A nonlinear discrete integrable coupling system and its infinite conservation laws*

Yu Fa-Jun(于发军)†

School of Mathematics and Systematic Sciences, Shenyang Normal University, Shenyang 110034, *China*

(Received 18 January 2012; revised manuscript received 22 May 2012)

We construct a nonlinear integrable coupling of discrete soliton hierarchy, and establish the infinite conservation laws (CLs) for the nonlinear integrable coupling of the lattice hierarchy. As an explicit application of the method proposed in the paper, the infinite conservation laws of the nonlinear integrable coupling of the Volterra lattice hierarchy are presented.

Keywords: nonlinear integrable coupling system, infinite conservation law, Volterra lattice hierarchy

PACS: 02.20.Sv, 02.30.Ik, 02.30.Jr **DOI:** 10.1088/1674-1056/21/11/110202

1. Introduction

As is well known, the conservation law (CL) plays an important role in mathematical physics. For instance, it describes the conservation of fundamental physical quantities, provides a method to study quantitative and qualitative properties of equations and their solutions, verifies the complete integrability of nonlinear partial differential equations, and is used to test numerical integrators.

Many methods have been developed to find an infinite number of conservation laws for the KdV equation.[1] Generally, an infinite number of conservation laws or conserved quantities for both continuous and discrete systems can be obtained from the Bäcklund transformation,[2] the scattering problem,[2,3] the formal solutions of eigenfunctions,[4] the quasi-diffierential operator based on the Sato theory,[5] the trace identity,[6] or the Lax pair.[7] It is easy to establish a variational formula for a discrete system. Applications of the semi-inverse method can be found in Ref. [8].

The existence of infinite conservation laws is an important indicator of integrability of the system. From a physical view, it is also very interesting to study the conservation laws for a lattice system. For a lattice equation

$$F(\dot{u}_n, \ddot{u}_n, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots) = 0, \tag{1}$$

if there exist functions ρ_n and J_n such that

$$\dot{\rho}_n|_{F=0} = J_n - J_{n+1}, \tag{2}$$

then Eq. (2) is called the conservation law of Eq. (1), where ρ_n is the conserved density and J_n is the associated flux. The J_n is bounded for all n and vanishes at the boundaries, and $\sum_n \rho_n = c$ is an integral of motion of lattice given by Eq. (1).

Integrable couplings were first proposed by Ma and Fuchsstiener, and some interesting integrable couplings and their Hamiltonian structures were obtained. A few ways to establish integrable couplings were presented by using perturbations,[9,10] enlarging spectral problems,[11,12] semidirect sums of Lie algebras,[13,14] and creating new Lie algebras.[15−20] Furthermore, Ma did a series of original works dedicated to considerably improving the Tu trace identity to construct a Hamiltonian structure for more general systems including integrable couplings, discrete integrable systems, and super-integrable systems with the variational identity.[21−23]

Integrable coupling system is presented using Virasoro symmetry algebra. In detail, let

$$u_{nt} = K(u_n) \tag{3}$$

be a known integrable system, and the following system

$$\begin{cases} u_{nt} = K(u_n), \\ v_{nt} = S(u_n, v_n), \end{cases} \tag{4}$$

*Project supported by the Postdoctoral Science Foundation of China (Grant No. 2011M500404) and the Program for Liaoning Excellent Talents in University, China (Grant No. LJQ2011119).

†Corresponding author. E-mail: yfajun@163.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

is called an integrable coupling of the system (3), if $v_{nt} = S(u_n, v_n)$ is also integrable and $S(u_n, v_n)$ contains explicitly u_n or derivatives of u_n with respect to the shift operator. The subscript t denotes the derivation with respect to time. However, many integrable coupling systems are linear integrable couplings in Refs. [9]–[23]. Indeed, the nonlinear integrable coupling systems are more important. In this paper, we begin with the Lax pairs from special non-semisimple matrix Lie algebras, and propose a scheme for constructing discrete nonlinear integrable couplings.

Concerning the CLs of differential-difference equations, many results have been achieved successfully. However, to our knowledge, it seems that examples of the CLs of integrable discrete equations in most literatures are just (1+1)- or (1+2)-dimensional cases.[5,6] For comparison, the less integrable couplings of discrete equations have been considered for their CLs.

In this paper, we first propose a method to construct the nonlinear integrable couplings of a discrete soliton hierarchy. Furthermore, we present the infinite conservation laws of nonlinear discrete integrable couplings. As an application, we obtain the infinite conservation laws for the nonlinear integrable couplings of the Volterra lattice hierarchy.

2. A nonlinear discrete integrable coupling and its conservation laws

In this section, we will give a new pair of $\boldsymbol{U}$ and $\boldsymbol{V}$ by Kronecker product, which can produce the nonlinear integrable coupling system. The new Lax pair $\boldsymbol{U}$ and $\boldsymbol{V}$ is given as follows:

$$\begin{aligned} \boldsymbol{U} &= \boldsymbol{I}_N \otimes \boldsymbol{U}_0 + \boldsymbol{T}_N \otimes \boldsymbol{U}_1, \\ \boldsymbol{V} &= \boldsymbol{I}_N \otimes \boldsymbol{V}_0 + \boldsymbol{T}_N \otimes \boldsymbol{V}_1, \end{aligned} \tag{5}$$

where $\boldsymbol{T}_N$ is an $N \times N$ matrix, and

$$\begin{aligned} (\boldsymbol{EV})\boldsymbol{U} - \boldsymbol{UV} = {} & \boldsymbol{I}_N \otimes (\boldsymbol{EV}_0)\boldsymbol{U}_0 - \boldsymbol{I}_N \otimes \boldsymbol{U}_0\boldsymbol{V}_0 \\ & + \boldsymbol{T}_N \otimes (\boldsymbol{EV}_0)\boldsymbol{U}_1 - \boldsymbol{T}_N \otimes \boldsymbol{U}_0\boldsymbol{V}_1 \\ & + \boldsymbol{T}_N \otimes (\boldsymbol{EV}_1)\boldsymbol{U}_0 - \boldsymbol{T}_N \otimes \boldsymbol{U}_1\boldsymbol{V}_0 \\ & + \boldsymbol{T}_N^2 \otimes (\boldsymbol{EV}_1)\boldsymbol{U}_1 - \boldsymbol{T}_N^2 \otimes \boldsymbol{U}_1\boldsymbol{V}_1, \end{aligned} \tag{6}$$

where $\boldsymbol{E}$ is the shift operator and $\boldsymbol{E}u_n = u_{n+1}$, $\boldsymbol{I}_N$ and $\boldsymbol{T}_N$ are the same ranks, and $\boldsymbol{T}_N^2 = \boldsymbol{T}_N$. Substituting the Lax pairs (5) and the stationary zero curvature equation

$$(\boldsymbol{EV})\boldsymbol{U} - \boldsymbol{UV} = 0 \tag{7}$$

into the discrete zero curvature equation, we obtain the new discrete zero curvature equations

$$\begin{cases} \boldsymbol{U}_{0t} - (\boldsymbol{EV}_0)\boldsymbol{U}_0 + \boldsymbol{U}_0\boldsymbol{V}_0 = 0, \\ \boldsymbol{U}_{1t} - (\boldsymbol{EV}_1)\boldsymbol{U}_0 + \boldsymbol{U}_0\boldsymbol{V}_1 - (\boldsymbol{EV}_0)\boldsymbol{U}_1 \\ +\boldsymbol{U}_1\boldsymbol{V}_0 - (\boldsymbol{EV}_1)\boldsymbol{U}_1 + \boldsymbol{U}_1\boldsymbol{V}_1 = 0. \end{cases} \tag{8}$$

This is a nonlinear integrable coupling because the matrix $(\boldsymbol{EV}_1)\boldsymbol{U}_1 - \boldsymbol{U}_1\boldsymbol{V}_1$ often produces the nonlinear terms.

Next let us shed light on the general idea of constructing coupling systems by a particular class of the Kronecker product. Consider a 2×2 Kronecker product and let $\boldsymbol{U}$ and $\boldsymbol{V}$ take the following forms:

$$\boldsymbol{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \boldsymbol{U}_0 + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \boldsymbol{U}_1, \tag{9}$$

$$\boldsymbol{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \boldsymbol{V}_0 + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \otimes \boldsymbol{V}_1, \tag{10}$$

then we obtain a new pair of $\boldsymbol{U}$ and $\boldsymbol{V}$

$$\begin{aligned} \boldsymbol{U} &= \begin{pmatrix} \boldsymbol{U}_0 & 0 \\ \boldsymbol{U}_1 & \boldsymbol{U}_0 + \boldsymbol{U}_1 \end{pmatrix}, \\ \boldsymbol{V} &= \begin{pmatrix} \boldsymbol{V}_0 & 0 \\ \boldsymbol{V}_1 & \boldsymbol{V}_0 + \boldsymbol{V}_1 \end{pmatrix}. \end{aligned} \tag{11}$$

Therefore, the corresponding enlarged stationary equation and enlarged zero curvature equation are equivalent to

$$\begin{cases} (\boldsymbol{EV}_0)\boldsymbol{U}_0 - \boldsymbol{U}_0\boldsymbol{V}_0 = 0, \\ (\boldsymbol{EV}_1)\boldsymbol{U}_0 - \boldsymbol{U}_0\boldsymbol{V}_1 + (\boldsymbol{EV}_0)\boldsymbol{U}_1 - \boldsymbol{U}_1\boldsymbol{V}_0 \\ +(\boldsymbol{EV}_1)\boldsymbol{U}_1 - \boldsymbol{U}_1\boldsymbol{V}_1 = 0, \end{cases} \tag{12}$$

and

$$\begin{cases} \boldsymbol{U}_{0t} - (\boldsymbol{EV}_0)\boldsymbol{U}_0 + \boldsymbol{U}_0\boldsymbol{V}_0 = 0, \\ \boldsymbol{U}_{1t} - (\boldsymbol{EV}_1)\boldsymbol{U}_0 + \boldsymbol{U}_0\boldsymbol{V}_1 - (\boldsymbol{EV}_0)\boldsymbol{U}_1 \\ +\boldsymbol{U}_1\boldsymbol{V}_0 - (\boldsymbol{EV}_1)\boldsymbol{U}_1 + \boldsymbol{U}_1\boldsymbol{V}_1 = 0. \end{cases} \tag{13}$$

From the physical point of view, it is also very important to obtain the conservation laws for a discrete integrable coupling system. Next, we present the definition of the conservation laws of a nonlinear integrable coupling system.

For a discrete integrable coupling system

$$\begin{cases} F_1(\dot{u}_n, \ddot{u}_n, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots) = 0, \\ F_2(\dot{u}_n, \ddot{u}_n, \ldots, u_{n-1}, u_n, u_{n+1}, \ldots, \dot{v}_n, \ldots, \\ v_{n-1}, v_n, \ldots,) = 0, \end{cases} \tag{14}$$

if there exist functions ρ_{1n}, ρ_{2n}, J_n, and K_n such that

$$\begin{cases} \dot{\rho}_{1n}|_{F_1=0} = J_n - J_{n+1}, \\ \dot{\rho}_{2n}|_{F_2=0} = K_n - K_{n+1} + G(J_n), \end{cases} \tag{15}$$

then Eq. (15) is called the conservation law of the integrable coupling equation (14), where ρ_{1n} and ρ_{2n} are the conserved densities, J_n and K_n are the associated fluxes, and $G(J_n)$ is a function of J_n. In addition, J_n and K_n are bounded for all n, and vanish at the boundaries. Then $\sum\limits_n \rho_{in} = c$ $(i = 1, 2)$ is an integral of motion of lattice (14).

3. A nonlinear integrable coupling of Volterra lattice hierarchy

In this section, we use the Kronecker product of Lie algebra to construct the nonlinear integrable couplings of the Volterra lattice hierarchy. Consider the following isospectral problem:

$$\boldsymbol{E}\psi_n = \boldsymbol{U}\psi_n, \psi_{nt} = \boldsymbol{V}\psi_n, \psi_n = \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \\ \psi_{3n} \\ \psi_{4n} \end{pmatrix}, \tag{16}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ take the forms given by Eqs. (9) and (10), and

$$\begin{aligned} \boldsymbol{U}_0 &= \begin{pmatrix} 1 & u_n \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \boldsymbol{U}_1 = \begin{pmatrix} 0 & v_n \\ 0 & 0 \end{pmatrix}, \\ \boldsymbol{V}_0 &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \boldsymbol{V}_1 = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}. \end{aligned} \tag{17}$$

Obviously, we obtain

$$\begin{aligned} \boldsymbol{U} &= \begin{pmatrix} 1 & u_n & 0 & 0 \\ \lambda^{-1} & 0 & 0 & 0 \\ 0 & v_n & 1 & u_n + v_n \\ 0 & 0 & \lambda^{-1} & 0 \end{pmatrix}, \\ \boldsymbol{V} &= \begin{pmatrix} a & b & 0 & 0 \\ c & -a & 0 & 0 \\ e & f & a+e & b+f \\ g & -e & c+g & -a-e \end{pmatrix}. \end{aligned} \tag{18}$$

From the stationary discrete zero curvature equation

$$(\boldsymbol{E}\boldsymbol{V}_0)\boldsymbol{U}_0 - \boldsymbol{U}_0\boldsymbol{V}_0 = 0, \tag{19}$$

$$\begin{aligned} &(\boldsymbol{E}\boldsymbol{V}_1)\boldsymbol{U}_0 - \boldsymbol{U}_0\boldsymbol{V}_1 + (\boldsymbol{E}\boldsymbol{V}_0)\boldsymbol{U}_1 - \boldsymbol{U}_1\boldsymbol{V}_0 \\ &+(\boldsymbol{E}\boldsymbol{V}_1)\boldsymbol{U}_1 - \boldsymbol{U}_1\boldsymbol{V}_1 = 0, \end{aligned} \tag{20}$$

we obtain

$$\begin{cases} a^{(1)} + \lambda^{-1}b^{(1)} - a - u_n c = 0, \\ u_n a^{(1)} - b + u_n a = 0, \\ c^{(1)} - \lambda^{-1}a^{(1)} - \lambda^{-1}a = 0, \\ u_n c^{(1)} - \lambda^{-1}b = 0, \\ e^{(1)} + \lambda^{-1}f^{(1)} - e - u_n g - v_n c - v_n g = 0, \\ v_n a^{(1)} + u_n e^{(1)} - f + u_n e + v_n a \\ +v_n e^{(1)} + v_n e = 0, \\ g^{(1)} - \lambda^{-1}e^{(1)} - \lambda^{-1}e = 0, \\ v_n c^{(1)} + u_n g^{(1)} - \lambda^{-1}f + v_n g^{(1)} = 0, \end{cases} \tag{21}$$

which equivalently leads to

$$\begin{cases} b = u_n(a^{(1)} + a), \\ c = \lambda^{-1}(a + a^{(-1)}), \\ a^{(1)} - a + \lambda^{-1}[u_{n+1}(a^{(2)} + a^{(1)}) \\ -u_n(a_n + a^{(-1)})] = 0, \\ f = u_n(e^{(1)} + e) + v_n(e^{(1)} + e) + v_n(a^{(1)} + a), \\ g = \lambda^{-1}(e + e^{(-1)}), \\ e^{(1)} - e + \lambda^{-1}[u_{n+1}(e^{(2)} + e^{(1)}) - u_n(e + e^{(-1)}) \\ +v_{n+1}(e^{(2)} + e^{(1)} + a^{(2)} + a^{(1)}) - v_n(e + e^{(-1)} \\ +a + a^{(-1)})] = 0. \end{cases}$$

Substituting

$$a = \sum_{m\geq 0} a_m\lambda^{-m},\ b = \sum_{m\geq 0} b_m\lambda^{-m},\ c = \sum_{m\geq 0} c_m\lambda^{-m},$$

$$e = \sum_{m\geq 0} e_m\lambda^{-m},\ f = \sum_{m\geq 0} f_m\lambda^{-m},\ g = \sum_{m\geq 0} g_m\lambda^{-m},$$

into Eq. (21), and setting $a_0 = 1/2$ and $e_0 = 1/2$, we find that all sets of functions $a_m^{(i)}$, $b_m^{(i)}$, $c_m^{(i)}$, $e_m^{(i)}$, $f_m^{(i)}$, and $g_m^{(i)}$ are separately determined. In particular, the first few sets are given by

$$\begin{cases} b_0 = u_n,\ c_0 = 0,\ a_1 = -u_n, \\ b_1 = -u_n(u_{n+1} + u_n),\ c_1 = 1, \\ f_0 = 2v_n + u_n,\ g_0 = 0, \\ e_1 = -u_n - 2v_n,\ g_1 = 1, \\ f_1 = -u_n(2v_{n+1} + 2v_n + u_{n+1} + u_n) \\ \quad -2v_n(u_{n+1} + u_n + v_{n+1} + v_n). \end{cases} \tag{22}$$

Considering

$$\boldsymbol{\Delta}_0 = \begin{pmatrix} 0 & -b_{m+1} \\ 0 & a_{m+1} + a_{m+1}^{(-1)} \end{pmatrix},\ \boldsymbol{V}_0 = \boldsymbol{V}_{0m} + \boldsymbol{\Delta}_0,$$

$$\boldsymbol{\Delta}_1 = \begin{pmatrix} 0 & -f_{m+1} \\ 0 & e_{m+1} + e_{m+1}^{(-1)} \end{pmatrix},\ \boldsymbol{V}_1 = (\lambda^{m+1}\boldsymbol{V}_1)_+ + \boldsymbol{\Delta}_1,$$

defining

$$\boldsymbol{V}_n^{(m)} = \begin{pmatrix} \boldsymbol{V}_0 & 0 \\ \boldsymbol{V}_1 & \boldsymbol{V}_0 + \boldsymbol{V}_1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Delta}_0 & 0 \\ \boldsymbol{\Delta}_1 & \boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_1 \end{pmatrix}, \tag{23}$$

and using Eqs. (19) and (20), we have

$$(\boldsymbol{E}\boldsymbol{V}_m)\boldsymbol{U} - \boldsymbol{U}\boldsymbol{V}_m = \begin{pmatrix} 0 & u_n(a_{n+1}^{(1)} - a_{m+1}^{(-1)}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & L_1 & 0 & L_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

with

$$L_1 = u_n\Big(e_{m+1}^{(1)} - e_{m+1}^{(-1)}\Big) + v_n\Big(a_{m+1}^{(1)} - a_{m+1}^{(-1)} + e_{m+1}^{(1)} - e_{m+1}^{(-1)}\Big).$$
$$L_2 = u_n\Big(a_{n+1}^{(1)} - a_{m+1}^{(-1)}\Big) + u_n\Big(e_{m+1}^{(1)} - e_{m+1}^{(-1)}\Big) + v_n\Big(a_{m+1}^{(1)} - a_{m+1}^{(-1)} + e_{m+1}^{(1)} - e_{m+1}^{(-1)}\Big).$$

Then, the enlarged discrete zero curvature equation (13) leads to the Volterra lattice equations (16) and (17). The hierarchy of coupling systems is produced for the Volterra lattice hierarchy as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = \boldsymbol{K}_m = \begin{pmatrix} u_n(a_{n+1}^{(1)} - a_{m+1}^{(-1)}) \\ u_n(e_{m+1}^{(1)} - e_{m+1}^{(-1)}) + v_n(a_{m+1}^{(1)} - a_{m+1}^{(-1)} + e_{m+1}^{(1)} - e_{m+1}^{(-1)}) \end{pmatrix}. \tag{25}$$

Using Eqs. (14) and (22), the first two systems in Eq. (24) with $m = 0$ are obtained

$$\begin{cases} u_{n,t_0} = u_n(u_{n-1} - u_{n+1}), \\ v_{n,t_0} = 2v_n(u_{n-1} - u_{n+1}) + 2u_n(v_{n-1} - v_{n+1}) \\ \qquad +2v_n(v_{n-1} - v_{n+1}) + u_n(u_{n-1} - u_{n+1}). \end{cases} \tag{26}$$

Obviously, the supplementary evolution equation of v_n is nonlinear with respect to v_n itself, so system (24) gives a nonlinear discrete integrable coupling of the Volterra lattice hierarchy. The nonlinear discrete integrable coupling system of a Volterra lattice soliton (given by Eq. (24)) is obtained using the Kronecker product. The above results are similar to those in Ref. [22]. However, an important problem is how to construct the infinite conservation laws for a nonlinear integrable coupling of the Volterra lattice hierarchy.

4. Infinite conservation laws for the nonlinear integrable coupling of Volterra lattice hierarchy

In this section, we will construct the infinite conservation laws for the nonlinear integrable coupling of the Volterra lattice hierarchy. For N different real eigenvalues λ_j, from the spectral problem

$$\begin{cases} \varphi_{1j}(n+1) = \varphi_{1j}(n) + \lambda_j^{-1}u_n\varphi_{1j}(n-1), \\ \varphi_{3j}(n+1) = \lambda_j^{-1}v_n\varphi_{1j}(n-1) + \varphi_{3j}(n) \\ \qquad +\lambda_j^{-1}(u_n + v_n)\varphi_{3j}(n-1), \end{cases} \tag{27}$$

and

$$\begin{cases} \varphi_{1j,\mathrm{t}}(n) = a\varphi_{1j}(n) + \lambda_j^{-1}b\varphi_{1j}(n-1), \\ \varphi_{3j,\mathrm{t}}(n) = e\varphi_{1j}(n) + \lambda_j^{-1}f\varphi_{1j}(n-1) \\ \qquad +(a+e)\varphi_{3j}(n) \\ \qquad +\lambda_j^{-1}(b+f)\varphi_{3j}(n-1), \end{cases} \tag{28}$$

the conservation law of the nonlinear integrable coupling of Volterra lattice system (24) can be obtained as

$$\begin{cases} -\ln(\theta_n)_{\mathrm{t}} = (E - E^{-1})(a + b\theta_{n-1}), \\ \ln(\Gamma_n)_{\mathrm{t}} = (E - E^{-1})\big[e + f\theta_{n-1} \\ \qquad +(a+e)\Gamma_n + (b+f)\Gamma_{n-1}\big], \end{cases} \tag{29}$$

by setting

$$\begin{cases} \theta_n = \theta(n) = \dfrac{\varphi_1(n)}{\varphi_1(n+1)}, \\ \Gamma_n = \Gamma(n) = \dfrac{\varphi_3(n)}{\varphi_1(n)}. \end{cases} \tag{30}$$

Especially, for the first two equations in the system (24), this law is rewritten as

$$\begin{cases} -\ln(\theta_n)_t = (E-E^{-1})[-u_n - u_n(u_{n+1}+u_n)\theta_{n-1}], \\ \ln(\Gamma_n)_t = (E-E^{-1})\big\{-u_n - 2v_n + [-u_n(2v_{n+1}+2v_n+u_{n+1}+u_n) - 2v_n(v_{n+1}+v_n) + u_{n+1}+u_n]\theta_{n-1} \\ \qquad -2(u_n+v_n)\Gamma_n + [-2u_n(v_{n+1}+v_n+u_{n+1}+u_n) - 2v_n(v_{n+1}+v_n) + u_{n+1}+u_n]\Gamma_{n-1}\big\}. \end{cases} \tag{31}$$

In order to obtain the infinite conservation laws, we first rewrite Eq. (27) as

$$\begin{cases} 1 = \theta_n + \lambda^{-1}u_n\theta_{n-1}\theta_n, \\ \Gamma_{n+1} = \lambda^{-1}v_n\theta_{n-1}\theta_n + \Gamma_n\theta_n \\ \qquad +\lambda^{-1}(u_n+v_n)\theta_{n-1}\theta_n\Gamma_{n-1}. \end{cases} \tag{32}$$

Then, expanding θ_n and Γ_n in terms of the power series of $1/\lambda$

$$\begin{cases} \theta_n = \sum\limits_{j=1}^{\infty}\dfrac{\theta_n^{(j)}}{\lambda^j}, \\ \Gamma_n = \sum\limits_{j=1}^{\infty}\dfrac{\Gamma_n^{(j)}}{\lambda^j}, \end{cases} \tag{33}$$

and substituting it into Eq. (31), we have a recursion formula for $\theta_n^{(j)}$ and $\Gamma_n^{(j)}$ as

$$\begin{cases} \delta_{j,0} = \theta_n^{(j)} + u_n\sum\limits_{i=1}^{j}\theta_{n-1}^{(i)}\theta_n^{(j+1-i)}, \\ \Gamma_{n+1}^{(j)} = v_n\sum\limits_{i=1}^{j}\theta_{n-1}^{(i)}\theta_n^{(j+1-i)} + \sum\limits_{i=1}^{j}\Gamma_n^{(i)}\theta_n^{(j-i)} \\ \qquad +(u_n+v_n)\sum\limits_{i+m+n=j}\theta_{n-1}^{(i)}\theta_n^{(m)}\Gamma_{n-1}^{(n)}, \end{cases} \tag{34}$$

with which it is easy to obtain all the coefficients $\theta_n^{(j)}$ and $\Gamma_n^{(j)}$. In the next step, due to $\theta_n^{(1)}=1$, $\Gamma_n^{(1)}=1$, and $\lambda_{\rm t}=0$, the conservation law (29) for the nonlinear integrable coupling of a Volterra lattice hierarchy is given by

$$\begin{cases} \Big[\sum\limits_{k=1}^{\infty}(-1)^k\dfrac{1}{k}\Big(\sum\limits_{j=1}^{\infty}\dfrac{\theta_n^{(j+1)}}{\lambda^j}\Big)^k\Big]_t = (E-E^{-1})\Big[-u_n - u_n(u_{n+1}+u_n)\sum\limits_{j=1}^{\infty}\dfrac{\theta_{n-1}^{(j)}}{\lambda^j}\Big], \\ \Big[\sum\limits_{k=1}^{\infty}(-1)^k\dfrac{1}{k}(\sum\limits_{j=1}^{\infty}\dfrac{\Gamma_n^{(j+1)}}{\lambda^j})^k\Big]_t = (E-E^{-1})\Big\{(-u_n-2v_n)\sum\limits_{j=1}^{\infty}\dfrac{\theta_n^{(j)}}{\lambda^j} + \big[-u_n(2v_{n+1}+2v_n+u_{n+1}+u_n) \\ \qquad -2v_n(v_{n+1}+v_n)+u_{n+1}+u_n\big]\sum\limits_{j=1}^{\infty}\dfrac{\theta_{n-1}^{(j)}}{\lambda^j} - 2(u_n+v_n)\sum\limits_{j=1}^{\infty}\dfrac{\Gamma_n^{(j)}}{\lambda^j} \\ \qquad +\big[-2u_n(v_{n+1}+v_n+u_{n+1}+u_n) - 2v_n(v_{n+1}+v_n) + u_{n+1} \\ \qquad +u_n\big]\sum\limits_{j=1}^{\infty}\dfrac{\Gamma_{n-1}^{(j)}}{\lambda^j}\Big\}. \end{cases} \tag{35}$$

Based on Eq. (35), an infinite number of conservation laws can be determined by equating the powers of $1/\lambda$. We list them as follows:

$$\begin{cases} u_{n,t} = (E-E^{-1})u_n, \\ v_{n,t} = (E-E^{-1})[v_n(-2u_n-2v_n) + u - n(-u-n-2v_n)], \end{cases} \tag{36}$$

$$\begin{cases} (u_n + \dfrac{1}{2}v_n^2)_{\rm t} = (E-E^{-1})[-u_n - u_n(u_{n+1}+u_n)], \\ [u_n(v_n+v_{n-1}) + \dfrac{1}{3}v_n^3]_{\rm t} = (E-E^{-1})[-u_n - 2v_n - u_n(2v_{n+1}+2v_n+u_{n+1}+u_n) - 2v_n(v_{n+1}+v_n) \\ \qquad +u_{n+1}+u_n-2u_n-2v_n-2u_n(v_{n+1}+v_n+u_{n+1}+u_n) \\ \qquad -2v_n(v_{n+1}+v_n)+u_{n+1}+u_n]. \end{cases} \tag{37}$$

According to the definition of the conservation laws of a nonlinear integrable coupling (15), we obtain the infinite conservation laws for the nonlinear integrable coupling of the Volterra lattice hierarchy given by Eqs. (27)–(37).

If system (26) has two integrals of motion

$$u_n = D(n), \quad v_n = C(n), \tag{38}$$

where $D(n)$ and $C(n)$ are arbitrary functions. We can obtain the physical meaning of these equations through investigation in the continual limit: $f_{n\pm1} = f \pm (\partial f)/(\partial n) + (1/2)(\partial^2 f)/(\partial n^2) \pm \cdots$. Based on system (26), we obtain

$$\frac{\partial u}{\partial t} \cong -\frac{\mathrm{d}D}{\mathrm{d}n}u + \frac{\partial}{\partial n}\Big(D\frac{\partial u}{\partial n}\Big) - \frac{D}{u}\Big(\frac{\partial u}{\partial n}\Big), \qquad (39)$$

and

$$\frac{\partial v}{\partial t} + C\Big(1+\frac{u}{v}\Big)\frac{\partial v}{\partial n} \cong -\frac{\mathrm{d}}{\mathrm{d}n}(D+C)v - \frac{\partial u}{\partial n}v^2 + 2\frac{\mathrm{d}C}{\mathrm{d}n}u + C\frac{\partial u}{\partial n}. \qquad (40)$$

Therefore, we conclude that the set of equations (26) describes the nonlinear diffusion wave processes in the dissipative discrete medium, and $D(n)$ and $C(n)$ are the diffusion coefficient and the wave velocity, respectively. Equations (39) and (40) can be used to describe the possible formation mechanisms for hydrodynamics, solid state physics, and plasma physics.

5. Conclusion

In this paper, we have considered the definition of the conservation laws of a nonlinear integrable couplings. We obtain the infinite conservation laws for the nonlinear integrable couplings of the Volterra lattice hierarchy. The obtained integrable couplings of the Volterra lattice equations and conservation laws can be used to describe the possible formation mechanisms for hydrodynamics, solid state physics, and plasma physics, respectively.

References

[1] Miura R M 1968 *J. Math. Phys.* **9** 1204

[2] Wadati M, Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419

[3] Zakharov V and Shabat A 1972 *Sov. Phys. JETP* **34** 62

[4] Konno K, Sanuki H and Ichikawa Y H 1974 *Prog. Theor. Phys.* **52** 886

[5] Kajiwara K, Matsukidaira J and Satsuma J 1990 *Phys. Lett.* A **146** 115

[6] Tsuchida T and Wadati M 1998 *J. Phys. Soc. Jpn.* **67** 1175

[7] Zhang D J and Chen D Y 2002 *Chaos Soliton. Fract.* **14** 573

[8] Zhu Z N, Xue W M, Wu X N and Zhu Z M 2002 *J. Phys.* A: *Math. Gen.* **35** 5079

[9] Ma W X and Fuchssteiner B 1996 *Chaos Soliton. Fract.* **7** 1227

[10] Ma W X 1992 *Chin. J. Cont. Math.* **13** 79

[11] Guo F K and Zhang Y F 2003 *J. Math. Phys.* **44** 5793

[12] Zhang Y F, Fan E G and Tam H 2006 *Phys. Lett.* A **359** 471

[13] Xia T C and You F C 2007 *Chin. Phys.* **16** 605

[14] Ma W X 2003 *Phys. Lett.* A **316** 72

[15] Fan E G 2000 *J. Math. Phys.* **41** 7769

[16] Ma W X, Xu X X and Zhang Y F 2006 *J. Math. Phys.* **47** 053501

[17] Ma W X, Xu X X and Zhang Y F 2006 *Phys. Lett.* A **351** 125

[18] Zhang Y F and Zhang H Q 2002 *J. Math. Phys.* **43** 466

[19] Yu F J and Zhang H Q 2006 *Phys. Lett.* A **353** 326

[20] Ma W X and Guo F K 1997 *Int. J. Theor. Phys.* **36** 697

[21] Ma W X and Chen M 2006 *J. Phys.* A: *Gen. Math.* **39** 10787

[22] Ma W X and Zhu Z N 2010 *Comput. Math. Appl.* **60** 2601

[23] Ma W X, He J S and Qin Z Y 2008 *J. Math. Phys.* **49** 033511