A new generalized fractional Dirac soliton hierarchy and its fractional Hamiltonian structure*

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Based on the differential forms and exterior derivatives of fractional orders, Wu first presented the generalized Tu formula to construct the generalized Hamiltonian structure of the fractional soliton equation. We apply the generalized Tu formula to calculate the fractional Dirac soliton equation hierarchy and its Hamiltonian structure. The method can be generalized to the other fractional soliton hierarchy.

Keywords: fractional calculus, generalized Tu formula, Dirac soliton hierarchy, Hamiltonian structure

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1. Introduction

Nobel Laureate Gerardus 't Hooft once remarked that discrete space–time is the most radical and logical viewpoint of reality. In discontinuous space–time, fractional calculus plays an important role, which can accurately describe many nonlinear phenomena in physics, such as kinetic theories,[1,2] statistical mechanics,[3,4] dynamics in complex media,[5,6] and many others.[7−13] The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov, and Riemann.

Fractional calculus is a generalization of the ordinary differentiation and integration to an arbitrary order.[14−17] Since Riewe[18,19] proposed the concept of non-conservation mechanics, the fractional conservation laws[20] and the fractional Hamiltonian equations were derived, and numerous encouraging results were effectively obtained and discussed.[21−24] Recently, the fractional action–like variational problems have gained attention in studying the dynamics of nonconservative systems.[25,26]

Tu has proposed a Lie algebra and a trace identity to construct the integrable system and the Hamiltonian structure of integrable systems,[27] which was called the Tu scheme by Ma.[28] From then on, many integrable systems and their Hamiltonian structures have been obtained.[29−32] Then, obtaining new fractional integrable hierarchies of the fractional soliton equations is an important and interesting work in soliton theory. Wu and Zhang firstly proposed the generalized Tu formula and searched for the Hamiltonian structures of the fractional AKNS hierarchy.[33]

In this paper, we consider a fractional generalized Hamiltonian system of the Dirac hierarchy by using the generalized Tu formula to investigate the Hamiltonian structure of the fractional Dirac soliton hierarchy.

2. Brief overview of fractional derivatives and integrals

The derivative of arbitrary real order α can be considered as an interpolation of the sequence of operators. We will use the notion suggested and used by Davis,[34] namely,

$$ {}_a\mathrm{D}_t^{\alpha} f(t), \tag{1}$$

where subscripts a and t denote the two limits related to the operation of fractional differentiation. Following Ross,[35] we will call them the terminals of fractional differentiation. The appearance of the terminals in the fractional symbol is essential; it helps to avoid ambiguities in the applications of fractional deriva-

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tives to real problems. The derivatives of arbitrary order are also called fractional derivatives.

The integral of arbitrary order $p > 0$ is

$$\begin{aligned} &{}_a\mathrm{D}_t^{-p} f(t) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} \\ &\quad + \frac{1}{\Gamma(p+k+1)} \int_a^t (t-\tau)^{p+m} f^{(m+1)}(\tau)\mathrm{d}\tau. \end{aligned} \quad (2)$$

Formula (2) immediately provides us with the asymptotic result of ${}_a\mathrm{D}_t^{-p} f(t)$ at $t = 0$.

The derivative of arbitrary order p is

$$\begin{aligned} &{}_a\mathrm{D}_t^{p} f(t) \\ &= \lim_{h\to\infty, nh\to t-a} f_n^{(p)}(t) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ &\quad + \frac{1}{\Gamma(-p+k+1)} \int_a^t (t-\tau)^{-p+m} f^{(m+1)}(\tau)\mathrm{d}\tau. \end{aligned} \quad (3)$$

Formula (3) has been obtained under the assumption that the derivatives f_t^k $(k = 1, 2, \ldots, m+1)$ are continuous in closed interval $[a, t]$ and that m is an integer number satisfying the condition $m > p + 1$. The smallest probable value for m is determined by the inequality $m < p < m + 1$.

From the pure mathematical point of view, such a class of functions is narrow. However, this class of functions is very important for applications. Understanding this fact is important for the proper use of the fractional calculus in applications, especially the Riemann–Liouville definition

$$\begin{aligned} {}_a\mathrm{D}_t^{p} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_a^t (t-\tau)^{-p+m} f(\tau)\mathrm{d}\tau, \\ (m \le p \le m+1) \end{aligned} \quad (4)$$

provides an excellent opportunity to weaken the conditions on function $f(t)$.

Let us consider how the Riemann–Liouville definition (4) appears as the result of the unification of the notions of integer-order integration and differentiation. Suppose that function $f(\tau)$ is continuous and integrable in every finite interval (a, t), and function $f(t)$ may have an integrable singularity of order $r < 1$ at point $\tau = a$

$$\lim_{\tau\to a} (\tau - a)^r f(t) = \text{const.}\ (\neq 0). \quad (5)$$

Then the integral

$$f^{-1}(t) = \int_a^t f(\tau)\mathrm{d}\tau \quad (6)$$

exists and has a finite value 0 as $t \to a$.

In the general case, we have the Cauchy formula

$$f^{-n}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)\mathrm{d}\tau. \quad (7)$$

To extend the n-fold integration to non-integer values of n, we can start with Cauchy formula (7) and replace the integer n in it by a real $p > 0$

$${}_a\mathrm{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)\mathrm{d}\tau. \quad (8)$$

In Eq. (7) the integer n must satisfy the condition $n \ge 1$. The condition for p is weak; for the existence of integral (8), we must have $p > 0$. Moreover, under certain reasonable assumptions,

$$\lim_{p\to 0} {}_a\mathrm{D}_t^{-p} f(t) = f(t). \quad (9)$$

So we can put ${}_a\mathrm{D}_t^{0} f(t) = f(t)$. If $f(t)$ is continuous for $t \ge a$, the integration of arbitrary real order defined by Eq. (8) has the following important property:

$${}_a\mathrm{D}_t^{-p}({}_a\mathrm{D}_t^{-q}) f(t) = {}_a\mathrm{D}_t^{-p-q} f(t). \quad (10)$$

We have

$$\begin{aligned} &{}_a\mathrm{D}_t^{-p}({}_a\mathrm{D}_t^{-q}) f(t) \\ &= \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1}\, {}_a\mathrm{D}_t^{-p} f(\tau)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(p+q)} \int_a^t (t-\zeta)^{p+q-1} f(\zeta)\mathrm{d}\zeta \\ &= {}_a\mathrm{D}_t^{-p-q} f(t). \end{aligned} \quad (11)$$

Obviously, we can interchange p and q to have

$$\begin{aligned} {}_a\mathrm{D}_t^{-p}({}_a\mathrm{D}_t^{-q}) f(t) &= {}_a\mathrm{D}_t^{-q}({}_a\mathrm{D}_t^{-p}) f(t) \\ &= {}_a\mathrm{D}_t^{-p-q} f(t). \end{aligned} \quad (12)$$

Representation (7) for the derivative of an integer order provides an opportunity for extending the notion of differentiation to non-integer n. So $k - a > 0$, this gives

$$\begin{aligned} {}_a\mathrm{D}_t^{k-a} = \frac{1}{\Gamma(a)} \frac{\mathrm{d}k}{\mathrm{d}t^k} \int_a^t (t-\tau)^{a-1} f(\tau)\mathrm{d}\tau, \\ (0 \le a \le 1), \end{aligned} \quad (13)$$

where the only substantial restriction for a is $a > 0$, which is necessary for the convergence of the integral in Eq. (13).

Let us consider some properties of the Riemann–Liouville fractional derivative. The first and maybe

the most important property of the Riemann–Liouville fractional derivative is that for $p > 0$ and $t > a$,

$$ {}_a\mathrm{D}_t^p({}_a\mathrm{D}_t^{-p})f(t) = f(t), \tag{14}$$

$$ {}_a\mathrm{D}_t^p({}_a\mathrm{D}_t^q)f(t) = {}_a\mathrm{D}_t^{p+q}f(t), \tag{15}$$

$$ {}_a\mathrm{D}_t^p(f(t)g(t)) = g(t)\,{}_a\mathrm{D}_t^p f(t) + f(t)\,{}_a\mathrm{D}_t^p g(t), \tag{16}$$

which means that the Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the same order.

Similar to the conventional integer-order differentiation and integration, the fractional differentiation and integration do not commute.

If the fractional derivative ${}_a\mathrm{D}_t^p f(t)$ $(k-1 \le p \le k)$ of function $f(t)$ is integrable, then

$$\begin{aligned} &{}_a\mathrm{D}_t^{-p}({}_a\mathrm{D}_t^p)f(t) \\ &= f(t) - \sum_{m=1}^{k}({}_a\mathrm{D}_t^{p-m})_{t=a}\frac{(t-a)^{p-m}}{\Gamma(p-m+1)}, \end{aligned} \tag{17}$$

and especially when $0 \le p \le 1$,

$$\begin{aligned} &{}_a\mathrm{D}_t^{-p}({}_a\mathrm{D}_t^p)f(t) \\ &= f(t) - ({}_a\mathrm{D}_t^{p-1})_{t=a}\frac{(t-a)^{p-1}}{\Gamma(p)}. \end{aligned} \tag{18}$$

3. Fractional exterior derivative and Hamiltonian equations

Since Adda proposed the fractional generalization of differential forms,[36,37] several versions of fractional exterior differential approaches and the applications related to different forms of fractional derivatives have appeared in the literature,[38,39] and the properties of the fractional derivatives have been discussed.[40]

The exterior derivative is defined as

$$\mathrm{d} = \sum_{m=1}^{n}\mathrm{d}x_m\frac{\partial}{\partial x_m}. \tag{19}$$

The exterior derivative maps k forms into $k+1$ forms and has the following algebraic results. Let γ and λ be k forms, and μ be an m form, then

$$\mathrm{d}(\gamma+\lambda) = \mathrm{d}\gamma + \mathrm{d}\lambda, \tag{20}$$

$$\mathrm{d}(\gamma\wedge\mu) = (\mathrm{d}\gamma)\wedge\mu + (-1)^k\gamma\wedge\mathrm{d}\mu, \tag{21}$$

$$\mathrm{d}(\mathrm{d}\gamma) = 0. \tag{22}$$

The last identity is called the Poincaré lemma. Form γ is called closed if $\mathrm{d}\gamma = 0$. Form γ is called exact if there exists a form μ such that $\mathrm{d}\mu = \gamma$. The order of μ is one less than that of γ. Exact forms are always closed, closed forms are not always exact.

Next, we introduce the fractional exterior derivative

$$\mathrm{d}^p = (\mathrm{d}x_i)^p\mathrm{D}_{x_i}^p. \tag{23}$$

A differential 1-form is defined as

$$\omega_p = F^i(x)(\mathrm{d}x_i)^p, \tag{24}$$

where the vector field $F^i(x)$ can be represented as $F^i(x) = -\mathrm{D}_{x_i}^p V$, and $V(x)$ is a continuously differentiable function. Using Eq. (23), the exact fractional form can be represented as

$$\omega_p = -\mathrm{d}^p V = -(\mathrm{d}x_i)^p\mathrm{D}_{x_i}^p V. \tag{25}$$

Note that Eq. (24) is a fractional generalization of the differential form (19). Obviously, the fractional 1-form ω_p can be closed when the differential 1-form $\omega = \omega_1$ is not closed.

We define the fractional functional

$$J[p,q] = \frac{1}{\Gamma(1+\alpha)}\int[p\mathrm{D}_t^\alpha q - H(t,p,q)](\mathrm{d}t)^\alpha, \tag{26}$$

then we can readily derive the generalized Poincare–Cartan 1-form, which reads

$$\omega = p\mathrm{d}^\alpha q - H(\mathrm{d}t)^\alpha. \tag{27}$$

From Eq. (27), we have

$$\begin{aligned} \mathrm{d}^\alpha\omega &= p_t^\alpha(\mathrm{d}t)^\alpha\wedge\mathrm{d}^\alpha q + \mathrm{d}^\alpha p\wedge\mathrm{d}^\alpha q \\ &\quad - \frac{\partial H}{\partial p}\mathrm{d}^\alpha p\wedge(\mathrm{d}t)^\alpha - \frac{\partial H}{\partial q}\mathrm{d}^\alpha q\wedge(\mathrm{d}t)^\alpha \\ &= \left[p_t^\alpha + \frac{\partial H}{\partial q}\right](\mathrm{d}t)^\alpha\wedge\mathrm{d}^\alpha q \\ &\quad + \left[\frac{\partial H}{\partial p}(\mathrm{d}t)^\alpha - \mathrm{d}^\alpha q\right]\wedge\mathrm{d}^\alpha p. \end{aligned} \tag{28}$$

In the above derivation, p and q are fractional differentiable functions with respect to t.

The fractional closed condition $\mathrm{d}^\alpha\omega = 0$ allows the following fractional Hamilton equations

$$q_t^\alpha = \frac{\partial H}{\partial p}, \quad p_t^\alpha = -\frac{\partial H}{\partial q}. \tag{29}$$

4. Generalized Tu formula

In this section, we introduce the generalized Tu formula.[33] If A_n is a Lie algebra, $\widetilde{A_n}$ is the corresponding loop algebra. Consider the fractional compatibility condition

$$\boldsymbol{\psi}_x^\alpha = \boldsymbol{U}\boldsymbol{\psi}, \quad \boldsymbol{\psi}_t^\beta = \boldsymbol{V}\boldsymbol{\psi}, \tag{30}$$

where the fractional derivative is in the sense of the modified derivative[24] or the local fractional derivative (3), and $\boldsymbol{\psi}$ is an n-dimensional function vector.

The compatibility condition of Eq. (30) leads to the generalized zero curvature equation

$$\boldsymbol{U}_t^{\beta} - \boldsymbol{V}_x^{\alpha} + [\boldsymbol{U}, \boldsymbol{V}] = 0. \tag{31}$$

When $\alpha = \beta = 1$, equation (31) reduces to the classical zero curvature equation. Set

$$\boldsymbol{U} = \boldsymbol{e}_0(\lambda) + \sum_{i}^{n} \boldsymbol{e}_i(\lambda) u_i, \quad \boldsymbol{e}_i \in \tilde{\boldsymbol{A}}_n, \quad 1 \leq i \leq n, \tag{32}$$

where $\boldsymbol{u}$ denotes a non-differentiable vector function. Define $\operatorname{rank}(\lambda)$=$\deg(\lambda)$, then $\operatorname{rank} \boldsymbol{e}_0(\lambda) = \zeta$, $\operatorname{rank} \boldsymbol{e}_i(\lambda) = \alpha_i$, $0 \leq i \leq n$. If the ranks of u_i are taken as $\zeta - \alpha_i, 1 \leq i \leq n$, then each term in $\boldsymbol{U}$ has the homogeneous rank α, which is denoted by

$$\operatorname{rank}(\boldsymbol{U}) = \operatorname{rank}\left(\frac{\partial \alpha}{\partial x^{\alpha}}\right) = \zeta. \tag{33}$$

Set $\boldsymbol{V} = \sum_{m \geq 0} \boldsymbol{V}_m \lambda^{-m}, (\boldsymbol{V}_m)_{\lambda} = 0, m \geq 0$ as a solution of the stationary zero curvature equation

$$-\boldsymbol{V}_x^{\alpha} + [\boldsymbol{U}, \boldsymbol{V}] = 0, \tag{34}$$

and $\operatorname{rank}(\boldsymbol{V}_m)_{\lambda}$ is assumed to be given as $\operatorname{rank}(\boldsymbol{V}_m)_{\lambda} = \xi, m \geq 0$, then each team in $\boldsymbol{V}$ has the same rank

$$\operatorname{rank}(\boldsymbol{V}) = \operatorname{rank}\left(\frac{\partial \beta}{\partial t^{\beta}}\right) = \xi. \tag{35}$$

We define that the form $f(\boldsymbol{A}, \boldsymbol{B})$ is proportional to the trace $\operatorname{Tr}(\boldsymbol{AB})$, $f(\boldsymbol{A}, \boldsymbol{B}) = \text{const.} \times \operatorname{Tr}(\boldsymbol{AB})$. For convenience, we take

$$f(\boldsymbol{A}, \boldsymbol{B}) = \operatorname{Tr}(\boldsymbol{AB}). \tag{36}$$

The fractional gradient $\boldsymbol{\nabla}_{\boldsymbol{B}} f(\boldsymbol{A}, \boldsymbol{B})$ of the functional $f(\boldsymbol{A}, \boldsymbol{B})$ is defined by

$$\begin{aligned} \frac{\partial}{\partial \epsilon} f(\boldsymbol{A}, \boldsymbol{B} + \epsilon \boldsymbol{C}) &= f(\delta_{\boldsymbol{B}} f(\boldsymbol{A}, \boldsymbol{B}), \boldsymbol{C}), \\ \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} &\in A_n, \end{aligned} \tag{37}$$

where $\delta_{\boldsymbol{B}}$ is the variational derivative with respect to $\boldsymbol{B}$.

We construct a function

$$\boldsymbol{W} = f(\boldsymbol{V}, \boldsymbol{U}_{\lambda}) + f(\boldsymbol{K}, \boldsymbol{V}_x^{\alpha} - [\boldsymbol{U}, \boldsymbol{V}]), \tag{38}$$

with the defined fractional variational derivative

$$\frac{\delta \boldsymbol{W}}{\delta \boldsymbol{K}} = \boldsymbol{V}_x^{\alpha} - [\boldsymbol{U}, \boldsymbol{V}], \quad \frac{\delta \boldsymbol{W}}{\delta \boldsymbol{V}} = \boldsymbol{U}_{\lambda} + [\boldsymbol{U}, \boldsymbol{V}]. \tag{39}$$

According to the above equations, we can obtain

$$[\boldsymbol{K}, \boldsymbol{V}]_x^{\alpha} = [\boldsymbol{U}_{\lambda}, \boldsymbol{V}] + [\boldsymbol{U}, [\boldsymbol{K}, \boldsymbol{V}]]. \tag{40}$$

The solution $\boldsymbol{V}$ of Eq. (39), which is of given homogeneous rank, is unique up to a constant multiplier, it proves that for any solution $\boldsymbol{V}$ of homogeneous rank, there exists a constant γ such that $\bar{\boldsymbol{V}} = \lambda^{\gamma} \boldsymbol{V}$ is also a solution of Eq. (39).

Consider the communication relationship

$$f([\boldsymbol{A}, \boldsymbol{B}], \boldsymbol{C}) = f(\boldsymbol{A}, [\boldsymbol{B}, \boldsymbol{C}]), \tag{41}$$

then, a fractional trace identity can be presented as follows:

$$\begin{aligned} \frac{\delta f(\boldsymbol{V}, \boldsymbol{U}_{\lambda})}{\delta u_i} &= f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}_{\lambda}}{\partial u_i}\right) + f\left([\boldsymbol{K}, \boldsymbol{V}], \frac{\partial \boldsymbol{U}_{\lambda}}{\partial u_i}\right) \\ &= f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}_{\lambda}}{\partial u_i}\right) + f\left(\boldsymbol{V}_{\lambda}, \frac{\partial \boldsymbol{U}_{\lambda}}{\partial u_i}\right) \\ &\quad + \frac{\gamma}{\lambda} f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u_i}\right) \\ &= \frac{\partial}{\partial \lambda} f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u_i}\right) + f\left(\boldsymbol{V}_{\lambda}, \frac{\partial \boldsymbol{U}_{\lambda}}{\partial u_i}\right) \\ &\quad + \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma}\right) f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u_i}\right) \\ &= \lambda^{-\gamma} \frac{\partial}{\partial \lambda}\left[\lambda^{\gamma} f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial u_i}\right)\right], \\ &\quad 0 \leq i \leq n. \end{aligned} \tag{42}$$

The fractional trace identity and the fractional soliton equation hierarchy are new equations in mathematical physics.

5. Fractional Dirac soliton hierarchy and its fractional Hamiltonian structure

To illustrate our method, we apply the fractional zero curvature equation and the fractional Hamiltonian structure to construct the generalized fractional Dirac equation hierarchy.

We consider the following generalized spectral problem

$$\begin{aligned} &\mathrm{D}_x^{\alpha} \boldsymbol{\psi} = \boldsymbol{U}(u, \lambda) \boldsymbol{\psi}, \\ &\boldsymbol{U}(u, \lambda) = \begin{pmatrix} q & \lambda + r \\ -\lambda + r & -q \end{pmatrix}, \quad \lambda_t^{\beta} = 0. \end{aligned} \tag{43}$$

Choose a simple subalgebra of A_1

$$\boldsymbol{e}_1(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\boldsymbol{e}_3(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{44}$$

which have the commutative relations

$$\begin{aligned} [\boldsymbol{e}_1(m), \boldsymbol{e}_2(n)] &= -2\boldsymbol{e}_3(m+n), \\ [\boldsymbol{e}_1(m), \boldsymbol{e}_3(n)] &= -2\boldsymbol{e}_2(m+n), \\ [\boldsymbol{e}_2(m), \boldsymbol{e}_3(n)] &= -2\boldsymbol{e}_1(m+n). \end{aligned}$$

Solving the stationary zero curvature equation

$$\mathrm{D}_x^\alpha \boldsymbol{V} = [\boldsymbol{U}, \boldsymbol{V}], \tag{45}$$

we have

$$\begin{aligned} &a_{ix}^\alpha = -2c_{i+1} + 2qb_i, \\ &b_{ix}^\alpha = -2rc_i + 2qa_i, \quad c_{ix}^\alpha = 2a_{i+1} - 2rb_i, \\ &a_0 = c_0 = 0, b_0 = -1, a_1 = -r, b_1 = 0, c_1 = -q, \\ &a_2 = -\frac{1}{2}q_x^\alpha, b_2 = -\frac{1}{2}q^2 - \frac{1}{2}r^2, c_2 = \frac{1}{2}r_x^\alpha, \ldots. \end{aligned} \tag{46}$$

From Eq. (46), we can obtain the following recursion relations

$$\begin{aligned} &\begin{pmatrix} 2c_{n+1} \\ 2a_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} -2q\mathrm{D}_x^{-\alpha}r & -\frac{1}{2}\mathrm{D}_x^\alpha + 2q\mathrm{D}_x^{-\alpha}q\mathrm{D}_x^\alpha \\ \frac{1}{2}\mathrm{D}_x^\alpha - 2r\mathrm{D}_x^{-\alpha}r & 2r\mathrm{D}_x^{-\alpha}q\mathrm{D}_x^\alpha \end{pmatrix} \begin{pmatrix} 2c_n \\ 2a_n \end{pmatrix} \\ &= \boldsymbol{L} \begin{pmatrix} 2c_n \\ 2a_n \end{pmatrix}, \\ b_n &= \mathrm{D}_x^{-\alpha}(-2rc_n + 2qa_n), \quad n = 1, 2, \ldots. \end{aligned} \tag{47}$$

Now, we set

$$\begin{aligned} \mathrm{D}_x^\alpha \boldsymbol{V}_+^{(n)} &= \sum_{m=0}^{n} a_m e_1(n-m) + b_m e_2(n-m) \\ &\quad + c_m e_3(n-m), \\ \boldsymbol{V}_-^{(n)} &= \lambda^n \boldsymbol{V} - \boldsymbol{V}_n^+, \end{aligned} \tag{48}$$

a direct calculation leads to

$$-\mathrm{D}_x^\alpha \boldsymbol{V}_+^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}] = 2c_{n+1}\boldsymbol{e}_1(0) - 2a_{n+1}\boldsymbol{e}_3(0). \tag{49}$$

The generalized zero curvature equation $\mathrm{D}_t^\beta \boldsymbol{U} - \mathrm{D}_x^\alpha \boldsymbol{V}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}] = 0$ leads to a system of evolution equations

$$\begin{aligned} \boldsymbol{U}_t^\beta &= \begin{pmatrix} q_t^\beta \\ r_t^\beta \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2c(n+1) \\ 2a(n+1) \end{pmatrix} \\ &= \boldsymbol{J}\boldsymbol{L} \begin{pmatrix} 2c(n) \\ 2a(n) \end{pmatrix} \\ &= \boldsymbol{J}\boldsymbol{L}^n \begin{pmatrix} -2q \\ -2r \end{pmatrix}, \end{aligned} \tag{50}$$

where J and L are the Hamiltonian operators.

When $n = 2$, we obtain the generalized fractional version of the Dirac nonlinear equation

$$\begin{aligned} \mathrm{D}_t^\beta q &= \frac{1}{2}\mathrm{D}_x^\alpha \mathrm{D}_x^\alpha r - q^2 r - r^3, \\ \mathrm{D}_t^\beta r &= -\frac{1}{2}\mathrm{D}_x^\alpha \mathrm{D}_x^\alpha q + q^3 + qr^2. \end{aligned} \tag{51}$$

For $\alpha = \beta = 1$, Eq. (51) can be reduced to the classical Dirac equation

$$\begin{aligned} q_t &= \frac{1}{2}r_{xx} - q^2 r - r^3, \\ r_t &= -\frac{1}{2}q_{xx} + q^3 + qr^2. \end{aligned} \tag{52}$$

Through a direct computation, by utilizing trace identity (42), we have

$$\begin{aligned} &f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial q}\right) = 2c, \\ &f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial r}\right) = 2a, \\ &f\left(\boldsymbol{V}, \frac{\partial \boldsymbol{U}}{\partial \lambda}\right) = -2b, \end{aligned} \tag{53}$$

$$\frac{\delta(-2b)}{\delta u} = \lambda^{-\gamma}\frac{\partial}{\partial \lambda}\lambda^\gamma \begin{pmatrix} 2c \\ 2a \end{pmatrix}. \tag{54}$$

Equating the coefficients of λ^{-n-2} on the two sides of Eq. (54) yields

$$\frac{\delta(-2b_{n+2})}{\delta u} = (\gamma - n - 1)\begin{pmatrix} 2c_{n+1} \\ 2a_{n+1} \end{pmatrix}. \tag{55}$$

To fix the constant γ, we simply set $n = 0$ and have $\gamma = 0$, then we obtain the fractional Hamiltonian function

$$\boldsymbol{H}_n = \frac{2b_{n+2}}{n+1}, \quad n = 0, 1, 2, \ldots. \tag{56}$$

The generalized fractional version of the Dirac equation hierarchy can be written as

$$\boldsymbol{U}_t^\beta = \begin{pmatrix} q_t^\beta \\ r_t^\beta \end{pmatrix} = \boldsymbol{J}\frac{\delta \boldsymbol{H}_n}{\delta u}. \tag{57}$$

6. Conclusion

A way to construct the generalized Hamiltonian structure and the application of the theory to the fractional soliton equation hierarchy have been studied. The key idea is to establish a new Dirac Hamiltonian system in the fractional form. In the near future, we will investigate how to prove it is Liouville integrable.

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