ * & -\hat{\boldsymbol{R}}_2\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-\sigma_1)\end{bmatrix}\\
&= \boldsymbol{\xi}^{\mathrm{T}}(t)\hat{\boldsymbol{\Phi}}\boldsymbol{\xi}(t),
\end{aligned}\tag{12}$$

where

$$\boldsymbol{\xi}(t) = \left[\boldsymbol{x}^{\mathrm{T}}(t), \boldsymbol{x}^{\mathrm{T}}(t-\tau(t)), \boldsymbol{x}^{\mathrm{T}}(t-\sigma(t)), \boldsymbol{x}^{\mathrm{T}}(t-\tau_1), \boldsymbol{x}^{\mathrm{T}}(t-\sigma_1)\right]^{\mathrm{T}},$$

$$\hat{\boldsymbol{\Phi}} = \begin{bmatrix} \hat{\boldsymbol{\Phi}}_{11} & \hat{\boldsymbol{\Phi}}_{12} & \hat{\boldsymbol{\Phi}}_{13} & \hat{\boldsymbol{Q}}_2 & \hat{\boldsymbol{R}}_2 \\ * & \hat{\boldsymbol{\Phi}}_{22} & \hat{\boldsymbol{\Phi}}_{23} & 0 & 0 \\ * & * & \hat{\boldsymbol{\Phi}}_{33} & 0 & 0 \\ * & * & * & -\hat{\boldsymbol{Q}}_1 - \hat{\boldsymbol{Q}}_2 & 0 \\ * & * & * & * & -\hat{\boldsymbol{R}}_1 - \hat{\boldsymbol{R}}_2 \end{bmatrix}, \tag{13}$$

with

$$\begin{aligned}
\hat{\boldsymbol{\Phi}}_{11} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{P}} + \hat{\boldsymbol{Q}}_1 + \hat{\boldsymbol{Q}}_3 + \hat{\boldsymbol{R}}_1 + \hat{\boldsymbol{R}}_3 \\
&\quad - \hat{\boldsymbol{Q}}_2 - \hat{\boldsymbol{R}}_2 + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{A}} + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{A}}, \\
\hat{\boldsymbol{\Phi}}_{12} &= \hat{\boldsymbol{P}}\bar{\boldsymbol{A}}_d + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{A}}_d + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{A}}_d, \\
\hat{\boldsymbol{\Phi}}_{13} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{B}}_d + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d, \\
\hat{\boldsymbol{\Phi}}_{22} &= (\tau_2 - 1)\hat{\boldsymbol{Q}}_3 + \tau_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{A}}_d + \sigma_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{A}}_d, \\
\hat{\boldsymbol{\Phi}}_{23} &= \tau_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d, \\
\hat{\boldsymbol{\Phi}}_{33} &= (\sigma_2 - 1)\hat{\boldsymbol{R}}_3 + \tau_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d.
\end{aligned}$$

Applying the Schur complement formula[34] to inequality (9), we can obtain $\hat{\boldsymbol{\Phi}} < 0$. Thus, $\dot{V}(t) < 0$ for all $\boldsymbol{\xi}(t) \neq 0$, which means that system (6) with $\boldsymbol{\omega}(t) = 0$ is asymptotically stable for any $\tau(t)$ and $\sigma(t)$ satisfying Eq. (2).

Next, assuming that $\boldsymbol{\phi}(t) = 0$, $t \in [-d, 0]$, we will show that the closed-loop system (6) has H_∞ performance γ. Define a performance index as

$$J = \int_0^\infty \left[\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t)\right] \mathrm{d}t. \tag{14}$$

Noting $\boldsymbol{\phi}(t) = 0$, $t \in [-d, 0]$, for any nonzero $\boldsymbol{\omega}(t) \in L_2[0, \infty)$, we have

$$\begin{aligned}
J &= \int_0^\infty \left[\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) + \dot{V}(t)\right] \mathrm{d}t \\
&\quad - \lim_{t\to\infty} V(t) \\
&\leq \int_0^\infty \left[\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) + \dot{V}(t)\right] \mathrm{d}t.
\end{aligned} \tag{15}$$

Then, along a similar argument as that in the derivation of Eq. (11), we can see that

$$\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2\boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) + \dot{V}(t) \leq \boldsymbol{\zeta}^{\mathrm{T}}(t)\tilde{\boldsymbol{\Phi}}\boldsymbol{\zeta}(t), \tag{16}$$

where

$$\boldsymbol{\zeta}(t) = \left[\boldsymbol{x}^{\mathrm{T}}(t), \boldsymbol{x}^{\mathrm{T}}(t-\tau(t)), \boldsymbol{x}^{\mathrm{T}}(t-\sigma(t)), \boldsymbol{\omega}^{\mathrm{T}}(t), \boldsymbol{x}^{\mathrm{T}}(t-\tau_1), \boldsymbol{x}^{\mathrm{T}}(t-\sigma_1)\right]^{\mathrm{T}},$$

$$\tilde{\boldsymbol{\Phi}} = \begin{bmatrix} \tilde{\boldsymbol{\Phi}}_{11} & \tilde{\boldsymbol{\Phi}}_{12} & \tilde{\boldsymbol{\Phi}}_{13} & \tilde{\boldsymbol{\Phi}}_{14} & -\hat{\boldsymbol{Q}}_2 & -\hat{\boldsymbol{R}}_2 \\ * & \tilde{\boldsymbol{\Phi}}_{22} & \tilde{\boldsymbol{\Phi}}_{23} & \tilde{\boldsymbol{\Phi}}_{24} & 0 & 0 \\ * & * & \tilde{\boldsymbol{\Phi}}_{33} & \tilde{\boldsymbol{\Phi}}_{34} & 0 & 0 \\ * & * & * & \tilde{\boldsymbol{\Phi}}_{44} & 0 & 0 \\ * & * & * & * & -\hat{\boldsymbol{Q}}_1 - \hat{\boldsymbol{Q}}_2 & 0 \\ * & * & * & * & * & -\hat{\boldsymbol{R}}_1 - \hat{\boldsymbol{R}}_2 \end{bmatrix}, \tag{17}$$

with

$$\begin{aligned}
\tilde{\boldsymbol{\Phi}}_{11} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{P}} + \hat{\boldsymbol{Q}}_1 + \hat{\boldsymbol{Q}}_3 + \hat{\boldsymbol{R}}_1 + \hat{\boldsymbol{R}}_3 \\
&\quad - \hat{\boldsymbol{Q}}_2 - \hat{\boldsymbol{R}}_2 + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{A}} + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{A}} \\
&\quad + (\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2), \\
\tilde{\boldsymbol{\Phi}}_{12} &= \hat{\boldsymbol{P}}\bar{\boldsymbol{A}}_d + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{A}}_d + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{A}}_d \\
&\quad + (\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}\boldsymbol{C}_{1d}, \\
\tilde{\boldsymbol{\Phi}}_{13} &= \hat{\boldsymbol{P}}\boldsymbol{\mathcal{B}}_d + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d \\
&\quad + (\boldsymbol{C}_1 + \boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2), \\
\tilde{\boldsymbol{\Phi}}_{14} &= \hat{\boldsymbol{P}}\bar{\boldsymbol{B}}_{\boldsymbol{\omega}} + \tau_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}} + \sigma_1^2\boldsymbol{\mathcal{A}}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}}, \\
\tilde{\boldsymbol{\Phi}}_{22} &= (\tau_2 - 1)\hat{\boldsymbol{Q}}_3 + \tau_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{A}}_d \\
&\quad + \sigma_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{A}}_d + \boldsymbol{C}_{1d}^{\mathrm{T}}\boldsymbol{C}_{1d}, \\
\tilde{\boldsymbol{\Phi}}_{23} &= \tau_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d + \boldsymbol{C}_{1d}^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2), \\
\tilde{\boldsymbol{\Phi}}_{24} &= \tau_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}} + \sigma_1^2\bar{\boldsymbol{A}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}}, \\
\tilde{\boldsymbol{\Phi}}_{33} &= (\sigma_2 - 1)\hat{\boldsymbol{R}}_3 + \tau_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{\mathcal{B}}_d + \sigma_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{\mathcal{B}}_d \\
&\quad + (\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}(\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2), \\
\tilde{\boldsymbol{\Phi}}_{34} &= \tau_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}} + \sigma_1^2\boldsymbol{\mathcal{B}}_d^{\mathrm{T}}\hat{\boldsymbol{R}}_2\bar{\boldsymbol{B}}_{\boldsymbol{\omega}},
\end{aligned}$$

$\tilde{\boldsymbol{\Phi}}_{44} = \tau_1^2 \bar{\boldsymbol{B}}_{\boldsymbol{\omega}}^{\mathrm{T}} \hat{\boldsymbol{Q}}_2 \bar{\boldsymbol{B}}_{\boldsymbol{\omega}} + \sigma_1^2 \bar{\boldsymbol{B}}_{\boldsymbol{\omega}}^{\mathrm{T}} \hat{\boldsymbol{R}}_2 \bar{\boldsymbol{B}}_{\boldsymbol{\omega}} - \gamma^2 \boldsymbol{I}$.

Then by using the Schur complement,[34] we can see that inequality (9) guarantees $\tilde{\boldsymbol{\Phi}} < 0$. Therefore, it follows from inequality (15) that

$$J = \int_0^\infty \left[\boldsymbol{z}^{\mathrm{T}}(t)\boldsymbol{z}(t) - \gamma^2 \boldsymbol{\omega}^{\mathrm{T}}(t)\boldsymbol{\omega}(t) \right] dt \leq 0. \quad (18)$$

Hence, $\|\boldsymbol{z}(t)\|_2 \leq \gamma \|\boldsymbol{\omega}\|_2$ is satisfied for any nonzero $\boldsymbol{\omega}(t) \in L_2[0, \infty)$. This completes the proof.

4. Robust H_∞ controller design

Theorem 1 establishes a delay-dependent sufficient condition for the existence of the H_∞ controller in terms of matrix inequality, which, however, contains coupled variables. In addition, there exist uncertainties included by the system parameters in inequality (9). This condition can not be directly applied to design the SOF controller. In the following, we will present some LMI-type conditions for the existence of the SOF controller.

4.1. The H_∞ control without uncertainties

We first consider the following nominal form of the closed-loop system (6) without system parameter uncertainties $\Delta \boldsymbol{A}(t)$, $\Delta \boldsymbol{A}_d(t)$, $\Delta \boldsymbol{B}(t)$, $\Delta \boldsymbol{B}_d(t)$, and $\Delta \boldsymbol{B}_{\boldsymbol{\omega}}(t)$:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t) + \boldsymbol{A}_d \boldsymbol{x}(t - \tau(t)) \\ \qquad\quad + (\boldsymbol{B}_d \boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t - \sigma(t)) + \boldsymbol{B}_{\boldsymbol{\omega}}\boldsymbol{\omega}(t), \\ \boldsymbol{z}(t) = (\boldsymbol{C}_1 + \boldsymbol{D}_1 \boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t) + \boldsymbol{C}_{1d}\boldsymbol{x}(t - \tau(t)) \\ \qquad\quad + (\boldsymbol{D}_{1d} \boldsymbol{K}\boldsymbol{C}_2)\boldsymbol{x}(t - \sigma(t)). \end{cases} \quad (19)$$

Theorem 2 For given scalars $\gamma > 0$, $\tau_1 > 0$, $\tau_2 > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, the closed-loop system (19) is robustly asymptotically stable with a H_∞ disturbance attenuation level γ if there exist matrices $\boldsymbol{X} > 0$, $\boldsymbol{Q}_1 > 0$, $\boldsymbol{Q}_2 > 0$, $\boldsymbol{Q}_3 > 0$, $\boldsymbol{R}_1 > 0$, $\boldsymbol{R}_2 > 0$, $\boldsymbol{R}_3 > 0$, and $\boldsymbol{Y}$ such that the following LMI holds:

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} & \boldsymbol{\Gamma}_{13} & \boldsymbol{\Gamma}_{14} & \boldsymbol{\Gamma}_{15} & \boldsymbol{\Gamma}_{16} & \boldsymbol{\Gamma}_{17} & -\boldsymbol{Q}_2 & -\boldsymbol{R}_2 \\ * & \boldsymbol{\Gamma}_{22} & 0 & 0 & \boldsymbol{\Gamma}_{25} & \boldsymbol{\Gamma}_{26} & \boldsymbol{\Gamma}_{27} & 0 & 0 \\ * & * & \boldsymbol{\Gamma}_{33} & 0 & \boldsymbol{\Gamma}_{35} & \boldsymbol{\Gamma}_{36} & \boldsymbol{\Gamma}_{37} & 0 & 0 \\ * & * & * & \boldsymbol{\Gamma}_{44} & \boldsymbol{\Gamma}_{45} & \boldsymbol{\Gamma}_{46} & 0 & 0 & 0 \\ * & * & * & * & \boldsymbol{\Gamma}_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \boldsymbol{\Gamma}_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \boldsymbol{\Gamma}_{77} & 0 & 0 \\ * & * & * & * & * & * & * & -\boldsymbol{Q}_1 - \boldsymbol{Q}_2 & 0 \\ * & * & * & * & * & * & * & * & -\boldsymbol{R}_1 - \boldsymbol{R}_2 \end{bmatrix} < 0, \quad (20)$$

where

$$\begin{aligned} \boldsymbol{\Gamma}_{11} &= (\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{Y}) + (\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{Y})^{\mathrm{T}} \\ &\quad + \boldsymbol{Q}_1 + \boldsymbol{Q}_3 + \boldsymbol{R}_1 + \boldsymbol{R}_3 - \boldsymbol{Q}_2 - \boldsymbol{R}_2, \\ \boldsymbol{\Gamma}_{12} &= \boldsymbol{A}_d \boldsymbol{X}, \quad \boldsymbol{\Gamma}_{13} = \boldsymbol{B}_d \boldsymbol{Y}, \\ \boldsymbol{\Gamma}_{14} &= \boldsymbol{B}_{\boldsymbol{\omega}}, \quad \boldsymbol{\Gamma}_{15} = \boldsymbol{\Gamma}_{16} = (\boldsymbol{A}\boldsymbol{X} + \boldsymbol{B}\boldsymbol{Y})^{\mathrm{T}}, \\ \boldsymbol{\Gamma}_{17} &= (\boldsymbol{C}_1 \boldsymbol{X} + \boldsymbol{D}_1 \boldsymbol{Y})^{\mathrm{T}}, \\ \boldsymbol{\Gamma}_{22} &= (\tau_2 - 1)\boldsymbol{Q}_3, \quad \boldsymbol{\Gamma}_{25} = \boldsymbol{\Gamma}_{26} = (\boldsymbol{A}_d \boldsymbol{X})^{\mathrm{T}}, \\ \boldsymbol{\Gamma}_{27} &= (\boldsymbol{C}_{1d} \boldsymbol{X})^{\mathrm{T}}, \quad \boldsymbol{\Gamma}_{33} = (\sigma_2 - 1)\boldsymbol{R}_3, \\ \boldsymbol{\Gamma}_{35} &= \boldsymbol{\Gamma}_{36} = (\boldsymbol{B}_d \boldsymbol{Y})^{\mathrm{T}}, \quad \boldsymbol{\Gamma}_{37} = (\boldsymbol{D}_{1d} \boldsymbol{Y})^{\mathrm{T}}, \\ \boldsymbol{\Gamma}_{44} &= -\gamma^2 \boldsymbol{I}, \quad \boldsymbol{\Gamma}_{45} = \boldsymbol{\Gamma}_{46} = \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}}, \\ \boldsymbol{\Gamma}_{55} &= \tau_1^{-2}(\boldsymbol{Q}_2 - 2\boldsymbol{X}), \\ \boldsymbol{\Gamma}_{66} &= \sigma_1^{-2}(\boldsymbol{R}_2 - 2\boldsymbol{X}), \quad \boldsymbol{\Gamma}_{77} = -\boldsymbol{X}. \end{aligned}$$

In this case, the desired SOF controller can be constructed as in Eq. (5) with

$$\boldsymbol{K} = \boldsymbol{Y}\boldsymbol{X}^{-1}\boldsymbol{C}_2^{\mathrm{T}}(\boldsymbol{C}_2\boldsymbol{C}_2^{\mathrm{T}})^{-1}. \quad (21)$$

Proof Without system parameter uncertainties $\Delta \boldsymbol{A}(t)$, $\Delta \boldsymbol{A}_d(t)$, $\Delta \boldsymbol{B}(t)$, $\Delta \boldsymbol{B}_d(t)$, and $\Delta \boldsymbol{B}_{\boldsymbol{\omega}}(t)$, it is easy to see that inequality (9) is equivalent to the following inequality by using the Schur complement formula:[34]

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} \hat{\boldsymbol{\Gamma}}_{11} & \hat{\boldsymbol{\Gamma}}_{12} & \hat{\boldsymbol{\Gamma}}_{13} & \hat{\boldsymbol{P}}\boldsymbol{B}_{\boldsymbol{\omega}} & \hat{\boldsymbol{\Gamma}}_{15} & \hat{\boldsymbol{\Gamma}}_{16} & \hat{\boldsymbol{\Gamma}}_{17} & -\hat{\boldsymbol{Q}}_2 & -\hat{\boldsymbol{R}}_2 \\ * & \hat{\boldsymbol{\Gamma}}_{22} & 0 & 0 & \boldsymbol{A}_d^{\mathrm{T}} & \boldsymbol{A}_d^{\mathrm{T}} & \boldsymbol{C}_{1d}^{\mathrm{T}} & 0 & 0 \\ * & * & \hat{\boldsymbol{\Gamma}}_{33} & 0 & \hat{\boldsymbol{\Gamma}}_{35} & \hat{\boldsymbol{\Gamma}}_{36} & \hat{\boldsymbol{\Gamma}}_{37} & 0 & 0 \\ * & * & * & -\gamma^2\boldsymbol{I} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & * & * & -\tau_1^{-2}\hat{\boldsymbol{Q}}_2^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\sigma_1^{-2}\hat{\boldsymbol{R}}_2^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\boldsymbol{I} & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\boldsymbol{\Gamma}}_{88} & 0 \\ * & * & * & * & * & * & * & * & \hat{\boldsymbol{\Gamma}}_{99} \end{bmatrix} < 0, \tag{22}$$

where

$$\begin{aligned} \hat{\boldsymbol{\Gamma}}_{11} &= \hat{\boldsymbol{P}}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}\boldsymbol{C}_2) + (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}\hat{\boldsymbol{P}} \\ &\quad + \hat{\boldsymbol{Q}}_1 + \hat{\boldsymbol{Q}}_3 + \hat{\boldsymbol{R}}_1 + \hat{\boldsymbol{R}}_3 - \hat{\boldsymbol{Q}}_2 - \hat{\boldsymbol{R}}_2, \\ \hat{\boldsymbol{\Gamma}}_{12} &= \hat{\boldsymbol{P}}\boldsymbol{A}_d, \quad \hat{\boldsymbol{\Gamma}}_{13} = \hat{\boldsymbol{P}}\boldsymbol{B}_d\boldsymbol{K}\boldsymbol{C}_2, \\ \hat{\boldsymbol{\Gamma}}_{15} &= \hat{\boldsymbol{\Gamma}}_{16} = (\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}, \\ \hat{\boldsymbol{\Gamma}}_{17} &= (\boldsymbol{C}_1+\boldsymbol{D}_1\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}, \\ \hat{\boldsymbol{\Gamma}}_{22} &= (\tau_2-1)\hat{\boldsymbol{Q}}_3, \quad \hat{\boldsymbol{\Gamma}}_{33} = (\sigma_2-1)\hat{\boldsymbol{R}}_3, \\ \hat{\boldsymbol{\Gamma}}_{35} &= \hat{\boldsymbol{\Gamma}}_{36} = (\boldsymbol{B}_d\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}, \quad \hat{\boldsymbol{\Gamma}}_{37} = (\boldsymbol{D}_{1d}\boldsymbol{K}\boldsymbol{C}_2)^{\mathrm{T}}, \\ \hat{\boldsymbol{\Gamma}}_{88} &= -\hat{\boldsymbol{Q}}_1 - \hat{\boldsymbol{Q}}_2, \quad \hat{\boldsymbol{\Gamma}}_{99} = -\hat{\boldsymbol{R}}_1 - \hat{\boldsymbol{R}}_2. \end{aligned} \tag{23}$$

We can see that inequality (22) holds if the following inequality holds:

$$\hat{\boldsymbol{P}}\hat{\boldsymbol{Q}}_2^{-1}\hat{\boldsymbol{P}} \geq 2\hat{\boldsymbol{P}} - \hat{\boldsymbol{Q}}_2, \hat{\boldsymbol{P}}\hat{\boldsymbol{R}}_2^{-1}\hat{\boldsymbol{P}} \geq 2\hat{\boldsymbol{P}} - \hat{\boldsymbol{R}}_2, \tag{24}$$

which is obtained from the following steps. Firstly, we pre- and post-multiply inequality (22) by $\mathrm{diag}\left\{\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\hat{\boldsymbol{P}},\hat{\boldsymbol{P}},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}\right\}$ and its transpose, respectively. Secondly, we replace the diagonal elements of the 5th and the 6th rows and columns with the right hand side of inequality (24) respectively.[35] We have

$$\tilde{\boldsymbol{\Gamma}} = \begin{bmatrix} \hat{\boldsymbol{\Gamma}}_{11} & \hat{\boldsymbol{\Gamma}}_{12} & \hat{\boldsymbol{\Gamma}}_{13} & \hat{\boldsymbol{P}}\boldsymbol{B}_{\boldsymbol{\omega}} & \hat{\boldsymbol{\Gamma}}_{15}\hat{\boldsymbol{P}} & \hat{\boldsymbol{\Gamma}}_{16}\hat{\boldsymbol{P}} & \hat{\boldsymbol{\Gamma}}_{17} & -\hat{\boldsymbol{Q}}_2 & -\hat{\boldsymbol{R}}_2 \\ * & \hat{\boldsymbol{\Gamma}}_{22} & 0 & 0 & \boldsymbol{A}_d^{\mathrm{T}}\hat{\boldsymbol{P}} & \boldsymbol{A}_d^{\mathrm{T}}\hat{\boldsymbol{P}} & \boldsymbol{C}_{1d}^{\mathrm{T}} & 0 & 0 \\ * & * & \hat{\boldsymbol{\Gamma}}_{33} & 0 & \hat{\boldsymbol{\Gamma}}_{35}\hat{\boldsymbol{P}} & \hat{\boldsymbol{\Gamma}}_{36}\hat{\boldsymbol{P}} & \hat{\boldsymbol{\Gamma}}_{37} & 0 & 0 \\ * & * & * & -\gamma^2\boldsymbol{I} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}}\hat{\boldsymbol{P}} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}}\hat{\boldsymbol{P}} & 0 & 0 & 0 \\ * & * & * & * & \tau_1^{-2}(\hat{\boldsymbol{Q}}_2 - 2\hat{\boldsymbol{P}}) & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \sigma_1^{-2}(\hat{\boldsymbol{R}}_2 - 2\hat{\boldsymbol{P}}) & 0 & 0 & 0 \\ * & * & * & * & * & * & -\boldsymbol{I} & 0 & 0 \\ * & * & * & * & * & * & * & \hat{\boldsymbol{\Gamma}}_{88} & 0 \\ * & * & * & * & * & * & * & * & \hat{\boldsymbol{\Gamma}}_{99} \end{bmatrix} < 0. \tag{25}$$

Then by defining $\boldsymbol{X} = \hat{\boldsymbol{P}}^{-1}$, $\boldsymbol{Y} = \boldsymbol{K}\boldsymbol{C}_2\boldsymbol{X}$, $\boldsymbol{Q}_1 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{Q}}_1\boldsymbol{X}$, $\boldsymbol{Q}_2 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{X}$, $\boldsymbol{Q}_3 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{Q}}_3\boldsymbol{X}$, $\boldsymbol{R}_1 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{R}}_1\boldsymbol{X}$, $\boldsymbol{R}_2 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{X}$, $\boldsymbol{R}_3 = \boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{R}}_3\boldsymbol{X}$, pre- and post-multiplying LMI (25) by $\mathrm{diag}\{\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}}\}$ and its transpose, respectively, we obtain

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12} & \boldsymbol{\Gamma}_{13} & \boldsymbol{B}_{\boldsymbol{\omega}} & \boldsymbol{\Gamma}_{15} & \boldsymbol{\Gamma}_{16} & \boldsymbol{\Gamma}_{17} & -\boldsymbol{Q}_2 & -\boldsymbol{R}_2 \\ * & \boldsymbol{\Gamma}_{22} & 0 & 0 & (\boldsymbol{A}_d\boldsymbol{X})^{\mathrm{T}} & (\boldsymbol{A}_d\boldsymbol{X})^{\mathrm{T}} & (\boldsymbol{C}_{1d}\boldsymbol{X})^{\mathrm{T}} & 0 & 0 \\ * & * & \boldsymbol{\Gamma}_{33} & 0 & (\boldsymbol{B}_d\boldsymbol{Y})^{\mathrm{T}} & (\boldsymbol{B}_d\boldsymbol{Y})^{\mathrm{T}} & (\boldsymbol{D}_{1d}\boldsymbol{Y})^{\mathrm{T}} & 0 & 0 \\ * & * & * & -\gamma^2\boldsymbol{I} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}} & \boldsymbol{B}_{\boldsymbol{\omega}}^{\mathrm{T}} & 0 & 0 & 0 \\ * & * & * & * & \boldsymbol{\Gamma}_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \boldsymbol{\Gamma}_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & -\boldsymbol{X} & 0 & 0 \\ * & * & * & * & * & * & * & \boldsymbol{\Gamma}_{88} & 0 \\ * & * & * & * & * & * & * & * & \boldsymbol{\Gamma}_{99} \end{bmatrix} < 0, \tag{26}$$

with

$$\begin{aligned}
\boldsymbol{\Gamma}_{11} &= (\boldsymbol{AX}+\boldsymbol{BY})+(\boldsymbol{AX}+\boldsymbol{BY})^{\mathrm{T}}\\
&\quad +\boldsymbol{Q}_1+\boldsymbol{Q}_3+\boldsymbol{R}_1+\boldsymbol{R}_3-\boldsymbol{Q}_2-\boldsymbol{R}_2,\\
\boldsymbol{\Gamma}_{12} &= \boldsymbol{A}_d\boldsymbol{X},\quad \boldsymbol{\Gamma}_{13}=\boldsymbol{B}_d\boldsymbol{Y},\\
\boldsymbol{\Gamma}_{15} &= \boldsymbol{\Gamma}_{16}=(\boldsymbol{AX}+\boldsymbol{BY})^{\mathrm{T}},\\
\boldsymbol{\Gamma}_{17} &= (\boldsymbol{C}_1\boldsymbol{X}+\boldsymbol{D}_1\boldsymbol{Y})^{\mathrm{T}},\\
\boldsymbol{\Gamma}_{22} &= (\tau_2-1)\boldsymbol{Q}_3,\quad \boldsymbol{\Gamma}_{33}=(\sigma_2-1)\boldsymbol{R}_3,\\
\boldsymbol{\Gamma}_{55} &= \tau_1^{-2}(\boldsymbol{Q}_2-2\boldsymbol{X}),\quad \boldsymbol{\Gamma}_{66}=\sigma_1^{-2}(\boldsymbol{R}_2-2\boldsymbol{X}),\\
\boldsymbol{\Gamma}_{88} &= -\boldsymbol{Q}_1-\boldsymbol{Q}_2,\ \boldsymbol{\Gamma}_{99}=-\boldsymbol{R}_1-\boldsymbol{R}_2,
\end{aligned}$$

which is the same as LMI (20). Therefore, LMI (20) ensures the robust asymptotical stability of system (19) with an H_∞ disturbance attenuation level γ for any $\tau(t)$ and $\sigma(t)$ satisfying LMI (2). This completes the proof.

4.2. Robust H_∞ control with uncertainties

Concerning the uncertainties appeared in system (6), using Theorem 1, we can obtain the following result.

Theorem 3 For given scalars $\gamma>0$, $\tau_1>0$, $\tau_2>0$, $\sigma_1>0$, $\sigma_2>0$, the closed-loop system (6) is robustly asymptotically stable with a H_∞ disturbance attenuation level γ if there exist a constant $\varepsilon>0$, matrices $\boldsymbol{X}>0$, $\boldsymbol{Q}_1>0$, $\boldsymbol{Q}_2>0$, $\boldsymbol{Q}_3>0$, $\boldsymbol{R}_1>0$, $\boldsymbol{R}_2>0$, $\boldsymbol{R}_3>0$ and matrix $\boldsymbol{Y}$ such that the following LMI holds:

$$\begin{bmatrix}
\boldsymbol{\Gamma} & \boldsymbol{\Pi}_1 & \boldsymbol{\Pi}_2\\
* & -\varepsilon\boldsymbol{I} & 0\\
* & * & -\varepsilon^{-1}\boldsymbol{I}
\end{bmatrix}<0, \tag{27}$$

where matrix $\boldsymbol{\Gamma}$ is the same as that defined in LMI (20), and

$$\begin{aligned}
\boldsymbol{\Pi}_1&=\left[\boldsymbol{M}^{\mathrm{T}},\ 0,\ 0,\ 0,\ \boldsymbol{M}^{\mathrm{T}},\ \boldsymbol{M}^{\mathrm{T}},\ 0,\ 0,\ 0\right]^{\mathrm{T}},\\
\boldsymbol{\Pi}_2&=[\boldsymbol{E}_1\boldsymbol{X}+\boldsymbol{E}_3\boldsymbol{Y},\ \boldsymbol{E}_2\boldsymbol{X},\ \boldsymbol{E}_4\boldsymbol{Y},\ \boldsymbol{E}_5,\ 0,\ 0,\ 0,\ 0,\ 0]^{\mathrm{T}}.
\end{aligned}$$

In this case, the desired SOF controller can be constructed as in Eq. (5) with

$$\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{X}^{-1}\boldsymbol{C}_2^{\mathrm{T}}(\boldsymbol{C}_2\boldsymbol{C}_2^{\mathrm{T}})^{-1}. \tag{28}$$

Proof Define

$$\begin{aligned}
\boldsymbol{D}&=\left[\boldsymbol{M}^{\mathrm{T}}\hat{\boldsymbol{P}},\ 0,\ 0,\ 0,\ \boldsymbol{M}^{\mathrm{T}}\hat{\boldsymbol{P}},\ \boldsymbol{M}^{\mathrm{T}}\hat{\boldsymbol{P}},\ 0,\ 0,\ 0\right]^{\mathrm{T}},\\
\boldsymbol{E}&=[\boldsymbol{E}_1+\boldsymbol{E}_3\boldsymbol{K},\ \boldsymbol{E}_2,\ \boldsymbol{E}_4\boldsymbol{K},\ \boldsymbol{E}_5,\ 0,\ 0,\ 0,\ 0,\ 0].
\end{aligned} \tag{29}$$

Recalling that $\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{F}(t)\leq\boldsymbol{I}$, we can easily obtain the following inequality:[36]

$$\begin{aligned}
&\tilde{\boldsymbol{\Gamma}}+\boldsymbol{D}\boldsymbol{F}(t)\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{D}^{\mathrm{T}}\\
&\leq\tilde{\boldsymbol{\Gamma}}+\varepsilon^{-1}\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}
\end{aligned} \tag{30}$$

for any scalar $\varepsilon>0$ and $\tilde{\boldsymbol{\Gamma}}$ defined in LMI (25).

Then, along a similar argument as that in the derivation of Theorem 2, we find that inequality (9) is inequivalent to the following inequality:

$$\tilde{\boldsymbol{\Gamma}}+\boldsymbol{D}\boldsymbol{F}(t)\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}\boldsymbol{F}^{\mathrm{T}}(t)\boldsymbol{D}^{\mathrm{T}}<0. \tag{31}$$

From inequality (30), we can see that if

$$\tilde{\boldsymbol{\Gamma}}+\varepsilon^{-1}\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}+\varepsilon\boldsymbol{E}^{\mathrm{T}}\boldsymbol{E}<0 \tag{32}$$

holds, then the robust asymptotical stability of system (6) with disturbance attenuation γ also holds according to Theorem 1. By using the Schur complement,[34] inequality (32) is equivalent to

$$\begin{bmatrix}
\tilde{\boldsymbol{\Gamma}} & \boldsymbol{D} & \boldsymbol{E}^{\mathrm{T}}\\
* & -\varepsilon\boldsymbol{I} & 0\\
* & * & -\varepsilon^{-1}\boldsymbol{I}
\end{bmatrix}<0. \tag{33}$$

Now, by defining $\boldsymbol{X}=\hat{\boldsymbol{P}}^{-1}$, $\boldsymbol{Y}=\boldsymbol{K}\boldsymbol{C}_2\boldsymbol{X}$, $\boldsymbol{Q}_1=\boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{Q}}_1\boldsymbol{X}$, $\boldsymbol{Q}_2=\boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{Q}}_2\boldsymbol{X}$, $\boldsymbol{R}_1=\boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{R}}_1\boldsymbol{X}$, $\boldsymbol{R}_2=\boldsymbol{X}^{\mathrm{T}}\hat{\boldsymbol{R}}_2\boldsymbol{X}$, then pre- and post-multiplying LMI (33) by

$$\mathrm{diag}\left\{\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{X}^{\mathrm{T}},\boldsymbol{I},\boldsymbol{I}\right\}$$

and its transpose, respectively, we have LMI (27). This completes the proof.

5. Numerical example

To illustrate the effectiveness and the flexibility of our theoretical analysis, a numerical example is presented. Consider the following uncertain linear system with heterogenous state and input time-varying delays:

$$\begin{aligned}
\dot{x}_1(t) &= 4x_1(t)+(2-0.06\sin(t))x_2(t)\\
&\quad +0.1x_1(t-\tau(t))\\
&\quad -(0.3+0.006\sin(t))x_2(t-\tau(t))\\
&\quad +5u(t)+(0.4-0.06\sin(t))u(t-\sigma(t))\\
&\quad +(0.5+0.06\sin(t))\omega(t),\\
\dot{x}_2(t) &= 2x_1(t)-3x_2(t)+0.2x_1(t-\tau(t))\\
&\quad -0.5x_2(t-\tau(t))+(0.5+0.03\cos(t))u(t)
\end{aligned}$$

$$
\begin{aligned}
&+0.2u(t-\sigma(t))+(0.09\cos(t)-1)\omega(t),\\
z_1(t) &= 0.1x_1(t)+0.1x_1(t-\tau(t))-u(t-\sigma(t)),\\
z_2(t) &= -0.2x_2(t)+0.1x_2(t-\tau(t))\\
&\quad -u(t)+u(t-\sigma(t)),\\
y_1(t) &= x_1(t)-0.7x_2(t),\\
y_2(t) &= -x_2(t),\\
\boldsymbol{x}(t) &= [3,-3]^{\mathrm{T}},\ t\in[-0.6,0], \qquad (34)
\end{aligned}
$$

which can be reformulated in the form of Eq. (1) with

$$
\begin{aligned}
&\boldsymbol{A}=\begin{pmatrix}4 & 2\\ 2 & -3\end{pmatrix},\ \boldsymbol{A}_d=\begin{pmatrix}0.1 & -0.3\\ 0.2 & -0.5\end{pmatrix},\\
&\boldsymbol{B}=\begin{pmatrix}5\\ 0.5\end{pmatrix},\ \boldsymbol{B}_d=\begin{pmatrix}0.4\\ 0.2\end{pmatrix},\\
&\boldsymbol{B}_{\boldsymbol{\omega}}=\begin{pmatrix}0.5\\ -1\end{pmatrix},\ \boldsymbol{C}=\begin{pmatrix}0.1 & 0\\ 0 & -0.2\end{pmatrix},\\
&\boldsymbol{C}_{1d}=\begin{pmatrix}0.1 & 0\\ 0 & 0.1\end{pmatrix},\ \boldsymbol{D}_1=\begin{pmatrix}0\\ -1\end{pmatrix},\\
&\boldsymbol{D}_{1d}=\begin{pmatrix}-1\\ 1\end{pmatrix},\quad \boldsymbol{C}_2=\begin{pmatrix}1 & -0.7\\ 0 & -1\end{pmatrix}, \qquad (35)
\end{aligned}
$$

and $\Delta\boldsymbol{A}(t)$, $\Delta\boldsymbol{A}_d(t)$, $\Delta\boldsymbol{B}(t)$, $\Delta\boldsymbol{B}_d(t)$, and $\Delta\boldsymbol{B}_{\boldsymbol{\omega}}(t)$ can be represented in the form of Eq. (3) with

$$
\begin{aligned}
&\boldsymbol{M}=\begin{pmatrix}-0.3 & 0\\ 0 & 0.3\end{pmatrix},\ \boldsymbol{E}_1=\begin{pmatrix}0 & 0.2\\ 0 & 0\end{pmatrix},\\
&\boldsymbol{E}_2=\begin{pmatrix}0 & 0.02\\ 0 & 0\end{pmatrix},\ \boldsymbol{E}_3=\begin{pmatrix}0\\ 0.1\end{pmatrix},\\
&\boldsymbol{E}_4=\begin{pmatrix}0.2\\ 0\end{pmatrix},\ \boldsymbol{E}_5=\begin{pmatrix}-0.2\\ 0.3\end{pmatrix},\\
&\boldsymbol{F}(t)=\begin{pmatrix}\sin(t) & 0\\ 0 & \cos(t)\end{pmatrix}. \qquad (36)
\end{aligned}
$$

The time-varying state and input delays are selected as $\tau(t)=0.3+0.3\cos(t)$ and $\sigma(t)=0.2+0.2\sin(t)$, respectively, with $\tau_1=0.6$, $\tau_2=0.3$, $\sigma_1=0.4$, and $\sigma_2=0.2$. Because the system uncertainties have been taken into account, Theorem 3 can be used to solve the robust H_∞ problem for the above system. For fixed $\gamma=1.6$ and $\varepsilon=1$, we verify using the solver of LMI Toolbox in Matlab that there is a set of feasible solutions to LMI (27) with the feedback gain matrix

$$
\boldsymbol{K}=\begin{bmatrix}-1.4152 & 1.4162\end{bmatrix}. \qquad (37)
$$

Therefore, we conclude from Theorem 3 that the robust H_∞ controller design problem has been solved. By choosing the exogenous disturbance as $\omega(t)=16\,\mathrm{e}^{-0.2t}\cos(2t)$, the simulation results for the two states of time-delay linear system (34) are shown in Fig. 1.

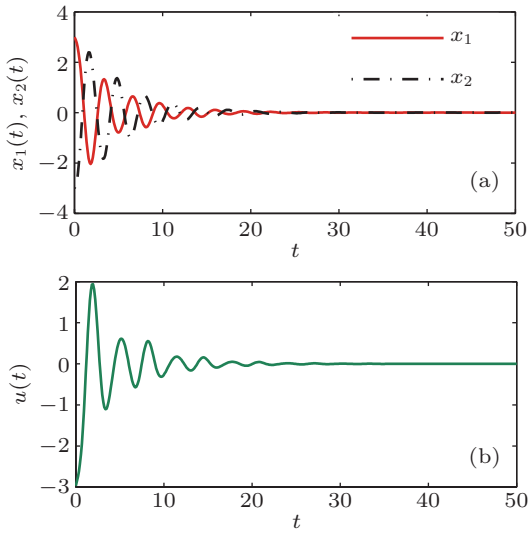


Fig. 1. (colour online) (a) State and (b) control input responses of the closed-loop system (34).

6. Discussion and conclusion

The robust H_∞ SOF controller design has been established for a novel class of uncertain linear time-delay systems. By constructing an appropriate Lyapunov–Krasovskii functional and using the Jensen integral inequality technique, we provide a delay-dependent stability criterion for the H_∞ performance of the closed-loop system. Based on such a stability criterion, the H_∞ SOF controller design algorithm is explicitly developed. The proposed controllers can be obtained by solving the convex optimization problems represented by linear matrix inequalities. Numerical simulations have been presented to demonstrate the effectiveness of the proposed control method.

It should be pointed out that unlike the existing methodologies involving free weighting matrices or model transformation, the main technique used is the Jensen integral inequality, which makes the results obtained in this paper naturally less conservative and more computationally efficient.[37] Moreover, the model proposed in this paper is very general because of the simultaneous consideration of heterogeneous time-varying state and input delays, unknown norm-bound uncertainties, and external deterministic noise. We

hope that our main results can be extended to solve other control problems (e.g., H_2 control, L_2-L_∞ control) for a wider class of time-delay systems, including general complex networks with communication time-delays.[38−42] These will be the subjects of our further investigation.

References

[1] Zames G 1981 *IEEE Trans. Autom. Control* **26** 301

[2] Doyle J C, Glover K, Khargonekar P P and Francis B A 1989 *IEEE Trans. Autom. Control* **34** 831

[3] Stoorvogel A A 1993 *IEEE Trans. Autom. Control* **38** 1358

[4] Wilson D A 1989 *IEEE Trans. Autom. Control* **34** 94

[5] Shi G, Zou Y and Yang C 1992 *Syst. Control Lett.* **18** 365

[6] Xie L and de Souza C E 1992 *IEEE Trans. Autom. Control* **37** 1188

[7] Gu K 1994 *IEEE Trans. Autom. Control* **39** 1320

[8] Yuan L, Achenie L E K and Jiang W 1996 *Syst. Control Lett.* **27** 199

[9] Coutinho D F, de Souza C E, Barbosa K A and Trofino A 2009 *SIAM J. Control Optim.* **48** 1452

[10] Zheng H Q and Jing Y W 2011 *Chin. Phys.* B **20** 060504

[11] Xie L, Fu M and de Souza C E 1992 *IEEE Trans. Autom. Control* **37** 1253

[12] de Souza C E, Fu M and Xie L 1993 *IEEE Trans. Autom. Control* **38** 459

[13] Peter K, Schöling I and Orlik B 2003 *IEEE Trans. Ind. Appl.* **39** 637

[14] Qiu J B, Feng G and Gao H J 2010 *IEEE Trans. Fuzzy Syst.* **18** 919

[15] Yang G H and Ye D 2010 *IET Control Theory Appl.* **4** 89

[16] Malek-Zavarei M and Jamshidi M 1987 *Time-delay Systems: Analysis, Optimation and Application* (Amsterdam: North-Holland)

[17] Gu K, Kharitonon V L and Chen J 2003 *Stability of Time-delay Systems* (Berlin: Springer)

[18] Bratsun D, Volfson D, Tsimring L S and Hasty J 2005 *Proc. Natl. Acad. Sci. USA* **102** 14593

[19] Wang J W, Zhang J J, Yuan Z J, Chen A M and Zhou T S 2008 *J. Biol. Rhythms* **23** 472

[20] Jia Z L 2010 *Chin. Phys.* B **19** 020504

[21] Xu S Y and Chen T W 2002 *IEEE Trans. Autom. Control* **47** 2089

[22] Li L and Jia Y 2009 *IET Control Theory Appl.* **3** 995

[23] Peng C, Yue D and Tian Y C 2009 *IEEE Trans. Fuzzy Syst.* **17** 890

[24] Liu J C, Zhang J, Zheng Y F and He M 2011 *J. Control Theory Appl.* **9** 189

[25] Tian E G, Yue D and Peng C 2012 *Int. J. Syst. Sci.* **43** 820

[26] Su H and Chu J 1999 *Int. J. Syst. Sci.* **30** 1093

[27] Li H Y, Chen B, Zhou Q and Lin C 2009 *Circuits Syst. Signal Process* **28** 169

[28] Senthilkumar T and Balasubramaniam P 2011 *Int. J. Syst. Sci.* **42** 877

[29] Alpaslan Parlakçi M N and Küçükdemiral İ B 2011 *Int. J. Robust. Nonlinear Control* **21** 974

[30] Mirkin B, Mirkin E L and Gutman P O 2009 *Control Signal Process* **23** 567

[31] Li D, Wang S L, Zhang X H and Yang D 2010 *Chin. Phys.* B **19** 010506

[32] Syrmos V L, Abdallah C T and Grigoriadis K *Automatica* **33** 125

[33] He Y, Wu M, Liu G P and She J H 2008 *IEEE Trans. Autom. Control* **53** 2372

[34] Boyd S, EI Ghaout L, Feron E and Balakrishnan V 1994 *Linear Matrix Inequalities in System and Control Theory* (Philadelphia: SIAM)

[35] de Oliveira M C, Bernussou J and Geromel J C 1999 *Syst. Control Lett.* **37** 261

[36] de Souza C E and Li X 1999 *Automatica* **35** 1313

[37] Zhu X L and Yang G H 2008 *IET Control Theory Appl.* **2** 524

[38] Lü J, Yu X, Chen G and Cheng D 2004 *IEEE Trans. Circuits Syst.* I *Reg. Papers* **51** 787

[39] Lü J and Chen G 2005 *IEEE Trans. Autom. Control* **50** 841

[40] Lü J, Yu X and Chen G 2004 *Physica* A **334** 281

[41] Wang P, Lü J and Ogorzalek M J 2012 *Neurocomputing* **78** 155

[42] Chen Y, Lü J, Han F and Yu X 2011 *Syst. Control Lett.* **60** 517