Exact solutions for nonlinear partial fractional differential equations

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In this article, we use the fractional complex transformation to convert nonlinear partial fractional differential equations to nonlinear ordinary differential equations. We use the improved (G'/G)-expansion function method to calculate the exact solutions to the time- and space-fractional derivative foam drainage equation and the time- and space-fractional derivative nonlinear KdV equation. This method is efficient and powerful for solving wide classes of nonlinear evolution fractional order equations.

Keywords: fractional calculus, complex transformation, modified Riemann–Liouville derivative, improved (G'/G)-expansion function method

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1. Introduction

The seeds of fractional calculus (that is the theory of integrals and derivatives of arbitrary real or complex orders) were planted over 300 years ago. Since then, many researchers have contributed to this field. Recently, it turns out that differential equations involving derivatives of non-integer orders can be adequate models for various physical phenomena.[1] For example, the nonlinear oscillation of an earthquake can be modeled with the fractional derivatives.[2] There has been some attempt to solve the linear problems with the multiple fractional derivatives (the so-called multi-term equations).[1,3] Little work has been done for the nonlinear problems, and only a few numerical schemes have been proposed to solve the nonlinear fractional differential equations. The applications have included classes of nonlinear equations with multi-order fractional derivatives; this motivates us to develop a numerical scheme for their solutions.[4] Numerical and analytical methods include the Adomian decomposition method,[5,6] the variational iteration method,[7] and the homotopy perturbation method.[8−10] Deng[11] has discussed the smoothness and the stability of the solutions for the nonlinear fractional differential equations. The numerical solutions of the nonlinear time fractional Fokker–Planck differential equations have been given in Ref. [12].

In this paper, we will implement the (G'/G)-expansion function method[13−15] to obtain the exact solutions for the following time- and space-fractional derivative nonlinear differential equations:

(i) the time- and space-fractional derivative foam drainage equation[16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}u\frac{\partial^{2\beta}u}{\partial x^{2\beta}} + 2u^2\,\frac{\partial^\beta u}{\partial x^\beta} + \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2, \quad t>0,\ \ 0<\alpha,\beta\le 1, \tag{1}$$

(ii) the nonlinear Korteweg–de Vries (KdV) equation with time- and space-fractional derivatives[17]

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au\frac{\partial^\beta u}{\partial x^\beta} + \frac{\partial^{3\beta}u}{\partial x^{3\beta}} = 0, \quad t>0, 0<\alpha,\beta\le 1. \tag{2}$$

2. Preliminaries

In recent years, in order to investigate the local behavior of fractional models, several local versions of fractional derivatives have been proposed, i.e., the Kolwankar–Gangal local fractional derivative,[18] Chen's fractal derivative,[19] Cresson's derivative,[20] and Jumarie's modified Riemann–

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Liouville derivative.[21,22] Jumarie's derivative is defined as

$$\begin{aligned} &\mathrm{D}_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \\ &\times \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) \mathrm{d}\xi, \\ &0 < \alpha < 1, \end{aligned} \tag{3}$$

where $f : \mathbf{R} \to \mathbf{R}$, $x \mapsto f(x)$ denotes a continuous (but not necessarily first-order-differentiable) function. We can obtain the following properties.

Property 1 Let $f(x)$ satisfy the definition of the modified Riemann–Liouville derivative and $f(x)$ be a $k\alpha$ order differentiable function. The generalized Taylor series is given by[21,23]

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{(\alpha k)!} f^{(\alpha k)}(x), \quad 0 < \alpha < 1. \tag{4}$$

Property 2 Assume that $f(x)$ denotes a continuous $\mathbf{R} \to \mathbf{R}$ function. We use the following equality for the integral with respect to $(\mathrm{d}x)^{\alpha}$:[22,23]

$$\begin{aligned} I_x^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) \mathrm{d}\xi \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi) (\mathrm{d}x)^{\alpha}, \quad 0 < \alpha \le 1. \end{aligned} \tag{5}$$

Property 3 Some useful formulas include

$$f^{(\alpha)}[x(t)] = \frac{\mathrm{d}f}{\mathrm{d}x} x^{(\alpha)}(t), \tag{6}$$

$$\mathrm{D}_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \tag{7}$$

$$\int (\mathrm{d}x)^{\beta} = x^{\beta}. \tag{8}$$

Function $f(x)$ should be differentiable with respect to $x(t)$, and $x(t)$ is fractional differentiable in Eq. (6). The above results are employed in the following sections. The modified Riemann–Liouville derivative has been successfully applied in the probability calculus,[24] the fractional Laplace problems,[25] the fractional variational approach with several variables,[26] the fractional variational iteration method,[27] the fractional variational approach with natural boundary conditions,[28] and the fractional Lie group method.[29]

3. The improved (G'/G)-expansion function method

Consider the following nonlinear partial fractional differential equation:

$$\begin{aligned} &F(u, \mathrm{D}_t^{\alpha} u, \mathrm{D}_x^{\beta} u, \mathrm{D}_y^{\gamma} u, \mathrm{D}_z^{\delta} u, \mathrm{D}_t^{\alpha} \mathrm{D}_t^{\alpha} u, \mathrm{D}_t^{\alpha} \mathrm{D}_x^{\beta} u, \\ &\mathrm{D}_x^{\beta} \mathrm{D}_x^{\beta} u, \mathrm{D}_x^{\beta} \mathrm{D}_y^{\gamma}, \mathrm{D}_y^{\gamma} \mathrm{D}_y^{\gamma} u, ...) = 0, \\ &0 < \alpha, \beta, \delta, \gamma < 1, \end{aligned} \tag{9}$$

where u is an unknown function, and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved.

In the following, we give the main steps of the improved (G'/G)-expansion method.

Step 1 Li *et al.*[30] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$\begin{aligned} &u(x, y, z, t) = u(\xi), \\ &\xi = \frac{K\ x^{\beta}}{\Gamma(\beta+1)} + \frac{N\ y^{\gamma}}{\Gamma(\gamma+1)} + \frac{M\ z^{\delta}}{\Gamma(\delta+1)} \\ &\quad + \frac{L t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{10}$$

where K L, M, and N are non zero arbitrary constants, permits us to reduce Eq. (9) to an ODE of $u = u(\xi)$ in the form

$$P(u, u', u'', u''', \ldots) = 0. \tag{11}$$

If possible, we should integrate Eq. (11) term by term one or more times.

Step 2 Suppose the solution of Eq. (11) can be expressed as a polynomial of (G'/G) in the form[13–15]

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'}{G} \right)^i, \quad \alpha_m \neq 0, \tag{12}$$

where α_i $(i = 0, \pm 1, ..., \pm m)$ are constants, while $G(\xi)$ satisfies the following second-order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{13}$$

with λ and μ being constants.

Step 3 The positive integer m can be determined by considering the homogeneous balance between the

highest order derivatives and the nonlinear terms appearing in Eq. (11). More precisely, we define the degree of $u(\xi)$ as $D[u(\xi)] = m$, which gives rise to the degrees of the other expressions as follows:

$$D\left[\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right] = m + q,$$
$$D\left[u^p\left(\frac{\mathrm{d}^q u}{\mathrm{d}\xi^q}\right)^s\right] = mp + s(q+m). \quad (14)$$

Therefore, we can obtain the value of m in Eq. (12).

Step 4 Substituting Eq. (12) into Eq. (11), using Eq. (13), collecting all terms with the same order of (G'/G) together, and then equating each coefficient of the resulting polynomial to zero , we obtain a set of algebraic equations for α_i $(i = 0, \pm1, ..., \pm m)$, λ, μ, K, L, M, and N.

Step 5 Since the general solutions to Eq. (13) are well known, then substituting α_i $(i = 0, \pm1, ..., \pm m)$, λ, μ , K, L, M, N and the general solutions of Eq. (13) into Eq. (12), we get more traveling wave solutions of the nonlinear partial fractional derivatives, Eq. (9).

4. Applications

In this section, we use the improved (G'/G)-expansion method to construct the exact solutions for some nonlinear partial fractional differential equations, i.e., the nonlinear foam drainage equation with time- and space-fractional derivatives and the nonlinear KdV equation with time- and space-fractional derivatives. Such equations have received wide attention.

4.1. The time- and space-fractional derivative nonlinear foam drainage equation

In this section, to demonstrate the effectiveness of our approach, we will apply the modified (G'/G)-expansion function method to construct the exact solutions for the time- and space-fractional derivative nonlinear foam drainage equation.[16]

We can see that the fractional complex transform

$$u(x,t) = U(\xi), \quad \xi = \frac{Kx^\beta}{\Gamma(1+\beta)} + \frac{Lt^\alpha}{\Gamma(1+\alpha)}, \quad (15)$$

where K and L are constants, permits us to reduce Eq. (1) into the following ODE:

$$-LU' + \frac{1}{2}K^2UU'' + 2KU^2U' + K^2U'^2 = 0. \quad (16)$$

Considering the homogeneous balance between the highest order derivative and the nonlinear term in Eq. (16), we deduce that $n = 1$. Thus, we get

$$u(\xi) = \alpha_1\left(\frac{G'}{G}\right) + \alpha_0 + b_1\left(\frac{G'}{G}\right)^{-1}, \quad (17)$$

where α_1, α_0, b_1, L, and K are arbitrary constants to be determined later. Substituting Eq. (17) into Eq. (16), collecting all the terms of powers of (G'/G), and setting each coefficient to zero, we get a system of algebraic equations. With the aid of Maple or Mathematica, we can solve this system of algebraic equations to obtain the following sets of solutions.

Case 1

$$\alpha_0 = \frac{K\lambda}{2},\ L = -K^3\mu + \frac{K^3\lambda^2}{4},\ a_1 = K,\ b_1 = 0, \quad (18)$$

where K, λ, and μ are arbitrary constants. Substituting Eq. (18) into Eq. (17) yields

$$u(\xi) = K\left(\frac{G'}{G}\right) + \frac{K\lambda}{2}. \quad (19)$$

From the solutions of Eqs. (13) and (19), we deduce the following types of traveling wave solutions for Eq. (1).

Family 1 If $\lambda^2 - 4\mu > 0$, then we have the hyperbolic solution

$$u(\xi) = \frac{K}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\,\xi)}\right), \quad (20)$$

Family 2 If $\lambda^2 - 4\mu < 0$, then we have the trigonometric solution

$$u(\xi) = \frac{K}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right), \quad (21)$$

where

$$\xi = \frac{Kx^{\beta}}{\Gamma(1+\beta)} + \frac{K^3(\lambda^2 - 4\mu)\ t^{\alpha}}{4\ \Gamma(1+\alpha)}. \tag{22}$$

In particular, if we set $B = 0$, $A \neq 0$, $\lambda > 0$, and $\mu = 0$ in Eq. (20), then we get the solitary wave solutions to Eq. (1)

$$u(\xi) = \frac{K\lambda}{2}\left[\coth\left(\frac{1}{2}\lambda\left[\frac{Kx^{\beta}}{\Gamma(1+\beta)} + \frac{K^3\lambda^2\ t^{\alpha}}{4\ \Gamma(1+\alpha)}\right]\right)\right]. \tag{23}$$

If $B \neq 0$, $A^2 < B^2$, $\lambda > 0$, and $\mu = 0$, then we get

$$u(\xi) = \frac{K\lambda}{2}\left[\tanh\left(\frac{1}{2}\lambda\left[\frac{Kx^{\beta}}{\Gamma(1+\beta)} + \frac{K^3\lambda^2\ t^{\alpha}}{4\ \Gamma(1+\alpha)}\right] + \xi_0\right)\right], \tag{24}$$

where $\xi_0 = \tanh^{-1}(A/B)$. Similarly we can calculate the solitary wave solutions to Eq. (21).

Remark 1 We neglect the case with $\lambda^2 - 4\mu = 0$, because if $\lambda^2 - 4\mu = 0$, we get $L = 0$. This yields a contradiction with L being a nonzero constant.

Case 2

$$\alpha_0 = -\frac{K\lambda}{2},\quad L = \frac{(\lambda^2 - 4\mu)K^3}{4},\quad a_1 = 0,\quad b_1 = -K\mu, \tag{25}$$

where K, λ, and μ are arbitrary constants. Substituting Eq. (25) into Eq. (17) yields

$$u(\xi) = -K\mu\left(\frac{G'}{G}\right)^{-1} - \frac{K\lambda}{2}. \tag{26}$$

From the solutions of Eqs. (13) and (26), we deduce the following types of traveling wave solutions of Eq. (1).

Family 1 If $\lambda^2 - 4\mu > 0$, then we have the hyperbolic solution

$$u(\xi) = -\frac{K\lambda}{2} - K\mu\left[\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2 - 4\mu)}\ \xi)}\right) - \frac{\lambda}{2}\right]^{-1}, \tag{27}$$

Family 2 If $\lambda^2 - 4\mu < 0$, then we have the trigonometric solution

$$u(\xi) = -\frac{K\lambda}{2} - K\mu\left[\frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi) + B \to (\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi)}\right) - \frac{\lambda}{2}\right]^{-1}, \tag{28}$$

where

$$\xi = \frac{Kx^{\beta}}{\Gamma(1+\beta)} + \frac{K^3(\lambda^2 - 4\mu)\ t^{\alpha}}{4\ \Gamma(1+\alpha)}. \tag{29}$$

Case 3

$$\alpha_0 = 0,\ L = -4\mu K^3,\ \lambda = 0,\ a_1 = K,\quad b_1 = -K\mu, \tag{30}$$

where K and μ are arbitrary constants. Substituting Eq. (30) into Eq. (17) yields

$$u(\xi) = K\left(\frac{G'}{G}\right) - K\mu\left(\frac{G'}{G}\right)^{-1}. \tag{31}$$

From the solutions to Eqs. (13) and (31), we deduce the following types of traveling wave solutions to Eq. (1).

Family 1 If $\mu < 0$, then we have the hyperbolic solution

$$u(\xi) = K\sqrt{-\mu}\left[\frac{A\cosh(\sqrt{-\mu}\ \xi) + B\sinh(\sqrt{-\mu}\ \xi)}{A\sinh(\sqrt{-\mu}\ \xi) + B\cosh(\sqrt{-\mu}\ \xi)}\right] + K\sqrt{-\mu}\left[\frac{A\sinh(\sqrt{-\mu}\ \xi) + B\cosh(\sqrt{-\mu}\ \xi)}{A\cosh(\sqrt{-\mu}\ \xi) + B\sinh(\sqrt{-\mu}\ \xi)}\right], \tag{32}$$

Family 2 If $\mu > 0$, then we have the trigonometric solution

$$u(\xi) = K\sqrt{\mu}\left[\frac{-A\sin(\sqrt{\mu}\xi) + B\cos(\sqrt{\mu}\xi)}{A\cos(\sqrt{\mu}\xi) + B\sin(\sqrt{\mu}\xi)}\right] - K\sqrt{\mu}\left[\frac{A\cos(\sqrt{\mu}\xi) + B\sin(\sqrt{\mu}\xi)}{-A\sin(\sqrt{\mu}\xi) + B\cos(\sqrt{\mu}\xi)}\right], \tag{33}$$

where

$$\xi = \frac{Kx^{\beta}}{\Gamma(1+\beta)} - \frac{4\mu K^3\ t^{\alpha}}{\Gamma(1+\alpha)}. \tag{34}$$

4.2. The nonlinear KdV equation with time- and space-fractional derivatives

In this section, to demonstrate the effectiveness of our approach, we use the complex transformation (15) to convert the nonlinear KdV equation with time- and space-fractional derivatives (2) to an ordinary differential equation. After that, we integrate twice to obtain

$$\frac{1}{2}LU^2 + \frac{a}{6}KU^3 + \frac{K^3}{2}[U']^2 + C_1U + C_2 = 0, \tag{35}$$

where C_1 and C_2 are the integration constants. Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (35), we have

$$\begin{aligned} U(\xi) = {} & \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2 \\ & + \alpha_3\left(\frac{G'}{G}\right)^{-1} + \alpha_4\left(\frac{G'}{G}\right)^{-2}, \end{aligned} \tag{36}$$

where α_0, α_1 α_2, b_1, b_2, L, and K are arbitrary constants to be determined later. Substituting Eqs. (13) and (36) into Eq. (35), collecting all the terms of powers of (G'/G), and setting each coefficient to zero, we get a system of algebraic equations. With the aid of Maple or Mathematica, we can solve this system of algebraic equations to obtain the following sets of solutions.

Case 1

$$\begin{aligned} L = {} & -8K^3\mu - K^3\lambda^2 - Ka\alpha_0, \\ \alpha_1 = {} & -\frac{12K^2\lambda}{a}, \quad \alpha_2 = -\frac{12K^2}{a}, \\ C_1 = {} & \frac{K}{2a}(24K^4\lambda^2\mu + a^2\alpha_0^2 + 2K^2a\lambda^2\alpha_0 \\ & + 48K^4\mu^2 + 16\alpha_0K^2a\mu), \\ C_2 = {} & -\frac{K}{6a^2}(a^3\alpha_0^3 + 432K^6\lambda^2\mu^2 + 144\alpha_0aK^4\mu^2 \\ & + 3\alpha_0^2a^2K^2\lambda^2 + 24\alpha_0^2a^2K^2\mu + 72\alpha_0aK^4\lambda^2\mu), \\ \alpha_3 = {} & \alpha_4 = 0, \end{aligned} \tag{37}$$

where α_0, K, a, λ, and μ are arbitrary constants. Substituting Eq. (37) into Eq. (36) yields

$$u(\xi) = \alpha_0 - \frac{12K^2\lambda}{a}\left(\frac{G'}{G}\right) - \frac{12K^2}{a}\left(\frac{G'}{G}\right)^2. \tag{38}$$

From the solutions to Eqs. (13) and (38), we deduce the following types of traveling wave solutions to Eq. (2).

Family 1 If $\lambda^2 - 4\mu > 0$, then we have the hyperbolic solution

$$u(\xi) = -\frac{3K^2(\lambda^2-4\mu)}{a}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}\right)^2 + \alpha_0 + \frac{3K^2\lambda^2}{a}, \tag{39}$$

Family 2 If $\lambda^2 - 4\mu < 0$, then we have the trigonometric solution

$$u(\xi) = -\frac{3K^2(4\mu-\lambda^2)}{a}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right)^2 + \alpha_0 + \frac{3K^2\lambda^2}{a}, \tag{40}$$

where

$$\xi = \frac{Kx^\beta}{\Gamma(1+\beta)} - \frac{(8K^3\mu + K^3\lambda^2 + Ka\ \alpha_0)\ t^\alpha}{\Gamma(1+\alpha)}. \tag{41}$$

Family 3 If $\lambda^2 - 4\mu = 0$, then we have the rational solution

$$u(\eta) = -\frac{12K^2B^2}{a(B\eta + A)^2} + \alpha_0 + \frac{3K^2\lambda^2}{a}, \tag{42}$$

where

$$\eta = \frac{Kx^\beta}{\Gamma(1+\beta)} - \frac{(3K^3\lambda^2 + Ka\ \alpha_0)\ t^\alpha}{\Gamma(1+\alpha)}. \tag{43}$$

In particular, if we set $B = 0, A \neq 0$, $\lambda > 0$, and $\mu = 0$, in Eq. (39), then we get the solitary wave solutions to Eq. (2) as follows:

$$\begin{aligned} u(\xi) = {} & -\frac{3K^2\lambda^2}{a}\coth^2\left(\frac{1}{2}\lambda\left[\frac{Kx^\beta}{\Gamma(1+\beta)}\right.\right. \\ & \left.\left.\frac{K^3\lambda^2\ t^\alpha}{4\ \Gamma(1+\alpha)}\right]\right) + \alpha_0 + \frac{3K^2\lambda^2}{a}. \end{aligned} \tag{44}$$

While if $B \neq 0$, $A^2 < B^2$, $\lambda > 0$, and $\mu = 0$, then we

get

$$u(\xi) = -\frac{3K^2\lambda^2}{a}\tanh^2\left(\frac{1}{2}\lambda\left[\frac{Kx^\beta}{\Gamma(1+\beta)} + \frac{K^3\lambda^2\ t^\alpha}{4\ \Gamma(1+\alpha)}\right] + \xi_0\right) + \alpha_0 + \frac{3K^2\lambda^2}{a}, \quad (45)$$

where $\xi_0 = \tanh^{-1}(A/B)$. Similarly we can calculate the solitary wave solutions to Eq. (40).

Case 2

$$\begin{aligned} L &= -8K^3\mu - K^3\lambda^2 - Ka\alpha_0, \\ \alpha_3 &= -\frac{12K^2\lambda\mu}{a}, \quad \alpha_4 = -\frac{12K^2\mu^2}{a}, \\ C_1 &= \frac{K}{2a}(24K^4\lambda^2\mu + a^2\alpha_0^2 + 2K^2a\lambda^2\alpha_0 \\ &\quad + 48K^4\mu^2 + 16\alpha_0K^2a\mu), \\ C_2 &= -\frac{K}{6a^2}(a^3\alpha_0^3 + 432K^6\lambda^2\mu^2 + 144\alpha_0aK^4\mu^2 \\ &\quad + 3\alpha_0^2a^2K^2\lambda^2 + 24\alpha_0^2a^2K^2\mu + 72\alpha_0aK^4\lambda^2\mu), \\ \alpha_1 &= \alpha_2 = 0, \end{aligned} \quad (46)$$

where α_0, K, a, λ, and μ are arbitrary constants. Substituting Eq. (46) into Eq. (36) yields

$$u(\xi) = \alpha_0 - \frac{12K^2\lambda\mu}{a}\left(\frac{G'}{G}\right)^{-1} - \frac{12K^2\mu^2}{a}\left(\frac{G'}{G}\right)^{-2}. \quad (47)$$

From the solutions to Eqs. (13) and (47), we deduce the following types of traveling wave solutions to Eq. (2).

Family 1 If $\lambda^2 - 4\mu > 0$, then we have the hyperbolic solution

$$\begin{aligned} u(\xi) &= \alpha_0 - \frac{12K^2\lambda\mu}{a}\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}\right) - \frac{\lambda}{2}\right]^{-1} \\ &\quad - \frac{12K^2\mu^2}{a}\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}{A\sinh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi) + B\cosh(\frac{1}{2}\sqrt{(\lambda^2-4\mu)}\ \xi)}\right) - \frac{\lambda}{2}\right]^{-2}. \end{aligned} \quad (48)$$

Family 2 If $\lambda^2 - 4\mu < 0$, then we have the trigonometric solution

$$\begin{aligned} u(\xi) &= \alpha_0 - \frac{12K^2\lambda\mu}{a}\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right) - \frac{\lambda}{2}\right]^{-1} \\ &\quad - \frac{12K^2\mu^2}{a}\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-A\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}{A\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi) + B\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi)}\right) - \frac{\lambda}{2}\right]^{-2}, \end{aligned} \quad (49)$$

where

$$\xi = \frac{Kx^\beta}{\Gamma(1+\beta)} - \frac{(8K^3\mu + K^3\lambda^2 + Ka\ \alpha_0)\ t^\alpha}{\Gamma(1+\alpha)}. \quad (50)$$

Family 3 If $\lambda^2 - 4\mu = 0$, then we have the rational solution

$$u(\eta) = \alpha_0 - \frac{12K^2\lambda\mu}{a}\left[\frac{2(B\eta + A)}{(2B - A\lambda) - \lambda B\eta}\right] - \frac{12K^2\mu^2}{a}\left[\frac{2(B\eta + A)}{(2B - A\lambda) - \lambda B\eta}\right]^2, \quad (51)$$

where

$$\eta = \frac{Kx^\beta}{\Gamma(1+\beta)} - \frac{(3K^3\lambda^2 + Ka\alpha_0)\ t^\alpha}{\Gamma(1+\alpha)}. \quad (52)$$

5. Conclusion

In this paper, we use the improved (G'/G) - expansion function method to calculate the exact solutions for the time- and space-fractional derivative foam drainage equation and the time- and space-fractional derivative nonlinear KdV equation. This method is reliable, simple and gives many new exact solutions for the nonlinear fractional differential equations. When the parameters take certain special values, the solitary waves are derived from the traveling waves. This method is very efficient and powerful in finding the exact solutions for the nonlinear fractional differential equations.

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