Linear superposition solutions to nonlinear wave equations

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(Received 12 April 2012; revised manuscript received 22 May 2012)

The solutions to a linear wave equation can satisfy the principle of superposition, i.e., the linear superposition of two or more known solutions is still a solution of the linear wave equation. We show in this article that many nonlinear wave equations possess exact traveling wave solutions involving hyperbolic, triangle, and exponential functions, and the suitable linear combinations of these known solutions can also constitute linear superposition solutions to some nonlinear wave equations with special structural characteristics. The linear superposition solutions to the generalized KdV equation $K(2,2,1)$, the Oliver water wave equation, and the $k(n,n)$ equation are given. The structure characteristic of the nonlinear wave equations having linear superposition solutions is analyzed, and the reason why the solutions with the forms of hyperbolic, triangle, and exponential functions can form the linear superposition solutions is also discussed.

Keywords: linear superposition solution, nonlinear wave equation, generalized KdV equation, Oliver water wave equation

PACS: 02.30.Jr **DOI:** 10.1088/1674-1056/21/11/110205

1. Introduction

Many physical phenomena in nature possess superposition properties, such as the superposition of mechanical waves, the superposition of electric fields, and the interference of light, all of which are well-known superposition cases in the classical physics.[1,2] In quantum mechanics, the superposition also exists, such as the superposition of motion states of microscopic particles.[3] All superposition phenomena satisfy the principle of superposition, i.e., the total effect caused by two or more physical quantities with the same physical meaning is equal to the sum of the effects caused by each quantity individually. The reason for these physical phenomena is that the partial differential equations describing these phenomena are linear. The wave equation is a kind of important partial differential equation used for describing various wave phenomena like water, light, and sound waves. The solution to the wave equation, i.e., the wave function, can characterize the evolution process of a physical quantity (such as the height of water wave or the pressure of sound wave) with space and time. We know that a solution to a wave equation represents a kind of wave. A wave equation can have many independent solutions, if the wave equation is linear, these independent solutions can constitute a linear superposition solution. For example, in classical physics, if $u_1(x,t)$, $u_2(x,t)$,..., $u_n(x,t)$ are the solutions to a wave equation, the linear superposition of them, $u(x,t) = c_1u_1(x,t)+c_2u_2(x,t)+\cdots+c_nu_n(x,t)$, will also be a solution to the equation. This means that the linear superposition of two or more known waves can form a new wave. For another example, in quantum mechanics, the Schrödinger equation is a wave equation concerning the motion of microscopic particles in a quantum system,[4] and the solution to the Schrödinger equation, $\psi(r,t)$, can describe the motion of the microscopic particles. Because the Schrödinger equation is a homogeneous linear differential equation concerning $\psi(r,t)$, the solution to the equation can satisfy the principle of linear superposition. Therefore, if $\psi_1,(r,t)$ and $\psi_2(r,t)$ are the solutions to the equation, which characterize two possible motion states of the particles, the linear superposition of them, $\psi(r,t) = c_1\psi_1(r,t)+c_2\psi_2(r,t)$, is also a solution to the equation, which characterizes another possible motion state of the particles.[3] This means that we can obtain a new motion state from the known motion states by way of linear superposition.

However, if a wave equation is nonlinear, the solutions to the equation cannot satisfy the principle of linear superposition. Nevertheless, the possibility that some specific forms of solutions to a nonlinear wave

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equation may be formally satisfied with this principle could not be absolutely excluded. In other words, a suitable linear combination of two or more specific solutions to a nonlinear wave equation may also be a solution.

In 2002, Khare and Sukhatme[5] found that for the Korteweg–de Vries (KdV) equation, the modified KdV equation, and the $\lambda\phi^4$ equation, the solutions obtained by linear combinations of known Jacobi elliptic function solutions are still the solutions to them. This means that the nonlinear wave equation could have linear superposition solutions. For example, the KdV equation has the cnoidal traveling wave solution

$$u(x,t) = -2\alpha^2 \mathrm{dn}^2(\xi_1, m) + \beta\alpha^2, \tag{1}$$

where $\xi_1 = \alpha\left(x - b_1\alpha^2 t\right)$; and α, β, and m are constants. This solution is a representative of the periodic wave and can characterize the wave of shallow water with a finite amplitude.

From expression (1), a linear superposition solution can be obtained as

$$u_p(x,t) = -2\alpha^2 \sum_{i=1}^{p} d_i^2 + \beta\alpha^2, \tag{2}$$

where

$$d_i = \mathrm{dn}\left(\xi_p + \frac{2(i-1)K(m)}{p}, m\right),$$

and $\xi_p = \alpha\left(x - b_p\alpha^2 t\right)$.

The linear superposition solutions similar to Eq. (2) are also obtained for the Kadomtsev–Petviashvili equation,[6] the nonlinear Schrödinger equation,[6] the (2+1)-dimensional general Schrödinger and Boussinesq equations.[7]

However, further research[8,9] has indicated that this kind of linear superposition solution (such as Eq. (2)) is not a new solution, but merely a novel way of rewriting known periodic solutions (such as expression (1)) using a different modulus parameter. Although the results of Refs. [5]–[7] do not give new solutions, the possibility that the nonlinear equations possess linear superposition solutions exists indeed.

Being enlightened by Refs. [5]–[7], we study the linear superposition of solutions to nonlinear wave equations and find that for many nonlinear wave equations, the suitable linear combinations of hyperbolic, triangle, and exponential function solutions could arrive at linear superposition solutions. In this article, we provide firstly some actual cases of linear superposition solutions to three nonlinear wave equations. Then, the structural characteristics of the nonlinear wave equations having linear superposition solutions are analyzed, and the reason why the solutions with forms of hyperbolic, triangle, and exponential functions can constitute the linear superposition solutions is discussed.

2. Actual cases of linear superposition solutions

Firstly, we provide some actual linear superposition solutions to the generalized KdV equation $K(2,2,1)$,[10] the Oliver water wave equation, and the $k(n,n)$ equation. Suppose these equations possess the traveling wave solution

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{3}$$

where v is the wave velocity.

2.1. Generalized KdV equation

The generalized KdV equation $K(2,2,1)$ is given by

$$u_t + \alpha(u^2)_x + \beta(u^2)_{3x} + \gamma u_{5x} = 0, \tag{4}$$

where α, β, and γ are real constants.

Using the ansatz method[11] for solving Eq. (4) (the concrete steps are omitted), we obtain the following traveling wave solutions.

Case I $\alpha/\beta < 0$. Some exact solutions are

$$u_1(x,t) = \sinh\left(\pm\sqrt{\frac{-\alpha}{4\beta}}\left(x - \frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \tag{5}$$

$$u_2(x,t) = \cosh\left(\sqrt{\frac{-\alpha}{4\beta}}\left(x - \frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \tag{6}$$

$$u_3(x,t) = \exp\left(\pm\sqrt{\frac{-\alpha}{4\beta}}\left(x - \frac{\gamma\alpha^2}{16\beta^2}t\right)\right). \tag{7}$$

The linear combination of $u_1(x,t)$ and $u_2(x,t)$ will yield

$$\begin{aligned} u &= c_1u_1(x,t) + c_2u_2(x,t) \\ &= c_1\sinh\left(\pm\sqrt{\frac{-\alpha}{4\beta}}\left(x - \frac{\gamma\alpha^2}{16\beta^2}t\right)\right) \\ &\quad + c_2\cosh\left(\sqrt{\frac{-\alpha}{4\beta}}\left(x - \frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \end{aligned} \tag{8}$$

where c_1 and c_2 are arbitrary non-zero constants. Expression (8) can satisfy Eq. (4) and is a solution to Eq. (4).

The linear combination of $u_1(x,t)$, $u_2(x,t)$, and $u_3(x,t)$ can yield

$$u = c_1u_1(x,t) + c_2u_2(x,t) + c_3u_3(x,t)$$

$$= c_1 \sinh\left(\pm\sqrt{\frac{-\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right) + c_2\cosh\left(\sqrt{\frac{-\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right) + c_3\exp\left(\pm\sqrt{\frac{-\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \quad (9)$$

where c_1, c_2, and c_3 are arbitrary non-zero constants. Expression (9) can also satisfy Eq. (4) and is a solution to Eq. (4).

It can be proved that all linear combinations of expressions (5)–(7) can satisfy Eq. (4), so the generalized KdV Eq. (4) has linear superposition solutions

$$u=\sum_{i=1}^{n} c_i u_j(x,t), \quad n=2,3, \quad (10)$$

where c_i is an arbitrary non-zero constant, and $u_j(x,t)$ is any of $u_1(x,t)$–$u_3(x,t)$.

Case II $\alpha/\beta>0$. Equation (4) has the triangle function periodic wave solutions

$$u_4(x,t)=\sin\left(\pm\sqrt{\frac{\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \quad (11)$$

$$u_5(x,t)=\cos\left(\sqrt{\frac{\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right). \quad (12)$$

The linear combination of $u_4(x,t)$ and $u_5(x,t)$ will yield

$$u = c_1u_4(x,t)+c_2u_5(x,t) = c_1\sin\left(\pm\sqrt{\frac{\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right) + c_2\cos\left(\sqrt{\frac{\alpha}{4\beta}}\left(x-\frac{\gamma\alpha^2}{16\beta^2}t\right)\right), \quad (13)$$

where c_1 and c_2 are arbitrary non-zero constants. Expression (13) can also satisfy Eq. (4) and is a solution to Eq. (4).

2.2. Oliver water wave equation

The Oliver water wave equation is[12]

$$u_t+u_x+k_1uu_x+k_2u_{3x}+k_3u_xu_{xx} +k_4uu_{3x}+k_5u_{5x}=0, \quad (14)$$

where k_1, $k_2,\ldots,k_5$ are real constants. The equation can be used for describing the water wave movement over a shallow horizontal bottom.

Case I when

$$v=1-\frac{k_1k_2}{k_3+k_4}+\frac{k_1^2k_5}{(k_3+k_4)^2},$$

Eq. (14) has the following traveling wave solutions:

$$u_1(x,t)=\sinh\left(\pm\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right), \quad (15)$$

$$u_2(x,t)=\cosh\left(\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right), \quad (16)$$

$$u_3(x,t)=\exp\left(\pm\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right), \quad (17)$$

where $\frac{-k_1}{k_3+k_4}>0$. The linear combination of expressions (15)–(17) will yield

$$u=\sum_{i=1}^{n} c_iu_j(x,t), \quad n=2,3, \quad (18)$$

where c_i is an arbitrary non-zero constant, and $u_j(x,t)$ is any of $u_1(x,t)$–$u_3(x,t)$. Expression (18) can satisfy Eq. (14) and is a solution to Eq. (14).

Case II when

$$v=1+k_1-(k_2+k_4)\frac{k_2}{k_3+k_4}+\frac{k_1^2k_5}{(k_3+k_4)^2},$$

the following traveling wave solutions are obtained:

$$u_1(x,t)=\sinh\left(\pm\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right)+1, \quad (19)$$

$$u_2(x,t)=\cosh\left(\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right)+1, \quad (20)$$

$$u_3(x,t)=\exp\left(\pm\sqrt{\frac{-k_1}{k_3+k_4}}(x-vt)\right)+1, \quad (21)$$

where $\frac{-k_1}{k_3+k_4}>0$. The linear combination of expressions (19)–(21) will yield

$$u=c_1u_1(x,t)+c_2u_2(x,t), \quad (22)$$

$$u=c_1u_1(x,t)+c_2u_3(x,t), \quad (23)$$

$$u=c_1u_2(x,t)+c_2u_3(x,t), \quad (24)$$

where c_1 and c_2 are non-zero constants. Under the condition of $c_1+c_2=1$, expressions (22)–(24) are solutions to Eq. (14).

2.3. The $k(n,n)$ equation

The $k(n,n)$ equation is[13]

$$u_t+\alpha(u^n)_x+\beta(u^n)_{3x}=0, \quad (25)$$

where α and β are real constants. This equation is an extended form of the standard KdV equation $u_t+\alpha uu_x+\beta u_{3x}=0$.

When $n=2$, solving Eq. (25) will arrive at the following traveling wave solutions:

$$u_1(x,t)=c\sinh\left(\pm\sqrt{\frac{-\alpha}{4\beta}}(x-vt)\right)+\frac{2v}{3\alpha}, \quad (26)$$

$$u_2(x,t) = c\cosh\left(\sqrt{\frac{-\alpha}{4\beta}}(x-vt)\right) + \frac{2v}{3\alpha}, \quad (27)$$

$$u_3(x,t) = c\exp\left(\pm\sqrt{\frac{-\alpha}{4\beta}}(x-vt)\right) + \frac{2v}{3\alpha}, \quad (28)$$

where $\alpha/\beta < 0$, and v and c are arbitrary non-zero constants.

The linear combination of expressions (26)–(28) will give

$$u = \sum_{i=1}^{n} c_i u_j(x,t), \quad n = 2,3, \quad (29)$$

where c_i is a non-zero constant, and $u_j(x,t)$ is any of $u_1(x,t)$–$u_3(x,t)$. When $0 < c_i < 1$ and $\sum_{i=1}^{n} c_i = 1$, expression (29) is a solution of Eq. (25).

In addition to the three nonlinear equations mentioned above, there are many other equations which also have linear superposition solutions. These equations include

1) the fifth-order KdV-like equation with square terms[11]

$$u_t + \alpha u^2 u_x + \beta u^2 u_{3x} + u_{5x} = 0, \quad (30)$$

2) the generalized KdV equation[14]

$$u_t + \alpha u u_t + u u_x + u_{xxx} = 0, \quad (31)$$

3) the fifth-order KdV-like equation[11]

$$u_t + \alpha u u_x + \beta u u_{3x} + u_{5x} = 0, \quad (32)$$

4) a nonlinear variant of the RLW equation[15,16]

$$u_t + \alpha u_x - k\left(u^2\right)_x + \beta\left(u^2\right)_{xxt} = 0, \quad (33)$$

5) the nonlinear evolution equation[10]

$$u_t - 2\alpha u u_x + 4\beta u_x u_{xx} + 2\beta u_{3x} - 2\gamma u_{5x} = 0, \quad (34)$$

6) the Boussinesq-like $B(2,2)$ equation[17]

$$u_{tt} + \alpha(u^2)_{xx} + \beta(u^2)_{xxxx} = 0, \quad (35)$$

7) the (2+1)-dimensional Nizhnik–Novikov–Veselov(NNV) equation[18]

$$\begin{aligned} &u_{xyt} + a(u_{xy}u_{yyy} + u_{yy}u_{xyy}) \\ &+9a(u_{xx}u_{xxy} + u_{xxx}u_{xy}) \\ &+du_{xxxxy} + eu_{xyyyy} = 0, \end{aligned} \quad (36)$$

8) and the (3+1)-dimensional Nizhnik–Novikov–Veselov equation[19]

$$\begin{aligned} &u_{yzt} + au_{yz}u_{yyy} + bu_{xz}u_{xyz} + cu_{yy}u_{yyz} \\ &+(9c-2b)u_{xx}u_{yzz} + (12c-2b-3a)u_{yz}u_{xxz} \\ &+du_{xxyzz} + eu_{yyyyz} = 0. \end{aligned} \quad (37)$$

3. Discussion

The results above reveal that the linear superposition solutions can be obtained by suitable linear combinations of known hyperbolic, triangle, and exponential function solutions. Why can the linear combinations of hyperbolic, triangle, and exponential function solutions constitute linear superposition solutions? And why can the nonlinear equations mentioned above have linear superposition solutions? By investigating the structures of the above nonlinear equations, we can find that all of them have two characteristics: (i) there are at least two nonlinear terms in each equation; and (ii) the powers of function u in the nonlinear terms are the same, whereas the orders of derivative are different.

For instance, Eq. (4) includes four terms, and the two nonlinear terms $\alpha(u^2)_x$ and $\beta(u^2)_{3x}$ have the same power for function u but different orders of derivative. Equation (14) contains three nonlinear terms k_1uu_x, $k_3u_xu_{xx}$, and k_4uu_{3x}, which also have the same power for function u but different orders of derivative. The third case, Eq. (25), includes two nonlinear terms $\alpha(u^n)_x$ and $\beta(u^n)_{3x}$, which have the same function u^n but different orders of derivative.

Why can a nonlinear equation having characteristics (i) and (ii) possess linear superposition solutions? We discuss this question by taking Eq. (4) as an example.

Substituting $u(x,t) = u(\xi)$ and $\xi = kx + vt$ into Eq. (4) yields

$$vu_\xi + \alpha k(u^2)_\xi + \beta k^3(u^2)_{3\xi} + \gamma k^5 u_{5\xi} = 0. \quad (38)$$

Suppose that Eq. (38) has two solutions, $u_1(\xi)$ and $u_2(\xi)$, and the linear combination of them is

$$u(\xi) = c_1u_1(\xi) + c_2u_2(\xi). \quad (39)$$

Generally speaking, c_1 and c_2 are arbitrary constants. In order to make the discussion simple and clear, we take $c_1 = c_2 = 1$. Substituting Eq. (39) into Eq. (38), we have

$$\begin{aligned} &v(u_1+u_2)_\xi + \alpha k\left((u_1+u_2)^2\right)_\xi \\ &+\beta k^3\left((u_1+u_2)^2\right)_{3\xi} \\ &+\gamma k^5(u_1+u_2)_{5\xi} = 0. \end{aligned} \quad (40)$$

Equation (40) can be rewritten as

$$\begin{aligned}&\left[v(u_1)_\xi+\alpha k(u_1^2)_\xi+\beta k^3(u_1^2)_{3\xi}+\gamma k^5(u_1)_{5\xi}\right]\\&+\left[v(u_2)_\xi+\alpha k(u_2^2)_\xi+\beta k^3(u_2^2)_{3\xi}+\gamma k^5(u_2)_{5\xi}\right]\\&+\left[2\alpha k(u_1)(u_2)_\xi+2\alpha k(u_2)(u_1)_\xi+2\beta k^3(u_1)(u_2)_{3\xi}\right.\\&\left.+2\beta k^3(u_2)(u_1)_{3\xi}\right]=0.\end{aligned}\tag{41}$$

Because u_1 and u_2 are solutions of Eq. (38), the first two terms of Eq. (41) are equal to zero, i.e.,

$$v(u_1)_\xi+\alpha k(u_1^2)_\xi+\beta k^3(u_1^2)_{3\xi}+\gamma k^5(u_1)_{5\xi}=0$$

and

$$v(u_2)_\xi+\alpha k(u_2^2)_\xi+\beta k^3(u_2^2)_{3\xi}+\gamma k^5(u_2)_{5\xi}=0.$$

In order to make Eq. (41) established, the third term of Eq. (41) must be equal to zero, i.e.,

$$\begin{aligned}&2\alpha k(u_1)(u_2)_\xi+2\alpha k(u_2)(u_1)_\xi+2\beta k^3(u_1)(u_2)_{3\xi}\\&+2\beta k^3(u_2)(u_1)_{3\xi}=0.\end{aligned}\tag{42}$$

We rewrite Eq. (42) into

$$\begin{aligned}&2ku_1\left[\alpha(u_2)_\xi+\beta k^2(u_2)_{3\xi}\right]\\&+2ku_2\left[\alpha(u_1)_\xi+\beta k^2(u_1)_{3\xi}\right]=0.\end{aligned}\tag{43}$$

To make Eq. (43) established, u_1 and u_2 must satisfy the following differential equation set:

$$\begin{cases}\alpha(u_2)_\xi+\beta k^2(u_2)_{3\xi}=0,\\ \alpha(u_1)_\xi+\beta k^2(u_1)_{3\xi}=0.\end{cases}\tag{44}$$

The set of differential Eq. (44) is composed of two integrable linear ordinary differential equations. By integrating each of them, we can obtain concrete function forms of u_1 and u_2, both of which are hyperbolic, triangle, and exponential functions. This means that only the hyperbolic, triangle, and exponential functions can make Eq. (43) satisfied. This result may be the answer to why the linear combinations of hyperbolic, triangle, and exponential function solutions can constitute the linear superposition solutions. Meanwhile, this result also shows us that only when the structure of a nonlinear equation possess the two characteristics (i) and (ii), we can obtain a set of differential equations concerning u_1 and u_2 like Eq. (44). If there is only one nonlinear term in the nonlinear equation or the powers of function u in the nonlinear terms are not the same, a set of differential equations like Eq. (44) could not be established.

A similar discussion can be done for the other nonlinear equations, such as the Oliver water wave equation (14) and the $k(n,n)$ equation (25), and a set of differential equations similar to Eq. (44) has also be obtained, so the same opinion as that mentioned above is obtained.

In short, as long as the structure of a nonlinear equation has characteristics (i) and (ii), the nonlinear wave equation will have linear superposition solutions composed by the solutions with forms of hyperbolic, triangle, and exponential functions.

4. Conclusion

Some specific solutions of nonlinear wave equations can satisfy the principle of linear superposition. The existence of linear superposition solutions depends on the structure of the nonlinear wave equation. The result and discussion in this article provide a helpful ideal for seeking linear superposition solutions to the nonlinear wave equation. We can naturally raise the question otf whether there are linear superposition solutions to the other nonlinear equations? If there are, what structural characteristics should the nonlinear equation have and what conditions should be satisfied with it? Research on these questions will be done in the future.

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