Novel stability criteria for fuzzy Hopfield neural networks based on an improved homogeneous matrix polynomials technique*

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The global stability problem of Takagi–Sugeno (T–S) fuzzy Hopfield neural networks (FHNNs) with time delays is investigated. Novel LMI-based stability criteria are obtained by using Lyapunov functional theory to guarantee the asymptotic stability of the FHNNs with less conservatism. Firstly, using both Finsler's lemma and an improved homogeneous matrix polynomial technique, and applying an affine parameter-dependent Lyapunov–Krasovskii functional, we obtain the convergent LMI-based stability criteria. Algebraic properties of the fuzzy membership functions in the unit simplex are considered in the process of stability analysis via the homogeneous matrix polynomials technique. Secondly, to further reduce the conservatism, a new right-hand-side slack variables introducing technique is also proposed in terms of LMIs, which is suitable to the homogeneous matrix polynomials setting. Finally, two illustrative examples are given to show the efficiency of the proposed approaches.

Keywords: Hopfield neural networks, linear matrix inequality, Takagi–Sugeno fuzzy model, homogeneous polynomially technique

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1. Introduction

Over the past decades, Hopfield neural networks (HNNs)[1] have been extensively studied due to their important applications in various fields such as combinatorial optimization, signal processing, image processing, and pattern recognition problems.[2,3] Indeed, these applications are built upon the stability of the equilibrium of neural networks. Thus, stability analysis is a necessary step for the design and applications of neural networks. Recently, it has been realized that the interactions between neurons are generally asynchronous, which inevitably result in time delays for both biological and artificial neural networks. Therefore, there exist amounts of stability results for various neural network with several kinds of time delays.[4−11]

On the other hand, fuzzy logic theory has shown to be an appealing and efficient approach to dealing with the analysis and synthesis problems for complex nonlinear systems. In Ref. [12], Takagi and Sugeno proposed an effective way to transform a nonlinear dynamic system to a set of linear sub-models via some fuzzy models by defining a linear input–output relationship as its consequence of individual plant rule. More importantly, it has been proved by the Stone–Weierstrass theorem that the fuzzy basis functions are capable of uniformly approximating to any real continuous function on a compact set to arbitrary accuracy.[13] This allows designers to take advantage of conventional linear systems to analyze the nonlinear systems based on the Takagi–Sugeno (T–S) fuzzy model.[14−17]

More recently, the standard T–S fuzzy model has been used for stability analysis of various kinds of nonlinear neural networks with time delays and some stability conditions have been presented in terms of linear matrix inequalities (LMIs). The problem of exponential stability for T–S fuzzy model, in which the consequent parts are composed of a set of stochastic HNNs with time-varying delays, has been addressed in Ref. [18], and the global asymptotic stability problem of fuzzy BAM neural networks with time-varying delays and parameter uncertainties has been investigated by means of the T–S fuzzy modeling approach

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in Ref. [19]. A new global stability criterion for T–S fuzzy Hopfield neural networks with time delays has also been provided in Ref. [20] by using a generalized Lyapunov functional and introducing a parameterized model transformation with free weighting matrices. Moreover, robust stability for uncertain delayed fuzzy Hopfield neural networks with markovian jumping parameters has also been investigated in Ref. [21] by considering both the lower and the upper values of the interval time delay. Delay-dependent robust exponential state estimation of Markovian jumping fuzzy Hopfield neural networks with mixed random time-varying delays and mean square exponential stability of stochastic fuzzy Hopfield neural networks with discrete and distributed time-varying delays have also been investigated in Refs. [22] and [23], respectively. Robust stability analysis of T–S uncertain stochastic fuzzy recurrent neural networks with mixed time-varying delays has been studied in Ref. [24]. More recently, novel delay-dependent stability conditions have been proposed in dealing with stability analysis of T–S fuzzy uncertain neural networks with time varying delays.[25] To the best of our knowledge, the common Lyapunov–Krasovskii functional (one Lyapunov matrix $\boldsymbol{P}$ for the overall fuzzy space) and conventional relaxed techniques were applied in the proof of Refs. [18]–[25] which tends to produce much conservatism.

In this paper, novel stability criteria for T–S fuzzy Hopfield neural networks with time delays are proposed through four kinds of new relaxed techniques: 1) A new affine parameter-dependent Lyapunov–Krasovskii functional is proposed to reduce the conservatism while the property of the membership functions is considered by introducing some free matrices. 2) Additional variables are introduced by using the Finsler's lemma to enlarge the design space to facilitate converting some nonlinear matrix inequality into finite sets of LMIs, and the difference equations that govern the behavior of the system are incorporated into the stability analysis conditions through the use of matrix Lagrange multipliers. 3) Convergent LMI-based stability criteria are derived by applying the generalization of Polya's theorem for the case of positive polynomials with matrix-valued coefficients. 4) A new right-hand-side slack variables introducing technique is also given to accelerate the convergence rate referred in kind 3). In summary, the conservatism is reduced by considering the information of fuzzy membership functions in the process of stability analysis.

Notations Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; $\boldsymbol{I}$ is the identity matrix with appropriate dimensions; "T" stands for the transpose of a matrix; for symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} > \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$) means that $\boldsymbol{X}$–$\boldsymbol{Y}$ is positive-definite (respectively, positive semi-definite); $\mathbb{Z}_+$ denotes the set of nonnegative integers $\{0, 1, 2, \ldots\}$ and $M! = M(M-1)(M-2)\ldots(2)(1)$ for $M \in \mathbb{Z}_+$; $\Delta_N = \{\alpha \in \mathbb{R}^N; \sum_{i=1}^N \alpha_i = 1; \alpha_i \geq 0\}$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2. System description and preliminaries

2.1. T–S fuzzy Hopfield neural networks with delays

Consider a class of nonlinear Hopfield neural networks (HNNs) with time delays described as follows:

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = -\boldsymbol{A} \times \boldsymbol{x}(t) + \boldsymbol{W} \times \boldsymbol{f}(\boldsymbol{x}(t-\tau)), \qquad (1)\\ &\boldsymbol{A} = \mathrm{diag}\{a_1(x_1(t)), a_2(x_2(t)), \ldots, a_n(x_n(t))\},\\ &\boldsymbol{W} = [w_{ij}(x(t))]_{n\times n}, \end{aligned}$$

where $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector, τ is the unknown time delay. $a_i(x_i(t)) \in \mathbb{R}$ and $w_{ij}(x(t)) \in \mathbb{R}$ are nonlinear functions about the state vector. $\boldsymbol{f}(\boldsymbol{x}(t)) = [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^{\mathrm{T}} \in \mathbb{R}^n$ is the neuron activation function with $f(0) = 0$. The initial condition associated with model (1) is $x_i(s) = \varphi_i(s), (i = 1, 2, \ldots, n)$, where $\varphi_i(s)$ is bounded and continuously differential on $[-\tau, 0]$.

The neuron activation functions $f_i(x_i(t))$ $(i = 1, 2, \ldots, n)$ are assumed to be nondecreasing, bounded and globally Lipschitz; that is,

Assumption A

$$0 \leq \frac{f_i(a) - f_i(b)}{a - b} \leq L_i, \quad i = 1, 2, \ldots, n, \qquad (2)$$

for all $a, b \in \mathbb{R}, a \neq b$ and denotes $\boldsymbol{L} = \mathrm{diag}\{L_1, L_2, \ldots, L_n\}$. It should be pointed out that Assumption A guarantees there is an equilibrium point for nonlinear HNNs (1). This can be easily verified by employing the well-known Brouwer's fixed point theorem.[18–25]

Unlike the usual linear HNNs investigated in Ref. [1], there exist some nonlinear terms $a_i(x_i(t))$

and $w_{ij}(x(t))$ in the system matrices $\boldsymbol{A}$ and $\boldsymbol{W}$ of the above nonlinear HNNs (1). Therefore, those results for stability analysis of the usual HNNs fails to work in this case. By generalizing the T–S fuzzy models approach to represent the complex nonlinear HNNs (1) as a set of sub-HNNs with constant system matrices,[18−25] these nonlinear terms $a_i(x_i(t))$ and $w_{ij}(x(t))$ are approximated by a series of IF-THEN fuzzy plant rules. Based on the T–S fuzzy model concept about nonlinear HNNs,[18−25] the complex nonlinear HNNs (1) could be represented by a set of sub-HNNs as follows:

Plant Rule i

IF $\theta_1(t)$ is M_{i1}, and ... and $\theta_m(t)$ is M_{im}, THEN

$\dot{\boldsymbol{x}}(t) = -\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{W}_i\boldsymbol{f}(\boldsymbol{x}(t-\tau)),\ \ i = 1, 2, \ldots, r$

where $\theta_j(t), (j = 1, 2, \ldots, m)$ are the fuzzy premise variables and $\boldsymbol{\theta}(t) = [\theta_1(t), \ldots, \theta_m(t)]^{\mathrm{T}}$. r is the number of model rules. M_{il} $(i \in \{1, 2, \ldots, r\}, l \in \{1, 2, \ldots, m\})$ are the fuzzy sets, and M_{il} $(i \in \{1, 2, \ldots, r\}, l \in \{1, 2, \ldots, m\})$ are usually nonnegative functions about $\theta(t)$. And $\boldsymbol{A}_i \in \mathbb{R}^{n\times n}$ is a diagonal system matrix for the i-th Plant Rule. $\boldsymbol{W}_i \in \mathbb{R}^{n\times n}$ is the connection weight matrix for the i-th Plant Rule.

Remark 1 It is worth noting that the neuron activation function $\boldsymbol{f}(\boldsymbol{x}(t))$ plays an important role in the dynamic characteristic of HNNs. Even for the original HNNs proposed in Ref. [1], the neuron activation function $\boldsymbol{f}(\boldsymbol{x}(t))$ is retained in its system dynamic equation. Similar to the purpose of the existing literature, e.g., Refs. [18]–[25], the T–S fuzzy models approach given in Ref. [12] are generalized for only representing the HNNs (1)'s nonlinear terms $a_i(x_i(t))$ and $w_{ij}(\boldsymbol{x}(t))$ by a set of sub-models with constant matrices, not for the neuron activation function $\boldsymbol{f}(\boldsymbol{x}(t))$.

By using the product of inference, singleton fuzzifier, and center-average defuzzifier, the overall output of the T–S fuzzy HNNs is obtained by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} \frac{w_i(\boldsymbol{\theta}(t))\left(-\boldsymbol{A}_i x(t) + \boldsymbol{W}_i\boldsymbol{f}(\boldsymbol{x}(t-\tau))\right)}{\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))} \\ &= \sum_{i=1}^{r} h_i(\boldsymbol{\theta}(t))\left(-\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{W}_i\boldsymbol{f}(x(t-\tau))\right), \quad (3)\end{aligned}$$

where the weighted functions of each fuzzy rule is denoted by $w_i(\boldsymbol{\theta}(t)) = \prod_{l=1}^{m} M_{il}(\boldsymbol{\theta}(t))$, and the averaged weight of each fuzzy rule is denoted by $h_i(\boldsymbol{\theta}(t)) = w_i(\boldsymbol{\theta}(t))/\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))$.

Since $M_{il}(i \in \{1, 2, \ldots, r\}, l \in \{1, 2, \ldots, m\})$ are usually designed as non-negative functions about $\boldsymbol{\theta}(t)$,[10] i.e., $M_{il}(\boldsymbol{\theta}(t)) \geq 0$ for all $\boldsymbol{\theta}(t)$, we have the fact that $w_i(\boldsymbol{\theta}(t)) = \prod_{l=1}^{m} M_{il}(\boldsymbol{\theta}(t)) \geq 0$ for all $\boldsymbol{\theta}(t)$ and

$$h_i(\boldsymbol{\theta}(t)) = \frac{w_i(\boldsymbol{\theta}(t))}{\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))} \geq 0$$

for all $\boldsymbol{\theta}(t)$. Furthermore, it is easy to see that

$$\begin{aligned}\sum_{i=1}^{r} h_i(\boldsymbol{\theta}(t)) = &\frac{w_1(\boldsymbol{\theta}(t))}{\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))} + \frac{w_2(\boldsymbol{\theta}(t))}{\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))} + \cdots \\ &+ \frac{w_r(\boldsymbol{\theta}(t))}{\sum_{j=1}^{r} w_j(\boldsymbol{\theta}(t))} = 1.\end{aligned}$$

Since the value of $\sum_{i=1}^{r} h_i(\boldsymbol{\theta}(t))$ is a constant number, the derivative of $\sum_{i=1}^{r} h_i(\boldsymbol{\theta}(t))$ is equal to 0, i.e., $\sum_{i=1}^{r} \dot{h}_i(\boldsymbol{\theta}(t)) = 0$. Therefore, we could assume the following sequel

$$h_i(\boldsymbol{\theta}(t)) \geq 0, \quad \sum_{i=1}^{r} h_i(\boldsymbol{\theta}(t)) = 1, \quad \sum_{i=1}^{r} \dot{h}_i(\boldsymbol{\theta}(t)) = 0. \quad (4)$$

2.2. Homogeneous matrix polynomials

The following definitions about homogeneous matrix polynomials are needed, which are consistent with those in Ref. [26].

A homogeneous matrix polynomial $\boldsymbol{P}(\boldsymbol{\alpha})$ of degree g can be generally written as

$$\begin{aligned}&\boldsymbol{P}(\boldsymbol{\alpha}) = \sum_{k\in\mathscr{H}(g)} \boldsymbol{\alpha}^k \boldsymbol{P}_k, \\ &\boldsymbol{\alpha}^k = \alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_N^{k_N}, k = k_1k_2\ldots k_N, \quad (5)\end{aligned}$$

where $\alpha_1^{k_1}\alpha_2^{k_2}\ldots\alpha_N^{k_N}, \alpha \in \triangle_N, k_i \in Z_+, i = 1, 2, \ldots, N$ are the monomials, and $\boldsymbol{P}_k \in \mathbb{R}^{n\times n}, \forall k \in \mathcal{K}(g)$ are matrix-valued coefficients. Here, by definitions, $\mathcal{K}(g)$ is the set of N-tuples obtained as all possible combinations of nonnegative integers $k_i, i = 1, 2, \ldots, N$, such that $k_1 + k_2 + \cdots + k_N = g$. Since the number of vertices is equal to N, the number of elements in $\mathcal{K}(g)$ is given by

$$J(g) = \frac{(N+g-1)!}{g!(N-1)!}. \quad (6)$$

To give an example, for homogeneous polynomials of degree $g = 4$ with $N = 2$ variables, the possible values of the partial degrees are $\mathcal{K}(4) = 04, 13, 22, 31, 40$, $J(4) = 5$, corresponding to the generic from $\boldsymbol{P}_4(\alpha) = \alpha_2^4\boldsymbol{P}_{04} + \alpha_1\alpha_2^3\boldsymbol{P}_{13} + \alpha_1^2\alpha_2^2\boldsymbol{P}_{22} + \alpha_1^3\alpha_2\boldsymbol{P}_{31} + \alpha_1^4\boldsymbol{P}_{40}$.

Consider the modified multinomial coefficients:

$$\mathscr{C}_i^l(d,a)=\begin{cases}\dfrac{d!}{k_1!\cdots(k_i-a)!\cdots k_r!}, & \text{if } k_i-a\in\mathbb{N},\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathscr{C}_{ij}^l(d,a,b)=\begin{cases}\dfrac{d!}{k_1!\cdots(k_i-a)!\cdots(k_j-b)!\cdots k_r!}, & \text{if } k_i-a\in\mathbb{N},\ k_j-b\in\mathbb{N},\\ 0, & \text{otherwise,}\end{cases}$$

all of which depend on $k_1k_2\cdots k_r=\mathcal{K}_l(d), l=1,\ldots,J(d)$, where $\mathcal{K}_l(d)$ is the l-th N-tuple of $\mathcal{K}(d)$ given by $J(d)$.

We end this section with a useful lemma.

Lemma 1 (Polya's Theorem[27]) Let $\boldsymbol{F}(\xi)\doteq\boldsymbol{F}(\alpha_1,\alpha_1,\ldots,\alpha_N)$ be a real homogenous polynomial which is positive $\forall\boldsymbol{\alpha}\in\Delta_N$. Then for a sufficiently large $d\in\mathbb{Z}_+$, the product $(\alpha_1+\alpha_2+\cdots+\alpha_N)^d\boldsymbol{F}(\boldsymbol{\alpha})$ has all its coefficients strictly positive, where Δ_N is defined as follows:

$$\Delta_N=\left\{\boldsymbol{\alpha}\in\mathbb{R}^N;\sum_{i=1}^N\alpha_i=1;\alpha_i\geq 0\right\}. \tag{7}$$

3. Main results

In this section, less conservative stability conditions for the underlying fuzzy HNNs (3) will be proposed by applying both an affine parameter-dependent Lyapunov–Krasovskii functional and a generalization of the Polya's theorem for the case of positive polynomials with matrix-valued coefficients. For simplicity, we use h_i instead of $h_i(\boldsymbol{\theta}(t))$ in the following sections.

Theorem 1 Under the Assumption A and $|\dot{h}_i|\leq\phi_i$ $(i=1,\ldots,r)$, the equilibrium point of system (3) is globally asymptotically stable if there exist symmetric positive matrices $\boldsymbol{P}_i>0,\boldsymbol{Q}>0,\boldsymbol{R}>0$, diagonal matrices $\boldsymbol{X}>0,\boldsymbol{Y}>0$, symmetric matrices $\boldsymbol{M}$, appropriately dimensional matrices $\boldsymbol{W}_{mn}^i$ $(m=1,\ldots,6,n=1,2)$, and $d\in\mathbb{Z}_+$, such that the following LMIs hold

$$\boldsymbol{P}_i+\boldsymbol{M}\geq 0,\quad i=1,2,\ldots,r; \tag{8}$$

$$\Gamma_l=\sum_{i=1}^r\mathscr{C}_i^l(d,2)T_i+\sum_{i=1}^{r-1}\sum_{j-i+1}^r\mathscr{C}_{ij}^l(d,1,1)T_{ij}<0, \tag{9}$$

$$k_1k_2\cdots k_r=\mathcal{K}_l(d+2),\quad l=1,2,\ldots,J(d+2),$$

where

$$T_i=\begin{bmatrix}T_i^{11}&T_i^{12}&T_i^{13}&T_i^{14}&T_i^{15}&T_i^{16}\\ *&T_i^{22}&T_i^{23}&T_i^{24}&T_i^{25}&T_i^{26}\\ *&*&Q-2X&T_i^{34}&T_i^{35}&T_i^{36}\\ *&*&*&T_i^{44}&T_i^{45}&T_i^{46}\\ *&*&*&*&T_i^{55}&T_i^{56}\\ *&*&*&*&*&T_i^{66}\end{bmatrix},$$

$$T_{ij}=\begin{bmatrix}T_{ij}^{11}&T_{ij}^{12}&T_{ij}^{13}&T_{ij}^{14}&T_{ij}^{15}&T_{ij}^{16}\\ *&T_{ij}^{22}&T_{ij}^{23}&T_{ij}^{24}&T_{ij}^{25}&T_{ij}^{26}\\ *&*&2Q-4X&T_{ij}^{34}&T_{ij}^{35}&T_{ij}^{36}\\ *&*&*&T_{ij}^{44}&T_{ij}^{45}&T_{ij}^{46}\\ *&*&*&*&T_{ij}^{55}&T_{ij}^{56}\\ *&*&*&*&*&T_{ij}^{66}\end{bmatrix},$$

$$\begin{aligned}
T_i^{11}&=\sum_{i=1}^r\phi_i(\boldsymbol{P}_i+\boldsymbol{M})-\boldsymbol{W}_{11}^i\boldsymbol{A}_i\\
&\quad+\boldsymbol{W}_{12}^i+(-\boldsymbol{W}_{11}^i\boldsymbol{A}_i+\boldsymbol{W}_{12}^i)^{\mathrm{T}},\\
T_i^{12}&=-\boldsymbol{W}_{12}^i+(-\boldsymbol{W}_{21}^i\boldsymbol{A}_i+\boldsymbol{W}_{22}^i)^{\mathrm{T}},\\
T_i^{13}&=\boldsymbol{X}\boldsymbol{L}+(-\boldsymbol{W}_{31}^i\boldsymbol{A}_i+\boldsymbol{W}_{32}^i)^{\mathrm{T}},\\
T_i^{14}&=\boldsymbol{W}_{11}^i\boldsymbol{W}_i+(-\boldsymbol{W}_{41}^i\boldsymbol{A}_i+\boldsymbol{W}_{42}^i)^{\mathrm{T}},\\
T_i^{15}&=\boldsymbol{P}_i-\boldsymbol{W}_{11}^i+(-\boldsymbol{W}_{51}^i\boldsymbol{A}_i+\boldsymbol{W}_{52}^i)^{\mathrm{T}},\\
T_i^{16}&=-\boldsymbol{W}_{12}^i+(-\boldsymbol{W}_{61}^i\boldsymbol{A}_i+\boldsymbol{W}_{62}^i)^{\mathrm{T}},\\
T_i^{22}&=-\boldsymbol{W}_{22}^i-\boldsymbol{W}_{22}^{i\,\mathrm{T}},\quad T_i^{23}=(-\boldsymbol{W}_{32}^i)^{\mathrm{T}},\\
T_i^{24}&=\boldsymbol{Y}\boldsymbol{L}+\boldsymbol{W}_{21}^i\boldsymbol{W}_i+(-\boldsymbol{W}_{42}^i)^{\mathrm{T}},\\
T_i^{25}&=-\boldsymbol{W}_{21}^i+(-\boldsymbol{W}_{52}^i)^{\mathrm{T}},\\
T_i^{26}&=-\boldsymbol{W}_{22}^i+(-\boldsymbol{W}_{62}^i)^{\mathrm{T}},\\
T_i^{34}&=\boldsymbol{W}_{31}^i\boldsymbol{W}_i,\quad T_i^{35}=-\boldsymbol{W}_{31}^i,\quad T_i^{36}=-\boldsymbol{W}_{32}^i,\\
T_i^{44}&=-\boldsymbol{Q}-2\boldsymbol{Y}+\boldsymbol{W}_{41}^i\boldsymbol{W}_i+(\boldsymbol{W}_{41}^i\boldsymbol{W}_i)^{\mathrm{T}},\\
T_i^{45}&=-\boldsymbol{W}_{41}^i+(\boldsymbol{W}_{51}^i\boldsymbol{W}_i)^{\mathrm{T}},\\
T_i^{46}&=-\boldsymbol{W}_{42}^i+(\boldsymbol{W}_{61}^i\boldsymbol{W}_i)^{\mathrm{T}},\\
T_i^{55}&=\tau^2\boldsymbol{R}-\boldsymbol{W}_{51}^i-(\boldsymbol{W}_{51}^i)^{\mathrm{T}},\\
T_i^{56}&=-\boldsymbol{W}_{52}^i-(\boldsymbol{W}_{61}^i)^{\mathrm{T}},\\
T_i^{66}&=-\boldsymbol{R}-\boldsymbol{W}_{62}^i-(\boldsymbol{W}_{62}^i)^{\mathrm{T}},\\
T_{ij}^{11}&==\sum_{i=1}^r2\phi_i(\boldsymbol{P}_i+\boldsymbol{M})-\boldsymbol{W}_{11}^j\boldsymbol{A}_i
\end{aligned}$$

$$
\begin{aligned}
&+\boldsymbol{W}_{12}^{j}+(-\boldsymbol{W}_{11}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{12}^{j})^{\mathrm{T}}\\
&-\boldsymbol{W}_{11}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{12}^{i}+(-\boldsymbol{W}_{11}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{12}^{i})^{\mathrm{T}},\\
T_{ij}^{12} &= -\boldsymbol{W}_{12}^{j}+(-\boldsymbol{W}_{21}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{22}^{j})^{\mathrm{T}}\\
&-\boldsymbol{W}_{12}^{i}+(-\boldsymbol{W}_{21}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{22}^{i})^{\mathrm{T}},\\
T_{ij}^{13} &= 2\boldsymbol{X}\boldsymbol{L}+(-\boldsymbol{W}_{31}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{32}^{j})^{\mathrm{T}}\\
&+(-\boldsymbol{W}_{31}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{32}^{i})^{\mathrm{T}},\\
T_{ij}^{14} &= \boldsymbol{W}_{11}^{j}\boldsymbol{W}_i+(-\boldsymbol{W}_{41}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{42}^{j})^{\mathrm{T}}\\
&+\boldsymbol{W}_{11}^{i}\boldsymbol{W}_j+(-\boldsymbol{W}_{41}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{42}^{i})^{\mathrm{T}},\\
T_{i}^{15} &= \boldsymbol{P}_i-\boldsymbol{W}_{11}^{j}+(-\boldsymbol{W}_{51}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{52}^{j})^{\mathrm{T}}\\
&+\boldsymbol{P}_j-\boldsymbol{W}_{11}^{i}+(-\boldsymbol{W}_{51}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{52}^{i})^{\mathrm{T}},\\
T_{ij}^{16} &= -\boldsymbol{W}_{12}^{j}+(-\boldsymbol{W}_{61}^{j}\boldsymbol{A}_i+\boldsymbol{W}_{62}^{j})^{\mathrm{T}}\\
&-\boldsymbol{W}_{12}^{i}+(-\boldsymbol{W}_{61}^{i}\boldsymbol{A}_j+\boldsymbol{W}_{62}^{i})^{\mathrm{T}},\\
T_{ij}^{22} &= -\boldsymbol{W}_{22}^{i}-\boldsymbol{W}_{22}^{i\ \mathrm{T}}-\boldsymbol{W}_{22}^{j}-\boldsymbol{W}_{22}^{j\ \mathrm{T}},\\
T_{ij}^{23} &= (-\boldsymbol{W}_{32}^{i})^{\mathrm{T}}+(-\boldsymbol{W}_{32}^{j})^{\mathrm{T}},\\
T_{ij}^{24} &= 2\boldsymbol{Y}\boldsymbol{L}+\boldsymbol{W}_{21}^{j}\boldsymbol{W}_i+(-\boldsymbol{W}_{42}^{j})^{\mathrm{T}}\\
&+\boldsymbol{W}_{21}^{i}\boldsymbol{W}_j+(-\boldsymbol{W}_{42}^{i})^{\mathrm{T}},\\
T_{ij}^{25} &= -\boldsymbol{W}_{21}^{j}+(-\boldsymbol{W}_{52}^{j})^{\mathrm{T}}-\boldsymbol{W}_{21}^{i}+(-\boldsymbol{W}_{52}^{i})^{\mathrm{T}},\\
T_{ij}^{26} &= -\boldsymbol{W}_{22}^{i}+(-\boldsymbol{W}_{62}^{i})^{\mathrm{T}}-\boldsymbol{W}_{22}^{j}+(-\boldsymbol{W}_{62}^{j})^{\mathrm{T}},\\
T_{ij}^{34} &= \boldsymbol{W}_{31}^{j}\boldsymbol{W}_i+\boldsymbol{W}_{31}^{i}\boldsymbol{W}_j,\\
T_{ij}^{35} &= -\boldsymbol{W}_{31}^{i}-\boldsymbol{W}_{31}^{j},\quad T_{ij}^{36}=-\boldsymbol{W}_{32}^{i}-\boldsymbol{W}_{32}^{j},\\
T_{ij}^{44} &= -2\boldsymbol{Q}-4\boldsymbol{Y}+\boldsymbol{W}_{41}^{j}\boldsymbol{W}_i+(\boldsymbol{W}_{41}^{j}\boldsymbol{W}_i)^{\mathrm{T}}\\
&+\boldsymbol{W}_{41}^{i}\boldsymbol{W}_j+(\boldsymbol{W}_{41}^{i}\boldsymbol{W}_j)^{\mathrm{T}},\\
T_{ij}^{45} &= -\boldsymbol{W}_{41}^{j}+(\boldsymbol{W}_{51}^{j}\boldsymbol{W}_i)^{\mathrm{T}}-\boldsymbol{W}_{41}^{i}+(\boldsymbol{W}_{51}^{i}\boldsymbol{W}_j)^{\mathrm{T}},\\
T_{ij}^{46} &= -\boldsymbol{W}_{42}^{j}+(\boldsymbol{W}_{61}^{j}\boldsymbol{W}_i)^{\mathrm{T}}-\boldsymbol{W}_{42}^{i}+(\boldsymbol{W}_{61}^{i}\boldsymbol{W}_j)^{\mathrm{T}},\\
T_{ij}^{55} &= 2\tau^2\boldsymbol{R}-\boldsymbol{W}_{51}^{i}-(\boldsymbol{W}_{51}^{i})^{\mathrm{T}}-\boldsymbol{W}_{51}^{j}-(\boldsymbol{W}_{51}^{j})^{\mathrm{T}},\\
T_{ij}^{56} &= -\boldsymbol{W}_{52}^{i}-(\boldsymbol{W}_{61}^{i})^{\mathrm{T}}-\boldsymbol{W}_{52}^{j}-(\boldsymbol{W}_{61}^{j})^{\mathrm{T}},\\
T_{ij}^{66} &= -2\boldsymbol{R}-\boldsymbol{W}_{62}^{i}-(\boldsymbol{W}_{62}^{i})^{\mathrm{T}}-\boldsymbol{W}_{62}^{j}-(\boldsymbol{W}_{62}^{j})^{\mathrm{T}}.
\end{aligned}
$$

Moreover, the conservatism will further be reduced as the value of d increases.

Proof Choosing an affine parameter-dependent Lyapunov–Krasovskii functional as follows:

$$V(t)=V_1(t)+V_2(t)+V_3(t), \tag{10}$$

where

$$
\begin{aligned}
V_1(t) &= \boldsymbol{x}(t)^{\mathrm{T}}\left(\sum_{i=1}^{r}h_i\boldsymbol{P}_i\right)\boldsymbol{x}(t),\\
V_2(t) &= \int_{t-\tau}^{t}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(s))\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{x}(s))\,\mathrm{d}s,\\
V_3(t) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\,\mathrm{d}\theta.
\end{aligned}
$$

Calculating the derivative of $V(t)$ along the trajectories of system (3), we have

$$
\begin{aligned}
\dot{V}(t) &= 2\boldsymbol{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{r}h_i\boldsymbol{P}_i\right)\dot{\boldsymbol{x}}(t)\\
&+\boldsymbol{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{r}\dot{h}_i\boldsymbol{P}_i\right)\boldsymbol{x}(t)\\
&+\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{x}(t))\\
&-\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{x}(t-\tau))\\
&+\tau^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\boldsymbol{x}}(t)\\
&-\tau\int_{t-\tau}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s.
\end{aligned}
\tag{11}
$$

Based on expression (4), it follows that $\sum_{i=1}^{r}\dot{h}_i\boldsymbol{M}=\bar{\boldsymbol{M}}=0$, where $\boldsymbol{M}$ is any symmetric matrix of proper dimension. By adding $\boldsymbol{x}^{\mathrm{T}}(t)\bar{\boldsymbol{M}}\boldsymbol{x}(t)$ to the right-hand side of Eq. (11) and recalling the assumption that $|\dot{h}_i|\leq\phi_i$ and inequality (8), the following inequality holds

$$
\begin{aligned}
\dot{V}(t) &\leq 2\boldsymbol{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{r}h_i\boldsymbol{P}_i\right)\dot{\boldsymbol{x}}(t)\\
&+\boldsymbol{x}^{\mathrm{T}}(t)\left(\sum_{i=1}^{r}\phi_i(\boldsymbol{P}_i+\boldsymbol{M})\right)\boldsymbol{x}(t)\\
&+\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{x}(t))\\
&-\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{x}(t-\tau))\\
&+\tau^2\dot{\boldsymbol{x}}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\boldsymbol{x}}(t)\\
&-\tau\int_{t-\tau}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s.
\end{aligned}
\tag{12}
$$

Using the Jensen's inequality,[28] we have

$$
\begin{aligned}
&-\tau\int_{t-\tau}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\\
&\leq-\left[\int_{t-\tau}^{t}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\right]^{\mathrm{T}}\boldsymbol{R}\left[\int_{t-\tau}^{t}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s\right].
\end{aligned}
\tag{13}
$$

From the two diagonal positive-definite matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ and using Assumption A, we have

$$2\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{X}\boldsymbol{L}\boldsymbol{x}(t)-2\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t))\boldsymbol{X}\boldsymbol{f}(\boldsymbol{x}(t))\geq 0, \tag{14}$$

$$
\begin{aligned}
&2\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{Y}\boldsymbol{L}\boldsymbol{x}(t-\tau)\\
&-2\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-\tau))\boldsymbol{Y}\boldsymbol{f}(\boldsymbol{x}(t-\tau))\geq 0,
\end{aligned}
\tag{15}
$$

From inequalities (12)–(15), we can obtain

$$\dot{V}(t)\leq\boldsymbol{\xi}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\xi}(t), \tag{16}$$

where $\boldsymbol{\xi}(t)=[\boldsymbol{x}^{\mathrm{T}}(t),\boldsymbol{x}^{\mathrm{T}}(t-\tau),\boldsymbol{f}^{\mathrm{T}}(x(t)),\boldsymbol{f}^{\mathrm{T}}(x(t-\tau)),\dot{\boldsymbol{x}}^{\mathrm{T}}(t),(\int_{t-\tau}^{t}\dot{\boldsymbol{x}}(s)\,\mathrm{d}s)^{\mathrm{T}}]^{\mathrm{T}}$ and

$$\boldsymbol{Q}=\begin{bmatrix} \sum_{i=1}^{r}\phi_i(\boldsymbol{P}_i+\boldsymbol{M}) & 0 & \boldsymbol{X}\boldsymbol{L} & 0 & \sum_{i=1}^{r}h_i\boldsymbol{P}_i & 0 \\ * & 0 & 0 & \boldsymbol{Y}\boldsymbol{L} & 0 & 0 \\ * & * & \boldsymbol{Q}-2\boldsymbol{X} & 0 & 0 & 0 \\ * & * & * & -\boldsymbol{Q}-2\boldsymbol{Y} & 0 & 0 \\ * & * & * & * & \tau^2\boldsymbol{R} & 0 \\ * & * & * & * & * & -\boldsymbol{R} \end{bmatrix}.$$

On the other hand, it is easy to see that the following equality holds

$$\boldsymbol{B}\boldsymbol{\xi}(t)=0, \tag{17}$$

where

$$\boldsymbol{B}=\begin{bmatrix} -\sum_{i=1}^{r}h_i\boldsymbol{A}_i & 0 & 0 & \sum_{i=1}^{r}h_i\boldsymbol{W}_i & -I & 0 \\ \boldsymbol{I} & -\boldsymbol{I} & 0 & 0 & 0 & -\boldsymbol{I} \end{bmatrix}.$$

Thus, by using the well-known Finsler lemma proposed in Ref. [29], $\boldsymbol{\xi}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{\xi}(t)<0$ with $\boldsymbol{B}\boldsymbol{\xi}(t)=0$ is equivalent to the following inequality

$$\Gamma(h)=\boldsymbol{Q}+\boldsymbol{W}\boldsymbol{B}+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}<0, \tag{18}$$

where $\boldsymbol{W}$ is an additional matrix variable which is affinely-dependent on the membership functions h_i and has the following structure

$$\boldsymbol{W}=\begin{bmatrix} \boldsymbol{W}_{11}^{\mathrm{T}}(t) & \boldsymbol{W}_{21}^{\mathrm{T}}(t) & \boldsymbol{W}_{31}^{\mathrm{T}}(t) & \boldsymbol{W}_{41}^{\mathrm{T}}(t) & \boldsymbol{W}_{51}^{\mathrm{T}}(t) & \boldsymbol{W}_{61}^{\mathrm{T}}(t) \\ \boldsymbol{W}_{12}^{\mathrm{T}}(t) & \boldsymbol{W}_{22}^{\mathrm{T}}(t) & \boldsymbol{W}_{32}^{\mathrm{T}}(t) & \boldsymbol{W}_{42}^{\mathrm{T}}(t) & \boldsymbol{W}_{52}^{\mathrm{T}}(t) & \boldsymbol{W}_{62}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}},$$

and $\boldsymbol{W}_{mn}(t)=\sum_{i=1}^{r}h_i\boldsymbol{W}_{mn}^{i},(m=1,\ldots,6;n=1,2)$.

Using the fact $\sum_{i=1}^{r}h_i=1$, Eq. (18) can be rewritten as

$$\Gamma(h)=\left(\sum_{i=1}^{r}h_i\right)^{d}(\boldsymbol{Q}+\boldsymbol{W}\boldsymbol{B}+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}})<0, \tag{19}$$

where $d\in\mathbb{Z}_+$ can be arbitrarily selected.

Reordering the expression of $\Gamma(h)$, we have

$$\begin{aligned}\Gamma(h)&=\left(\sum_{i=1}^{r}h_i\right)^{d}(\boldsymbol{Q}+\boldsymbol{W}\boldsymbol{B}+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}})\\&=\sum_{l=1}^{J(d+2)}h_1^{k_1}h_2^{k_2}\cdots h_r^{k_r}\left(\sum_{i=1}^{r}\mathscr{C}_i^{l}(d,2)T_i+\sum_{i=1}^{r-1}\sum_{j-i+1}^{r}\mathscr{C}_{ij}^{l}(d,1,1)T_{ij}\right)\\&=\sum_{l=1}^{J(d+2)}h_1^{k_1}h_2^{k_2}\cdots h_r^{k_r}\Gamma_l,\\&\quad k_1k_2\cdots k_r=\mathcal{K}_l(d+2),\\&\quad l=1,2,\ldots,J(d+2),\end{aligned} \tag{20}$$

where T_i and T_{ij} have been defined before, and

$$\Gamma_l=\sum_{i=1}^{r}\mathscr{C}_i^{l}(d,2)T_i+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\mathscr{C}_{ij}^{l}(d,1,1)T_{ij}.$$

Using the Lemma 1, we have

$$\Gamma(h)<0\Leftrightarrow\Gamma_l<0,\ \exists\ \text{a sufficient large } d\in\mathbb{Z}_+. \tag{21}$$

From expressions (19)–(21), we can obtain that the equilibrium point of system (3) is globally asymptotically stable if we have $\Gamma_l<0$ for all $l=1,2,\ldots,J(d+2)$. On the other hand, the LMIs (8) and (9) could guarantee that the equilibrium point of system (3) is globally asymptotically stable. Moreover, from expression (21), the conservatism will further be reduced as the value of d increases, and LMI-based stability conditions (8) and (9) will become sufficient and necessary conditions for ensuring $\Gamma(h)<0$ when d approaches a sufficient large value. This completes the proof.

Remark 2 In Theorem 1, novel stability criteria for the underlying fuzzy HNNs (3) is proposed by using both the Finsler's lemma and the Polya's The-

orem. The differential equations that govern the behavior of the system are seen as dynamic constraints. These dynamic constraints, which naturally involve the state derivative, are incorporated into the stability analysis condition through the use of matrix Lagrange multipliers. As the value of d increases, the conservatism will be further reduced and convergent LMI-based stability criteria will also be attained while the computational burden is increasing as a trade-off. Moreover, to accelerate the convergence rate, i.e., attaining the same or less conservative stability conditions with smaller d, new relaxed stability conditions will also be given by using a kind of novel right-hand-side slack variable introducing technique in Theorem 2.

Theorem 2 Under the Assumption A and $|\dot{h}_i| \leq \phi_i$ $(i = 1, \ldots, r)$, the equilibrium point of system (3) is globally asymptotically stable if there exist symmetric positive matrices $\boldsymbol{P}_i > 0, \boldsymbol{Q} > 0, \boldsymbol{R} > 0, \boldsymbol{X} > 0, \boldsymbol{Y} > 0$, symmetric matrices $\boldsymbol{M}$, matrices $\boldsymbol{W}_{mn}^i$ $(m = 1, \ldots, 6; n = 1, 2), \boldsymbol{E}_{ii}^{k_1 k_2 \cdots k_r}$ and matrices $\boldsymbol{E}_{ij}^{k_1 k_2 \cdots k_r} = (\boldsymbol{E}_{ji}^{k_1 k_2 \cdots k_r})^{\mathrm{T}}$, and $d \in \mathbb{Z}_+$, such that the following LMIs hold

$$\begin{aligned}
&\Gamma_l + \sum_{i=1}^{r} \boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} \\
&\quad + \mathtt{He}\left(\sum_{1 \leq i < j \leq r} \boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} \right) < 0, \qquad (22) \\
&k_1 k_2 \cdots k_r = \mathcal{K}_l(d+2), l = 1, 2, \ldots, J(d+2); \\
&\boldsymbol{E}^{k_1 k_2 \cdots k_r} = \left[\boldsymbol{E}_{ij}^{k_1 k_2 \cdots k_r} \right]_{r \times r} > 0, \qquad (23) \\
&k_1 k_2 \cdots k_r = \mathcal{K}_l(d) \text{ for } 1 = 1, 2, \ldots, J(d);
\end{aligned}$$

where $\boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} = 0$ for $k_i - 2 < 0$, $\boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} = 0$ for $k_i - 1 < 0$ or $k_j - 1 < 0$, and $T_l, l = 1, \ldots, J(d+2)$ are the same as that in Eq. (10).

Proof By pre- and post-multiplying expression (23) by $[\, h_1 I \;\; h_2 I \;\; \cdots \;\; h_r I \,]$ and its transpose, we then have

$$\begin{aligned}
&\sum_{1 \leq i \leq r} h_i^2 \boldsymbol{E}_{ii}^{k_1 k_2 \cdots k_r} + \mathtt{He}\left(\sum_{1 \leq i < j \leq r} h_i h_j \boldsymbol{E}_{ij}^{k_1 k_2 \cdots k_r} \right) > 0, \\
&k_1 k_2 \cdots k_r = \mathcal{K}_l(d), \; l = 1, \ldots, J(d). \qquad (24)
\end{aligned}$$

Multiplying expression (22) by $h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r}, k = \mathcal{K}_l(d+2)$, it follows that

$$\begin{aligned}
&h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r} \left(\Gamma_l + \sum_{i=1}^{r} \boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} + \mathtt{He}\left(\sum_{1 \leq i < j \leq r} \boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} \right) \right) \\
&= h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r} \Gamma_l + h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r} \sum_{i=1}^{r} \boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} \\
&\quad + h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r} \mathtt{He}\left(\sum_{1 \leq i < j \leq r} \boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} \right) < 0, \\
&\quad k_1 k_2 \cdots k_r = \mathcal{K}_l(d+2), \; l = 1, 2, \ldots, J(d+2). \qquad (25)
\end{aligned}$$

By summing inequality (25) from $l = 1$ to $J(d+2)$, we can obtain

$$\begin{aligned}
&\sum_{l=1}^{J(d+2)} h_1^{k_1} h_2^{k_2} \cdots h_r^{k_r} \Gamma_l + \sum_{l=1}^{J(d+2)} \sum_{i=1}^{r} (h_1^{k_1} \cdots h_i^{k_i-2} \cdots h_r^{k_r})(h_i^2) \boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} \\
&\quad + \sum_{l=1}^{J(d+2)} \mathtt{He}\left(\sum_{1 \leq i < j \leq r} (h_1^{k_1} \cdots h_i^{k_i-1} \cdots h_j^{k_j-1} \cdots h_r^{k_r})(h_i h_j) \boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} \right) < 0. \qquad (26)
\end{aligned}$$

On the other hand, combining with the fact $\boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} = 0$ for $k_i - 2 < 0$, $\boldsymbol{E}_{ij}^{k_1 \cdots (k_i-1) \cdots (k_j-1) \cdots k_r} = 0$ for $k_i - 1 < 0$ or $k_j - 1 < 0$, we have

$$\sum_{l=1}^{J(d+2)} \sum_{i=1}^{r} (h_1^{k_1} \cdots h_i^{k_i-2} \cdots h_r^{k_r})(h_i^2) \boldsymbol{E}_{ii}^{k_1 \cdots (k_i-2) \cdots k_r} +$$

$$\begin{aligned}
&\sum_{l=1}^{J(d+2)} \mathtt{He}\left(\sum_{1\le i<j\le r}(h_1^{k_1}\cdots h_i^{k_i-1}\cdots h_j^{k_j-1}\cdots h_r^{k_r})(h_ih_j)\boldsymbol{E}_{ij}^{k_1\cdots(k_i-1)\cdots(k_j-1)\cdots k_r}\right)\\
&=\sum_{l=1}^{J(d)}(h_1^{k_1}\cdots h_i^{k_i}\cdots h_r^{k_r})\sum_{i=1}^{r}(h_i^2)\boldsymbol{E}_{ii}^{k_1\cdots(k_i)\cdots k_r}\\
&\quad+\sum_{l=1}^{J(d)}(h_1^{k_1}\cdots h_i^{k_i}\cdots h_j^{k_j}\cdots h_r^{k_r})\mathtt{He}\left(\sum_{1\le i<j\le r}(h_ih_j)\boldsymbol{E}_{ij}^{k_1\cdots(k_i)\cdots(k_j)\cdots k_r}\right)\\
&=\sum_{l=1}^{J(d)}(h_1^{k_1}\cdots h_r^{k_r})\left(\sum_{i=1}^{r}(h_i^2)\boldsymbol{E}_{ii}^{k_1\cdots k_r}+\mathtt{He}\left(\sum_{1\le i<j\le r}(h_ih_j)\boldsymbol{E}_{ij}^{k_1\cdots k_r}\right)\right). \qquad (27)
\end{aligned}$$

Combining expressions (24), (26), and (27), we have

$$\sum_{l=1}^{J(d+2)} h_1^{k_1}h_2^{k_2}\cdots h_r^{k_r}\Gamma_l<0, \qquad (28)$$

which implies that the equilibrium point of system (3) is globally asymptotically stable. Thus, the proof is completed.

4. Numerical examples

Example 1[20] Consider the plant rule Eq. (3) with $r=2$. The fuzzy T–S fuzzy model of fuzzy Hopfield neural network is of the following form.

Plant Rules

Rule 1: IF $\theta_1(t)$ are M_{k1}, THEN $\dot{\boldsymbol{x}}(t)=-\boldsymbol{A}_1\boldsymbol{x}(t)+\boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}(t-\tau))$,

Rule 2: IF $\theta_2(t)$ are M_{k2}, THEN $\dot{\boldsymbol{x}}(t)=-\boldsymbol{A}_2\boldsymbol{x}(t)+\boldsymbol{W}_2\boldsymbol{f}(\boldsymbol{x}(t-\tau))$,

with $\boldsymbol{f}(\boldsymbol{x})=\tanh(\boldsymbol{x})$. The membership functions for rules 1 and 2 are $M_{k1}=1/\mathrm{e}^{-2\theta_1(t)}$, $M_{k2}=1-M_{k1}$, and

$$\boldsymbol{A}_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \boldsymbol{W}_1=\begin{bmatrix}0.1&0\\0&0.3\end{bmatrix},$$
$$\boldsymbol{A}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \boldsymbol{W}_2=\begin{bmatrix}0.88&0.30\\0.26&-0.25\end{bmatrix}.$$

Obviously, Assumption A is satisfied, $\boldsymbol{L}=\mathrm{diag}\{1,1\}$. In Ref. [20], the allowable value of time delay is $\tau=5$ s. By using the Matlab LMI toolbox, we solve the LMIs (9) and (10) with $d=2$, the allowable maximum value of time delay for the assumption of $\phi_1=\phi_2=10^2$ is $\tau=7.9$ s. Therefore, it is easy to see that Theorem 1 could provide less conservative results than the existing one. Moreover, the response of state variables with $\tau=7.9$ s and $x(0)=(0.9\ \ -0.4)^{\mathrm{T}}$ are shown in Fig. 1. It can be seen from Fig. 1 that the underlying system is globally asymptotically stable.

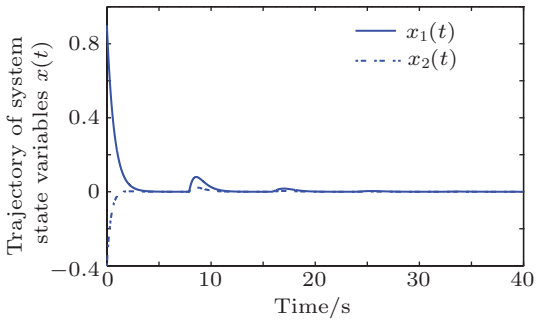


Fig. 1. (colour online) State trajectory of system state variables for Example 1.

Example 2 Consider the following Hopfield neural network with nonlinear terms in its system matrices

$$\begin{cases}\dot{x}_1(t)=-3x_1(t)+2\sin(x_1(t))+0.2f_1(x_1(t-\tau)),\\ \dot{x}_2(t)=-2x_2(t)+0.6f_2(x_2(t-\tau)),\\ \dot{x}_3(t)=-6x_3(t)+2f_3(x_3(t-\tau)),\end{cases} \qquad (29)$$

where $f_i(x_i(t))=\tanh(x_i(t))$.

It is easy to see that there is one nonlinear term $\sin(x_1(t))$ in the above HNNs. In order to represent system (29) by a set of usual sub-HNNs without nonlinear terms in system matrices A and W, two fuzzy sets $M_{11}(x_1(t))$ and $M_{21}(x_1(t))$ are selected as the well-known triangle-type membership functions with two pairs of parameters respectively. These curves of $M_{11}(x_1(t))$ and $M_{21}(x_1(t))$ are shown in Fig. 2. In Fig. 2, the interval of $[-\pi,\ \pi]$ are departed as 9 fuzzy numbers (i.e., $-\pi$, $-\frac{3}{4}\pi$, $-\frac{1}{2}\pi$, $-\frac{1}{4}\pi$, 0, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$, $\frac{3}{4}\pi,\pi$) and 17 fuzzy numbers (i.e., $-\pi$, $-\frac{7}{8}\pi$, $-\frac{3}{4}\pi$, $-\frac{5}{8}\pi$, $-\frac{1}{2}\pi$, $-\frac{3}{8}\pi$, $-\frac{1}{4}\pi$, $-\frac{1}{8}\pi$, 0, $\frac{1}{8}\pi$, $\frac{1}{4}\pi$, $\frac{3}{8}\pi$, $\frac{1}{2}\pi$, $\frac{5}{8}\pi$, $\frac{3}{4}\pi$, $\frac{7}{8}\pi,\pi$) respectively.

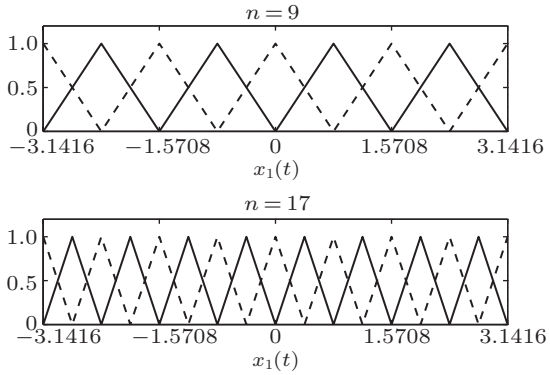


Fig. 2. Curves of two pairs of membership functions (solid line for $M_{11}(x_1(t))$ and dashed line for $M_{21}(x_1(t))$) for Example 2.

The approximation results of two pairs of membership functions ($M_{11}(x_1(t))$ and $M_{21}(x_1(t))$) with different fuzzy numbers ($n = 9$ and $n = 17$, respectively) are shown in Fig. 3. From Fig. 3, we can see that the nonlinear terms $\sin(x_1(t))$ are well approximated by the above two pairs of membership functions. Moreover, as the number of the underlying fuzzy numbers increases, i.e., $n = 9 \to n = 17$, the better approximation result could be obtained.

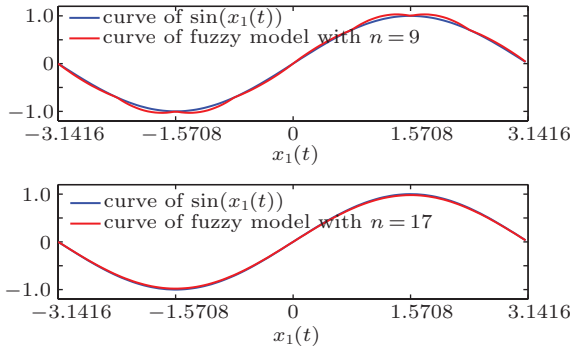


Fig. 3. (colour online) Approximation results of two pairs of membership functions ($M_{11}(x_1(t))$ and $M_{21}(x_1(t))$) with different fuzzy numbers.

Using the Stone–Weierstrass theorem, it has been proved that the fuzzy basis functions are capable of uniformly approximating to any real continuous function on a compact set to arbitrary accuracy.[13] However, higher accuracy (i.e., larger number of fuzzy numbers in fuzzy modeling) usually leads to much computational burden. Actually, designers often select the smallest number of fuzzy numbers while the approximation results satisfies some desired requirements. In this paper, in order to represent system (29) by a T–S fuzzy model with as less fuzzy numbers as possible while a desired accuracy in engineering applications should be also satisfied, two fuzzy sets with three fuzzy numbers (i.e., $-\pi$, 0, π, and $n = 3$) are chosen as follows:

$$M_{11}(x_1(t)) = 1 - \frac{|x_1(t)|}{\pi},\ M_{21}(x_1(t)) = \frac{|x_1(t)|}{\pi}. \quad (30)$$

By using the above membership functions $M_{11}(x_1(t))$ and $M_{21}(x_1(t))$, the nonlinear term $\sin(x_1(t))$ could be replaced by the nonlinear function $M_{11}(x_1(t)) \times x_1(t) + M_{21}(x_1(t)) \times 0$. Within the interval $x_1(m) \in [-\pi, \pi]$, the curves of $\sin(x_1(t))$ and $M_{11}(x_1(t)) \times x_1(t) + M_{21}(x_1(t)) \times 0$ are shown in Fig. 4. From Fig. 4, it can be seen that the nonlinear term $\sin(x_1(t))$ could be approximated by the nonlinear function $M_{11}(x_1(t)) \times x_1(t) + M_{21}(x_1(t)) \times 0$ with a desired accuracy in engineering applications.

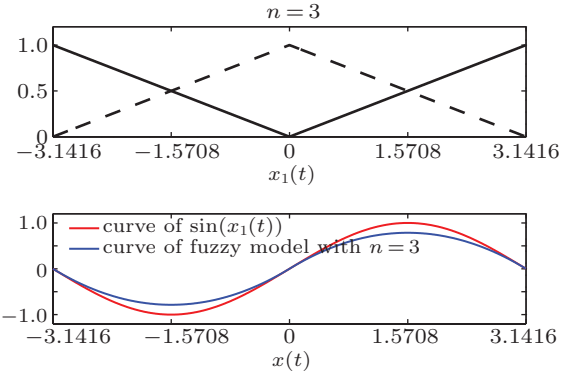


Fig. 4. (colour online) Approximation results of ($M_{11}(x_1(t))$ and $M_{21}(x_1(t))$) as the form of Eq. (30).

Thus, a T–S fuzzy model with the following plant rules well represents the dynamics of system (29).

Plant Rule 1

IF $x_1(t)$ is $M_{11}(x_1(t))$, THEN

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{W}_1\boldsymbol{f}(\boldsymbol{x}(t-\tau)); \quad (31)$$

Plant Rule 2

IF $x_1(t)$ is $M_{21}(x_1(t))$, THEN

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{W}_2\boldsymbol{f}(\boldsymbol{x}(t-\tau)); \quad (32)$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$\boldsymbol{f}(\boldsymbol{x}(t-\tau)) = [f_1(x_1(t-\tau))\ f_2(x_2(t-\tau))\ f_3(x_3(t-\tau))]^{\mathrm{T}}.$$

Given the assumption that $\phi_i = \phi_2 = 0.5$, these feasible values about τ with different d are calculated by using Theorem 1 and Theorem 2 with different values of d, i.e., $d = 1$, $d = 3$, $d = 5$, and $d = 7$ respectively. Indeed, it can be seen that Theorem 2 with $d = 7$ provides least conservative result than other methods. Furthermore, as the value of d increases, both the two results of Theorem 1 and Theorem 2 tend to become bigger, i.e., these two Theorems provide convergent stability criterion for the underlying fuzzy HNNs. Let $\tau = 9.6$ s, the globally asymptotically stability of the Hopfield neural network (29) could be ensured by applying Theorem 2 with $d = 7$. On the other hand, the response of state variables with $\tau = 9.6$ s and $\boldsymbol{x}(0) = (0.6 \quad -0.9 \quad 0.4)^{\mathrm{T}}$ are shown in Fig. 5. It can be seen from Fig. 5 that the Hopfield neural network (29) is globally asymptotically stable.

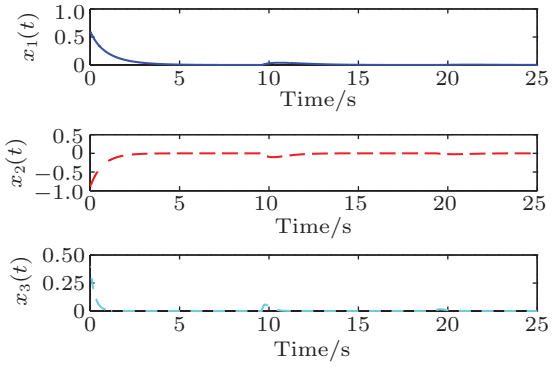


Fig. 5. (colour online) State trajectory of the system state variables for Example 2.

5. Conclusion

In this paper, less conservative stability criteria are derived to guarantee the stability of the equilibrium point for fuzzy Hopfield neural networks with time delays. The proposed stability criteria are given in terms of LMIs and can be readily solved via standard numerical software. Several kinds of relaxed techniques are developed to reduce the conservatism. Two numerical examples have also been provided to demonstrate the effectiveness of the proposed approach.

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