Atom-loss-induced quantum optical bi-stability switch*

Wu Bao-Jun(吴宝俊)† and Cui Fu-Cheng(崔傅成)

Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, *China*

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We investigate the nonlinear dynamics of a system composed of a cigar-shaped Bose–Einstein condensate and an optical cavity with the two sides coupled dispersively. By adopting discrete-mode approximation for the condensate, taking atom loss as a necessary part of the model to analyze the evolution of the system, while using trial and error method to find out steady states of the system as a reference, numerical simulation demonstrates that with a constant pump, atom loss will trigger a quantum optical bi-stability switch, which predicts a new interesting phenomenon for experiments to verify.

Keywords: atom loss, discrete-mode approximation, optical bi-stability switch

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1. Introduction

Many investigations have been devoted to the combination of ultra-cold atomic physics and cavity quantum electrodynamics since the coupling of a Bose–Einstein condensate (BEC), as a single matter-wave field mode, to a single cavity mode in a high finesse optical cavity was achieved experimentally.[1,2] This type of combined system showed many interesting phenomena, for example, $\sqrt{N}$ scaling of the atom–photon coupling,[1,2] vacuum Rabi splitting,[1,2] optical bi-stability,[3,4] cavity enhanced super-radiance of a BEC,[5] and a profound study found a map between the atom ensemble–cavity system and the canonical opto-mechanical system.[6,7] In theory, the cavity can be utilized to characterize the properties of the matter-wave field, and it has been experimentally adopted to study the statistics of an atom laser by detecting single atoms falling through a cavity.[8−10]

Intrinsic nonlinearity, which is the most remarkable characteristic of this atom ensemble–cavity system, can be considered responsible for many previous experimental and theoretical works,[3,6,7,11−18] and especially, the dispersive optical bi-stability.[3,11−13] If we only focus on the mechanical motion of the atoms and restrict it to the large detuning limit, the atom–photon interaction is of a dispersive nature. The atom–photon interaction provides a potential (an optical lattice) to the atoms, and also shifts the cavity mode frequency. Because of this, the atom ensemble can affect or even determine the potential acting on itself, which is different from the usual atom–atom interactions.

Based on this point of view, an extraordinary theoretical work, which developed the discrete-mode approximation (DMA) for the condensate, with the central idea that the dynamics of the condensate is confined to some discrete modes, has been applied to the nonlinear dynamics of a cigar-shaped Bose–Einstein condensate in an optical cavity.[19] Experimentally, this kind of system has been investigated (though not a cigar-shaped BEC) by Brennecke *et al.*,[7] with the main result showing that in the weak atom–atom interaction and weak excitation (or non-depletion) limit, the condensate-cavity system maps perfectly onto the generic cavity opto-mechanical system. However, the theoretical work is not restricted to the weak excitation limit, and provides a comparison between the DMA and the full description of the Gross–Pitaevskii (GP) equation, in which we find that the finite-mode approximation is quantitatively good in the case of weak atom–atom interaction. Furthermore, the study shows that the DMA can also be used to find out the steady states of the system, analyze their stability, and elucidate both the possibility of breakdown of adiabaticity and optical bi-stability in the system.[19]

However, we must point out that despite the DMA method framework being already built by the

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†Corresponding author. E-mail: dabaoski@126.com

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formal theoretical work, the atom loss of BEC, which is an important and realistic nature of the system, has not yet been taken into consideration.

In our work, we follow up the previous theoretical method, adopt a discrete-mode approximation for the condensate, take atom loss as a necessary part of the model to analyze the evolution of the system, and meanwhile use a trial and error method to find out steady states as a reference. Numerical simulation demonstrates that in a constant pump situation, atom loss will trigger a quantum optical bi-stability switch, which predicts a new interesting phenomenon for experiments to verify.

This paper is organized as follows. In Section 2, the system and the general formalism are described, and the discrete-mode approximation is introduced. In Section 3, we consider a constant pump situation with atom loss added in the system, and introduce a trail and error method to find out the steady states as a comparison; the evolution of this system predicts an optical bi-stability switch. Our results are summarized in Section 4.

2. General formalism

We study the cigar-shaped Bose–Einstein condensate, which is located inside an optical cavity with its axial direction parallel to that of the cavity. The internal transition frequency of the two-level atoms is $\omega_{\rm a}$ and the frequency of the relevant cavity mode interacting with the atoms is $\omega_{\rm c}$. The cavity mode is driven by a pump laser with frequency $\omega_{\rm p}$ and amplitude η. $\Delta_{\rm a} = \omega_{\rm a} - \omega_{\rm p}$ and $\Delta_{\rm c} = \omega_{\rm c} - \omega_{\rm p}$ indicate the atom–pump and cavity–pump detunings, respectively. In the large detuning limit, after transferring into the rotating frame at the pump frequency, the Hamiltonian for the condensate-cavity system can be written as ($\hbar = 1$ throughout in this paper),[7]

$$\begin{aligned} H = & \int {\rm d}x \hat{\Psi}^\dagger(x) \bigg(-\frac{1}{2m}\frac{{\rm d}^2}{{\rm d}^2 x} + V_{\rm ext}(x) \\ & + U_0 \hat{a}^\dagger \hat{a} \cos^2(kx) + \frac{g_{\rm 1D}}{2} \hat{\Psi}^\dagger(x) \hat{\Psi}(x) \bigg) \hat{\Psi}(x) \\ & + \Delta_{\rm c} \hat{a}^\dagger \hat{a} + \eta(\hat{a} + \hat{a}^\dagger) + H_\kappa, \end{aligned} \tag{1}$$

where $\hat{\Psi}^\dagger$ and $\hat{a}^\dagger$ denote the creation operators for the atoms (with mass m) and the cavity photons, respectively. $V_{\rm ext}(x) = (1/2) m \omega_\parallel^2 x^2$ is the harmonic potential in the axial direction with characteristic frequency $\omega_\parallel$, and $g_{\rm 1D}$ is the effective atom–atom interaction strength in a transversely tight-confining trap.[20] The atom–cavity photon interaction provides both the harmonic potential $V_{\rm ext}$ (which may also owe to atom–photon interaction, as in an optical trap) and an additional potential $U_0 \hat{a}^\dagger \hat{a} \cos^2(kx)$ for the atoms. The difference between them is that the former is static, while the latter, being proportional to the intra cavity photon number, may be dynamical and quantized.[21] $U_0 = -g_0^2/\Delta_{\rm a}$ represents the maximal light shift per photon that an atom may experience (at an antinode), with g_0 being the atom–photon coupling constant. We assume that the transverse radius of the condensate is much smaller than the cavity mode waist, and therefore neglect the variation of g_0 in the transverse direction. The cavity mode function in the axial direction is $\cos(kx)$, with the wave vector $k = 2\pi/\lambda$. H_κ means cavity photon decay with rate κ.

We may obtain the mean-field Hamiltonian for the condensate-cavity system as[19]

$$\begin{aligned} {\rm i} \frac{\partial \Psi(x,t)}{\partial t} = & \bigg(-\frac{1}{2m}\frac{{\rm d}^2}{{\rm d}^2 x} + V_{\rm ext}(x) + U_0 |\alpha|^2 \cos^2(kx) \\ & + g_{\rm 1D} |\Psi(x,t)|^2 \bigg) \Psi(x,t), \end{aligned} \tag{2}$$

and the equation of motion for the cavity field is

$$\begin{aligned} \frac{\partial \alpha}{\partial t} = & -{\rm i} \left(\Delta_{\rm c} + U_0 \int {\rm d}x |\Psi(x,t)|^2 \cos^2(kx) \right) \alpha \\ & - \kappa \alpha + \eta. \end{aligned} \tag{3}$$

Technically, we need to convert the above equations into their dimensionless form. The characteristic length scale is $\xi = \lambda/2\pi$ and energy scale of the system is $\omega_{\rm r} = k^2/2m$. We thus rescale position and time as $x = \tilde{x}\xi$ and $t = \tilde{t}/\omega_{\rm r}$, and the condensate wavefunction is $\Psi(x,t) = \sqrt{N/\xi}\tilde{\Psi}(\tilde{x},\tilde{t})$ so that the scaled wavefunction is normalized to unity (N is the total atom number). The above equations then transform to

$$\begin{aligned} {\rm i} \frac{\partial \tilde{\Psi}(\tilde{x},\tilde{t})}{\partial \tilde{t}} = & \bigg(-\frac{{\rm d}^2}{{\rm d}^2 \tilde{x}} + \omega^2 \tilde{x}^2 + \tilde{U}_0 |\alpha|^2 \cos^2 \tilde{x} \\ & + g |\tilde{\Psi}(\tilde{x},\tilde{t})|^2 \bigg) \tilde{\Psi}(\tilde{x},\tilde{t}), \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{\partial \alpha}{\partial \tilde{t}} = & -{\rm i} \left(\tilde{\Delta}_{\rm c} + N\tilde{U}_0 \int {\rm d}\tilde{x} |\tilde{\Psi}(\tilde{x},\tilde{t})|^2 \cos^2 \tilde{x} \right) \alpha \\ & - \tilde{\kappa}\alpha + \tilde{\eta}. \end{aligned} \tag{5}$$

Here we define the rescaled dimensionless quantities $(\tilde{U}_0, \tilde{\Delta}_{\rm c}, \tilde{\kappa}, \tilde{\eta}) = (U_0, \Delta_{\rm c}, \kappa, \eta)/\omega_{\rm r}$, the rescaled harmonic frequency $\omega = \sqrt{m\omega_\parallel^2 \xi^2/2\omega_{\rm r}}$, and the atom–atom interaction strength $g = N g_{\rm 1D}/\omega_{\rm r}\xi$.

Since the cavity decay is almost three orders of magnitude faster than the condensate motion,[7] we

can assume that the cavity field follows the condensate adiabatically and find out the photon number as

$$n_{\rm ph} = |\alpha(t)|^2 = \frac{\eta^2}{\kappa^2 + (\Delta_{\rm c} + NU_0\langle\cos^2\tilde{x}\rangle)^2}, \quad (6)$$

where $\langle\cos^2\tilde{x}\rangle$ is defined as $\int {\rm d}\tilde{x}|\tilde{\Psi}(\tilde{x},\tilde{t})|^2\cos^2\tilde{x}$, which means the overlap between the cavity mode intensity and the condensate density distribution. Equation (6) gives the dependence of the intra cavity optical lattice depth on the condensate. The cavity–pump detuning is shifted by the condensate to an effective value $\Delta_{\rm eff} = \Delta_{\rm c} + NU_0\langle\cos^2\tilde{x}\rangle$.

In Ref. [19], the authors have proven the validity of the discrete-mode approximation (DMA), so we can just follow up the previous work, define the discrete modes as $\phi_0 = \Psi_{\rm g}(x)$, $\phi_1 = \sqrt{2}\cos(2kx)\Psi_{\rm g}(x)$, and $\phi_2 = \sqrt{2}\cos(4kx)\Psi_{\rm g}(x)$, and neglect the atom–atom interaction, then substitute $\Psi(x) = \sum_{i=0}^{2}\hat{c}_i\phi_i(x)$ to get the Hamiltonian under the DMA

$$\begin{aligned} H \;=\; & 4\omega_{\rm r}\hat{c}_1^\dagger\hat{c}_1 + 16\omega_{\rm r}\hat{c}_2^\dagger\hat{c}_2 + \frac{U_0}{4}a^\dagger a\big(\sqrt{2}(\hat{c}_0^\dagger\hat{c}_1 + \hat{c}_1^\dagger\hat{c}_0) \\ & + \big(\hat{c}_1^\dagger\hat{c}_2 + c_2^\dagger\hat{c}_1 + 2N\big) + \Delta_{\rm c}a^\dagger a + \eta(a + a^\dagger) \\ & + H_\kappa. \end{aligned} \quad (7)$$

After taking the mean-field approximation $\hat{c}_i \sim \sqrt{N}Z_i$, $a \sim \alpha$, and integrating out the cavity field, the equation of motion for the condensate under the DMA is

$${\rm i}\frac{\rm d}{{\rm d}\tilde{t}}Z = H(n_{\rm ph})Z = (H_1 + n_{\rm ph}H_2)\,Z, \quad (8)$$

with $Z = (Z_0, Z_1, Z_2)^{\rm T}$, and

$$H_1 = \begin{pmatrix} 0 & & \\ & 4 & \\ & & 16 \end{pmatrix}, \quad H_2 = \frac{\tilde{U}_0}{4}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (9)$$

The photon number is

$$n_{\rm ph} = |\alpha|^2 = \frac{\tilde{\eta}^2}{\tilde{\kappa}^2 + (\tilde{\Delta}_{\rm c}' + NZ^\dagger H_2 Z)^2}, \quad (10)$$

where $\tilde{\Delta}_{\rm c}' = \tilde{\Delta}_{\rm c} + N\tilde{U}_0/2$ is the rescaled effective cavity–pump detuning with the condensate in its ground state [$Z = (1,0,0)^{\rm T}$]. We may define the general effective cavity–pump detuning $\tilde{\Delta}_{\rm eff} = \tilde{\Delta}_{\rm c}' + NZ^\dagger H_2 Z$. Thus we get the mathematical form under the DMA of Eq. (6).

3. Constant pump situation with atom loss added

In this section, we consider a BEC-cavity system with a constant pump acting on, and apply the formalism developed in the preceding section to deal with the problem where atom loss takes part in.

We prepare the system that a laser beam with constant frequency and constant amplitude project onto the cavity, and assume that the BEC is in a steady state before atom loss turns on. Here the question is how to find out such a steady state for the given parameters of the system.

We see that in the framework of the DMA, a steady state corresponds to a solution of Eq. (8) that satisfies the condition

$$Z_{\rm s}(\tilde{t}) = Z_{\rm s}\,{\rm e}^{-{\rm i}E_0\tilde{t}}. \quad (11)$$

Or equivalently,

$$H(n_{\rm ph})Z_{\rm s} = (H_1 + n_{\rm ph}H_2)\,Z_{\rm s} = E_0 Z_{\rm s}. \quad (12)$$

This is a nonlinear eigenvalue problem which cannot be solved by linear algebra because of the dependence of the Hamiltonian H on the eigenstate $Z_{\rm s}$ through $n_{\rm ph}$.

We introduce a trial and error method, which is illustrated in Fig. 1, to solve this problem. The main idea of this method is to choose an arbitrary trial photon number $n_{\rm tr}$, solve the ground state of the Hamiltonian $H(n_{\rm tr})$, $Z_{\rm tr}$, and substitute it back into Eq. (10) to gain an output photon number $n_{\rm out}$, then make a comparison between $n_{\rm tr}$ and $n_{\rm out}$. If $n_{\rm out} = n_{\rm tr}$, the solution is self-consistent and a steady state is found. Since $n_{\rm out}$ is a continuous function of $n_{\rm tr}$ and has the same upper and lower bound with $n_{\rm tr}$, $0 < n_{\rm out} < \tilde{\eta}^2/\tilde{\kappa}^2$, we may simply scan $n_{\rm tr}$ from 0 to $\tilde{\eta}^2/\tilde{\kappa}^2$. From the principle of continuity, there exists at least one point that satisfies $n_{\rm out} = n_{\rm tr}$, which guarantees the existence of steady-state solution.

In Fig. 1, we depict the output photon number $n_{\rm out}$ as a function of the trial photon number $n_{\rm tr}$ for different atom number N while other parameters are held at constant. For each curve, each of its intersections with the solid line $n_{\rm out} = n_{\rm tr}$ corresponds to a steady state.

Note that due to the nonlinearity of the system, there might exist more than one steady state solution for a given set of parameters, i.e., there might be more than one intersection point of a specific curve with the solid line.

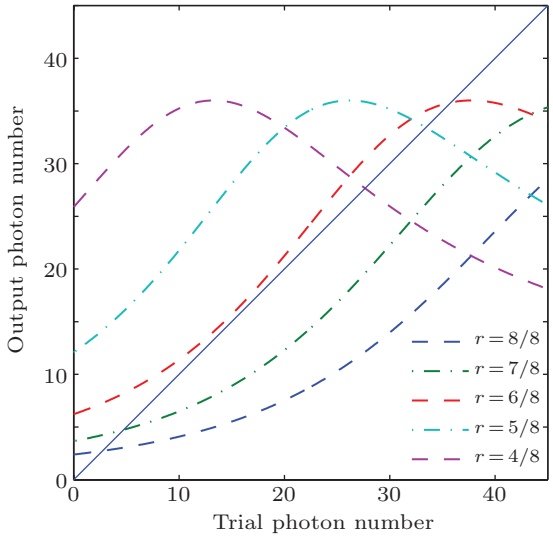


Fig. 1. (colour online) The output photon number $n_{\rm out}$ as a function of the trial photon number $n_{\rm tr}$. The intersection points of the curves with the solid line $y = x$ correspond to steady states. Each curve represents a specific N while other parameters remain fixed. The parameters are taken as: $U_0 = 0.125$, $\kappa = 100$, $\Delta = -250$, $\eta = 600$, $N_{\rm i} = 10000$.

Using the trial and error method, we can map out all the steady states for every set of parameters of the system. Hence the initial steady state of the BEC can be easily found.

Now we put atom loss, which is treated as a controllable quantity, into the system. In theory, we may add a pure imaginary time-dependent term into the Hamiltonian to represent atom loss phenomenologically. By choosing an appropriate mathematical wavefunction transformation, this term can be canceled out, with the consequence that N, which was a constant in the normalization condition $\int {\rm d}x|\Psi(x,t)|^2 = N$, becomes a time-dependent one.

However, without loss of generality, we will skip such a specific mathematical derivation, and simply regard N as a time-dependent quantity in Eq. (6) and in Eq. (10). By which means equation (6) becomes

$$n_{\rm ph} = |\alpha(t)|^2 = \frac{\eta^2}{\kappa^2 + (\Delta_{\rm c} + N(t)U_0\langle\cos^2\tilde{x}\rangle)^2}, \quad (13)$$

which shows an explicit dependence of $n_{\rm ph}$ with respect to $N(t)$.

Note that even we consider the atom loss to be controllable, we choose exponential decay of atoms instead of other mathematical forms in our paper for it is easier to achieve in experiment. Thus we have

$$N(t) = N_{\rm i}\,{\rm e}^{-\gamma t}, \quad (14)$$

where $N_{\rm i}$ is the initial atom number and γ is decay rate of the atoms.

The numerical simulation under the DMA of evolution of the system, with constant pump and exponential decay of atoms is illustrated in Fig. 2. Note that the evolution of the normalized population on the atomic modes ϕ_0 is not depicted in Fig. 2(a) since it is too far from zero point, while the normalized populations on the atomic modes ϕ_2 is too small to analyze.

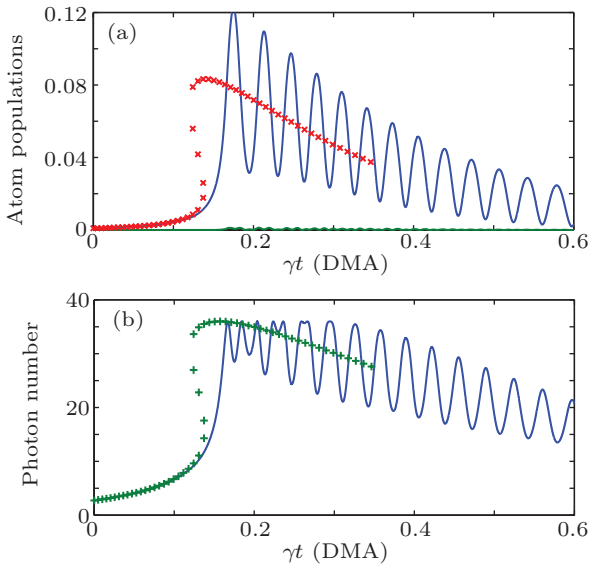


Fig. 2. (colour online) The evolution of the normalized populations on the atomic modes ϕ_1 and ϕ_2 as a function of γt. The dotted line, which represents the steady state points calculated by the trial and error method for every $N(t)$ that corresponds to γt, is drawn as a reference. (b) The photon number as a function of γt. The dotted line, which represents the photon number at steady states calculated by the trial and error method for every $N(t)$ that corresponds to γt, is drawn as a reference. The parameters are taken as: $U_0 = 0.125$, $\kappa = 100$, $\Delta = -250$, $\eta = 600$, $N_{\rm i} = 10000$.

From Fig. 2(a) we see that starting from an initial steady state, as atoms loss exponentially, the system first evolves slowly enough to match an adiabatic process, but as soon as it reaches the bi-stable point, the system suddenly jumps into the higher branch that continues from the other steady-state point which corresponds to the same $N(t)$. Then a fierce oscillation appears, until the population on this atomic mode vanishes due to atom loss. Figure 2(b) shows the same progress with photon number, and confirms the existence of such a jump and oscillation movement of the system.

This kind of movement, where the system jumps from one branch to the other branch at the bi-stable points, is what we call bi-stability switch. Since atom loss was never taken into consideration in previous works, our numerical simulation under the DMA predicts a new effect for experiments to verify. If this

phenomenon is true, it can be considered as a strong evidence that the DMA method is really a powerful tool that may replace the traditional G-P method to deal with the nonlinear dynamics of a system which is composed of a Bose–Einstein condensate and an optical cavity.

4. Conclusion

We follow up the previous investigation on the nonlinear dynamics of a BEC-cavity system. With the idea that the BEC-cavity system is characterized by the nonlinearity that the optical lattice acting on the condensate depends on the condensate itself, we treat the lattice potential as a dynamical variable instead of a static quantity or simply a time-dependent parameter.

For analyzing the BEC-cavity system with a constant pump acting on it and atom loss taking part in it, we apply the trial and error method to find out steady states and use the DMA methods to study the evolution of the system. Numerical simulation predicts a bi-stability switch of the system for experiments to verify, which can be considered as a proof of the utility of the DMA method.

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