Chaos in a fractional-order micro-electro-mechanical resonator and its suppression

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The present paper investigates the existence of chaos in a non-autonomous fractional-order micro-electro-mechanical resonator system (FOMEMRS). Using the maximal Lyapunov exponent criterion, we show that the FOMEMRS exhibits chaos. Strange attractors of the system are plotted to validate its chaotic behavior. Afterward, a novel fractional finite-time controller is introduced to suppress the chaos of the FOMEMRS with model uncertainties and external disturbances in a given finite time. Using the latest version of the fractional Lyapunov theory, the finite time stability and robustness of the proposed scheme are proved. Finally, we present some computer simulations to illustrate the usefulness and applicability of the proposed method.

Keywords: micro-electro-mechanical resonator, chaotic behavior, fractional calculus, fractional finite-time controller

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1. Introduction

Recently, there has been an accelerating level of interest in micro-engineering and technologies. Micro-engineering is changing the nature of the majority of human-made structures, devices, and systems. High-performance micro-scale systems have been widely used in medicine, biotechnology, robotics, and so on. Moreover, a broad spectrum of micro-electro-mechanical systems (MEMS) has been extensively implemented as actuators and sensors. Particularly, micro-electro-mechanical resonators have been applied to sense electromagnetic field, acceleration, pressure, and temperature.[1]

Due to relatively large structural deformations and displacement-dependent excitations, nonlinearities exist ubiquitously in MEMS resonators. Other micro-electro-mechanical devices, such as micro-motors,[2] micro-actuators,[3] and capacitive circuit elements,[4] also have nonlinear dynamics. Up to now, many scholars have performed researches on various nonlinear dynamic phenomena, including bending of the frequency-response curve and the jump phenomenon in MEMS resonators.[5−7] The existence of nonlinearities may also cause chaotic behaviors.[8] In Ref. [9], complex nonlinear behavior of electrostatically actuated microstructures has been reported. In Ref. [10], the chaotic motion of a micro-electro-mechanical resonant system with non-linearity from capacitors close to the specific resonant separatrix has been addressed. Chavarette *et al.*[11] have developed an optimal linear feedback controller to change the chaotic movement of the system proposed in Ref. [10] into a stable orbit. In Ref. [12] chaotic behavior of a micro-electro-mechanical oscillator governed by the nonlinear Mathieu equation has been investigated both numerically and experimentally. Chaotic motion of a micro-electro-mechanical cantilever beam under both open and closed loop control has been reported in Ref. [13]. Recently, Haghighi and Markazi[14] have numerically studied the chaotic dynamics of a micro-electro-mechanical resonator with electrostatic forces on both sides. In Ref. [15], Yau *et al.* have developed a robust fuzzy sliding mode control scheme to control chaos in a micro-electro-mechanical resonator with system uncertainties.

Fractional calculus is a 300-year-old mathematical topic which generalizes the concept of integer-order differentiation/integration to an arbitrary (non-integer) order one. Although it has a long history, for many years it had not been used in physics and engineering. However, during the last 20 years or so, fractional calculus has attracted the increasing attention of physicists, chemists, and engineers from an application point of view.[16,17] It has been found that many systems in interdisciplinary fields, such

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as heat diffusion systems,[18] electrical circuits,[19] finance systems,[20] population models,[21] and rub-impact rotor systems,[22] can display fractional-order dynamics.

In Ref. [23] the chaotic dynamics of a fractional-order Volta system has been investigated. Wu and Wang[24] have studied the dynamics of a fractional-order Rikitake-type system. The chaotic behaviors in fractional-order rotational mechanical systems have been addressed in Ref. [25]. In Ref. [26], the chaotic behaviors of a fractional-order nonlinear damped Mathieu system and of a nonlinear nano resonator system have been investigated numerically. Recently, Aghababa[27,28] has introduced fractional sliding mode controllers for synchronization of fractional-order uncertain chaotic systems.

In this paper, we investigate the existence of the chaos in a non-autonomous fractional-order micro-electro-mechanical resonator system (FOMEMRS). We use the maximal Lyapunov exponent (MLE) criterion to show that the nonlinear behaviors of the FOMEMRS are chaotic. Then, we propose a novel fractional finite-time controller to suppress the chaos of the FOMEMRS in a given finite time. The effects of model uncertainties and external disturbances are taken into account. We use the fractional Lyapunov stability theory[29] to prove the robustness and finite-time stability of the proposed scheme. Some numerical simulations are provided to illustrate the effectiveness and robustness of the proposed control technique

2. Chaos in micro-electro-mechanical resonators

2.1. Mathematical model of an MEMS resonator

A schematic picture of an electrostatically actuated micro-beam is plotted in Fig. 1, where d is the initial width of the gap and z is the vertical displacement of the beam. In this system, an electrical driving voltage on the resonator, as an external driving force, is applied via an electrical driving voltage that produces electrostatic excitation with DC-bias voltage between electrodes and the resonator: $V_i = V_\mathrm{b} + V_\mathrm{AC}\sin\Omega t$, where V_b is the bias voltage and V_AC and Ω are the AC amplitude and frequency, respectively. The amplitude of the AC driving voltage is usually much lower than the bias voltage, yielding the nondimensional equation of motion[14]

$$\ddot{x} + \mu\dot{x} + \alpha x + \beta x^3 = \gamma\left[\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right] + \frac{A}{(1-x)^2}\sin\omega t, \quad (1)$$

where $x = z/d$, $\omega = \Omega/\omega_0$, ω_0 is the purely elastic natural frequency and $A = 2\gamma V_\mathrm{AC}/V_\mathrm{b}$. More details about the nonlinear dynamics of system (1) can be found in Ref. [14].

Assuming $x_1 = x$ and $x_2 = \dot{x}$, non-autonomous system (1) is transformed into the following normal form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\mu x_2 - \alpha x_1 - \beta x_1^3 + \gamma\left[\dfrac{1}{(1-x_1)^2} - \dfrac{1}{(1+x_1)^2}\right] + \dfrac{A}{(1-x_1)^2}\sin\omega t. \end{cases} \quad (2)$$

It has been shown in Ref. [14] that system (2) exhibits complex and rich dynamics for values of V_AC in the range $0 < V_\mathrm{AC} < 0.47$ and constant values of $\alpha = 1$, $\beta = 12$, $\gamma = 0.338$, $\mu = 0.01$, $V_\mathrm{b} = 3.8$, and $\omega = 0.5$. The strange attractor of the MEMRS (2) with initial conditions of $x_1(t) = x_2(t) = 0$ and $V_\mathrm{AC} = 0.4$ is illustrated in Fig. 2. It is obvious that system (2) is chaotic.

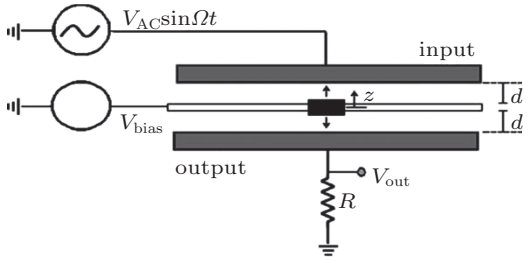


Fig. 1. A schematic picture of the MEMRS.[14]

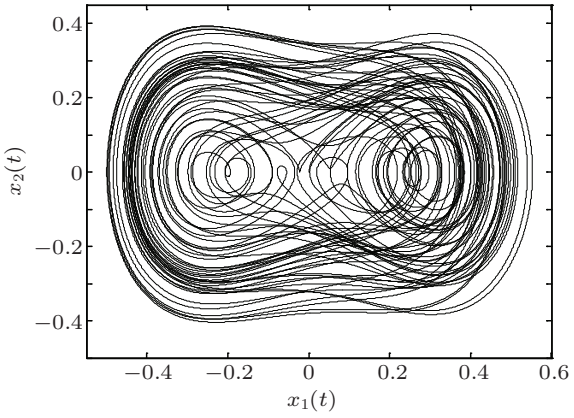


Fig. 2. The strange attractor of the MEMRS (2).

2.2. Fractional-order model of the MEMS resonator

Definition 1[16] The α-th order Riemann–Liouville fractional integration of function $f(t)$ is given by

$$ {}_{t_0}I_t^{\alpha} f(t) = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \tag{3} $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2[16] Let $m-1 < \alpha \leq m$, $m \in N$, the Riemann–Liouville fractional derivative of order α of function $f(t)$ is defined as follows:

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{\mathrm{d}^{\alpha} f(t)}{\mathrm{d}t^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau. \tag{4} $$

Definition 3[16] The Caputo fractional derivative of order α of a continuous function $f(t) : R^+ \to R$ is defined as follows:

$$ {}_{t_0}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, & m-1 < \alpha < m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \alpha = m, \end{cases} \tag{5} $$

where m is the smallest integer number larger than α.

Based on the definition of the fractional order differential equations and using Eq. (2), the following fractional order model for the MEMS resonator is proposed

$$ \begin{cases} D^q x_1 = x_2, \\ D^q x_2 = -\mu x_2 - \alpha x_1 - \beta x_1^3 + \gamma \left[\dfrac{1}{(1-x_1)^2} - \dfrac{1}{(1+x_1)^2} \right] + \dfrac{A}{(1-x_1)^2} \sin \omega t, \end{cases} \tag{6} $$

where $q \in (0,1)$ is the fractional order of the system.

In order to investigate the existence of the chaos for FOMEMRS (6), we use the maximal Lyapunov exponent criterion and apply the numerical method introduced in Ref. [30] to calculate the system's MLE. The values of the system parameters are chosen as $\alpha = 1$, $\beta = 12$, $\gamma = 0.338$, $\mu = 0.01$, $V_{\mathrm{b}} = 3.8$, and $\omega = 0.5$. The initial conditions of the system are chosen as $x_1(0) = x_2(0) = 0$. First, the parameter V_{AC} is set to 0.4. The MLEs of system (6) for different values of the fractional order q are shown in Fig. 3. It is seen that when $0.35 < q < 0.99$, system (6) has positive MLE, indicating that the behavior of this system is chaotic. The strange attractors of fractional-order system (6) for some randomly selected values of q are illustrated in Fig. 4. One can see that system (6) exhibits rich and chaotic dynamics. Next, we choose $q = 0.9$ and $0.01 < V_{\mathrm{AC}} < 0.47$. Figure 5 reveals the MLEs of system (6) for different values of V_{AC}. One can see that when $0 < V_{\mathrm{AC}} < 47$, system (6) has a positive MLE, which implies that non-autonomous fractional-order system (6) possesses chaotic motions. The strange attractors of fractional-order system (6) for some randomly selected values of V_{AC} are plotted in Fig. 6. It can be seen that system (6) exhibits chaos.

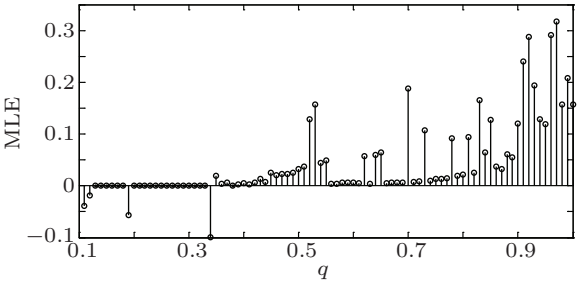


Fig. 3. The MLEs of system (6) for different values of the fractional order q.

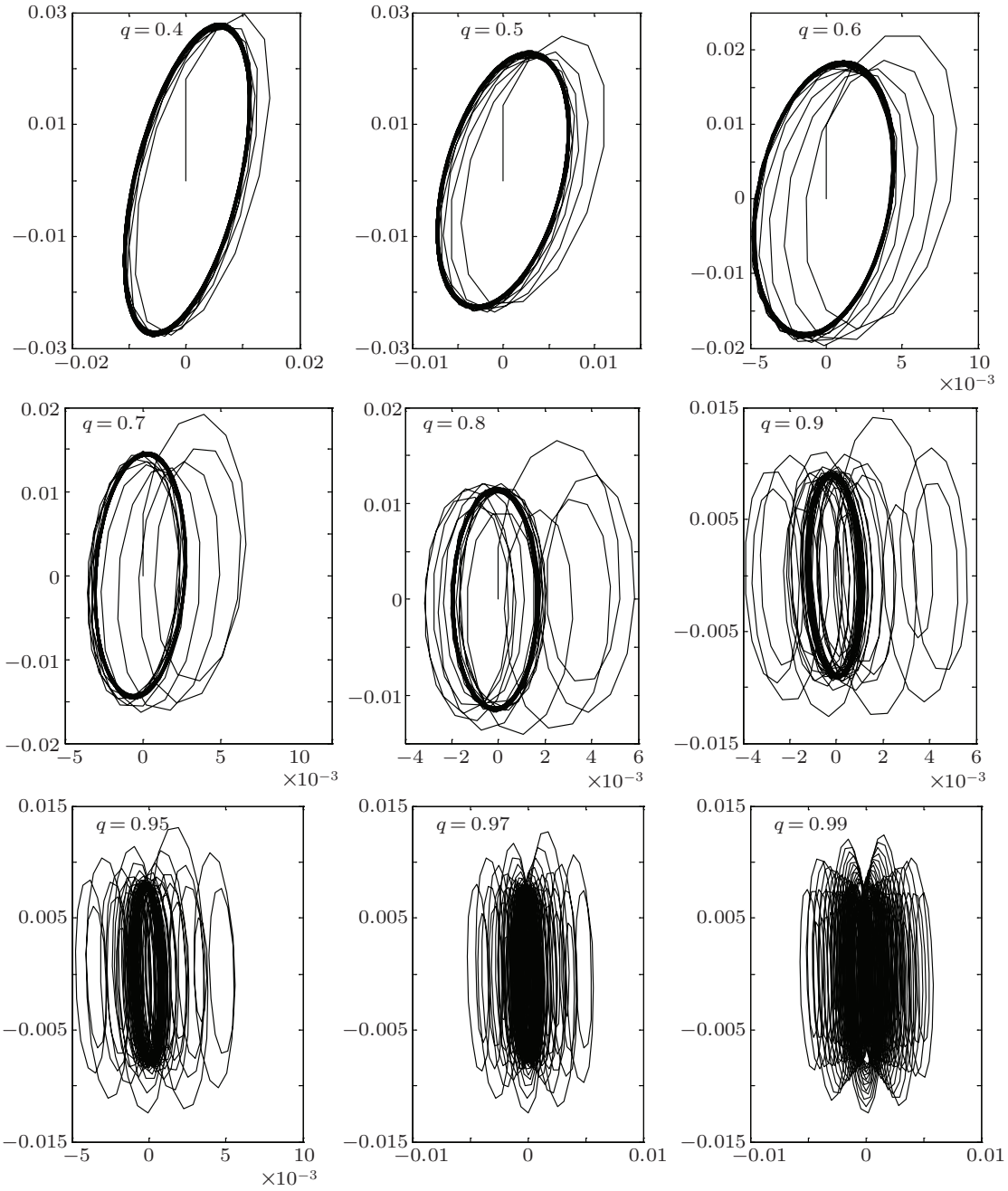


Fig. 4. The strange attractors of system (6) for different values of the fractional order q.

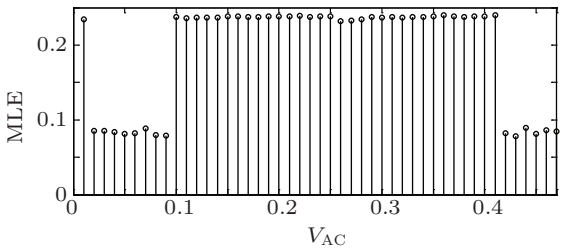


Fig. 5. The MLEs of system (6) for different values of V_{AC}.

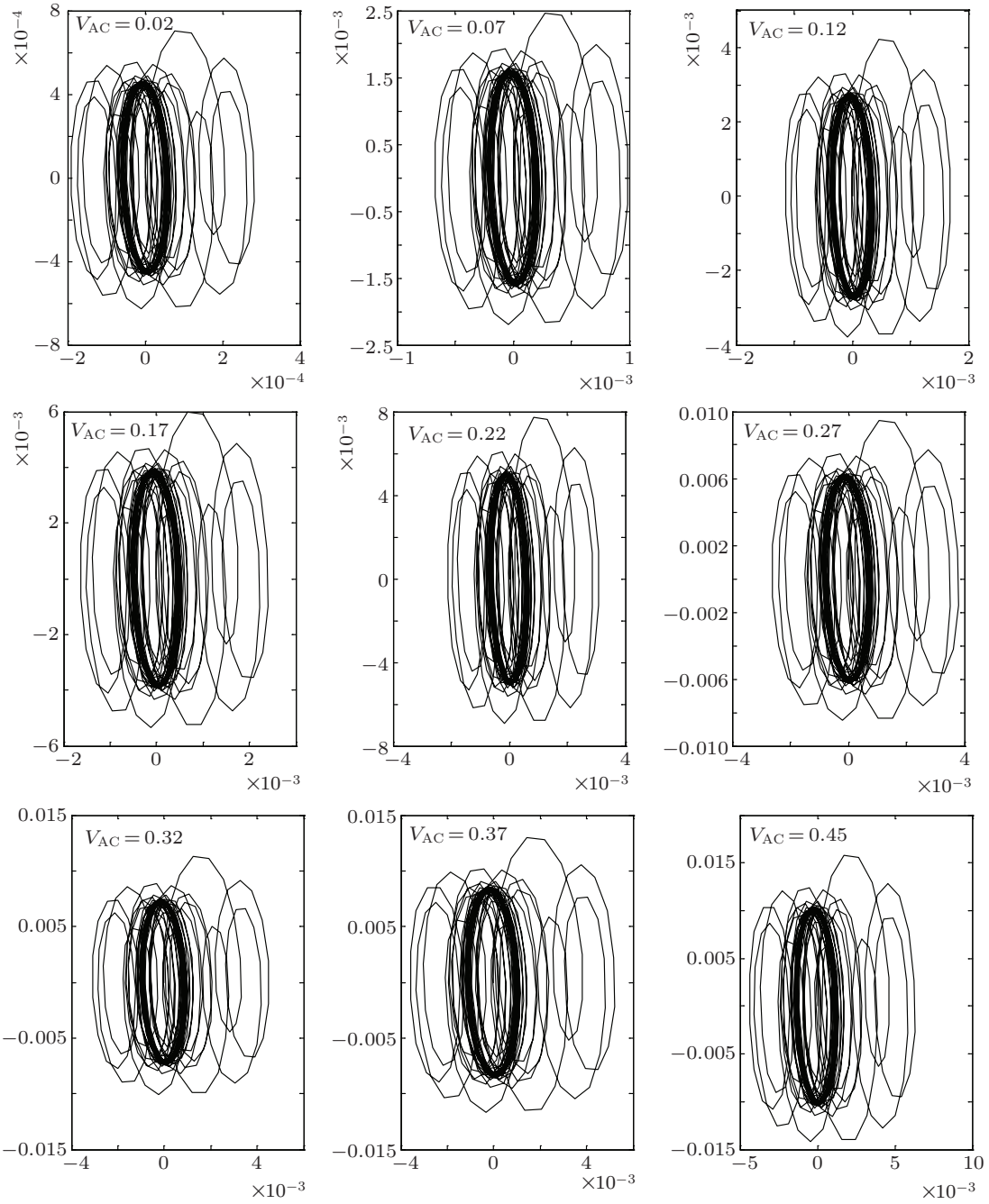


Fig. 6. The strange attractors of system (6) for different values of V_{AC}.

3. The chaos suppression of the FOMEMRS

3.1. Design of a finitetime fractional controller for chaos suppression of FOMEMRS

Theorem 1[29] Let $x = 0$ be an equilibrium point for the non-autonomous fractional-order system

$$D^{\alpha}x\,(t) = f(x,t), \tag{7}$$

where $f(x,t)$ satisfies the Lipschitz condition with Lipschitz constant $l > 0$ and $\alpha \in (0,1)$ is the fractional order of the system. Assume that there exists a Lyapunov function $V(t,x(t))$ which satisfies the following conditions

$$\alpha_1 \|x\|^a \leq V(t,x) \leq \alpha_2 \|x\|, \tag{8}$$

$$\dot{V}\,(t,x) \leq -\alpha_3 \|x\|, \tag{9}$$

where α_1, α_2, α_3, and a are positive constants. Then the equilibrium point of non-autonomous system (7) is Miattag–Leffler (asymptotically) stable.

Since in real-world applications model uncertainties and external disturbances affect the dynamics of the system, the following uncertain FOMEMRS with a control input is taken into account in this paper

$$\begin{cases} D^q x_1 = x_2, \\ D^q x_2 = -\mu x_2 - \alpha x_1 - \beta x_1^3 \\ \qquad + \gamma \left[\dfrac{1}{(1-x_1)^2} - \dfrac{1}{(1+x_1)^2} \right] \\ \qquad + \dfrac{A}{(1-x_1)^2} \sin \omega t + \Delta f(X) + d(t) + u(t), \end{cases} \tag{10}$$

where $X(t) = [x_1, x_2]^{\mathrm{T}}$ is the state vector of the system and $\Delta f(x)$, $d(t)$, and $u(t)$ represent the system model uncertainty, external disturbance, and control input, respectively.

Assumption 1 It is assumed that the uncertainty term and external disturbance are unknown, but bounded as follows:

$$\left\| D^{1-\alpha} \Delta f(x) \right\|_1 \le a_1, \qquad \left\| D^{1-\alpha} d(t) \right\|_1 \le a_2, \tag{11}$$

where a_1 and a_2 are given positive constants.

Our main purpose is to design a proper control input $u(t)$ such that it suppresses the chaotic behavior of uncertain fractional-order system (10) in a given finite time. To reach this goal, we propose the following control input

$$\begin{aligned} u(t) = -f(x,t) - D^{q-1} \big[& \big(a_1 + l_1 |x_1|^c + a_2 + l_2 |x_2|^c \\ & + \operatorname{sign}(x_1) D^{1-q}(x_2) + k_1 |x_1| \big) \operatorname{sign}(x_2) \\ & - k_2 x_2 \big], \end{aligned} \tag{12}$$

where

$$\begin{aligned} f(x,t) = & -\mu x_2 - \alpha x_1 - \beta x_1^3 \\ & + \gamma \left[\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2} \right] \\ & + \frac{A}{(1-x_1)^2} \sin \omega t, \end{aligned}$$

l_1, l_2, k_1, and k_2 are positive constants and $0 < c < 1$ is a real number.

Theorem 2 Consider non-autonomous uncertain chaotic FOMEMRS (10). If this system is controlled by control law (12), then its chaotic behavior will be suppressed in finite time.

Proof Choosing a Lyapunov function candidate in the form of $V(t) = \|X(t)\|_1$ and taking its time derivative, we have

$$\begin{aligned} \dot{V}(t) &= \operatorname{sign}(x_1) \dot{x}_1 + \operatorname{sign}(x_2) \dot{x}_2 \\ &= \operatorname{sign}(x_1) D^{1-q} (D^q x_1) \\ &\quad + \operatorname{sign}(x_2) D^{1-q} (D^q x_2). \end{aligned} \tag{13}$$

Substituting $D^q x_1$ and $D^q x_2$ from Eq. (10) into Eq. (13), one has

$$\begin{aligned} \dot{V}(t) = & \operatorname{sign}(x_1) D^{1-q}(x_2) \\ & + \operatorname{sign}(x_2) D^{1-q} \Big[-\mu x_2 - \alpha x_1 - \beta x_1^3 \\ & + \gamma \left(\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2} \right) \\ & + \frac{A}{(1-x_1)^2} \sin \omega t + \Delta f(x) \\ & + d(t) + u(t) \Big]. \end{aligned} \tag{14}$$

Using Assumption 1, one can obtain

$$\begin{aligned} \dot{V}(t) \le & \operatorname{sign}(x_1) D^{1-q}(x_2) \\ & + \operatorname{sign}(x_2) D^{1-q} \Big[-\mu x_2 - \alpha x_1 - \beta x_1^3 \\ & + \gamma \left(\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2} \right) \\ & + \frac{A}{(1-x_1)^2} \sin \omega t + u(t) \Big] + a_1 + a_2. \end{aligned} \tag{15}$$

Introducing control law (10) into the right-hand side of inequality (15), one can obtain

$$\dot{V}(t) \le -\left(k_1 |x_1| + k_2 |x_2| + l_1 |x_1|^c + l_2 |x_2|^c \right). \tag{16}$$

Using the fact $-\left(|x_1|^c + |x_2|^c \right) \le -\left(|x_1| + |x_2| \right)^c$,[31] we have

$$\begin{aligned} \dot{V}(t) &\le -\left(m \left(|x_1| + |x_2| \right) + n \left(|x_1| + |x_2| \right)^c \right) \\ &= -m \|X(t)\|_1 - n \|X(t)\|_1^c \\ &\le -m \|X(t)\|_1, \end{aligned} \tag{17}$$

where $m = \min\{k_1, k_2\}$ and $n = \min\{l_1, l_2\}$ are two positive constants.

As a result, according to Theorem 1, the state trajectories of uncertain FOMEMRS (10) will converge to zero and its chaotic behavior will be suppressed asymptotically. Using a similar approach in Refs. [27] and [28], we show that the chaos suppression takes place in finite time.

From inequality (17), we have

$$\dot{V}(t) = \frac{\mathrm{d} \|X(t)\|_1}{\mathrm{d}t} \le -m \|X(t)\|_1 - n \|X(t)\|_1^c. \tag{18}$$

Using some simple calculations, we obtain

$$\begin{aligned} \mathrm{d}t &\leq -\frac{\mathrm{d}\left\|X\left(t\right)\right\|_1}{m\left\|X\left(t\right)\right\|_1+n\left\|X\left(t\right)\right\|_1^c} \\ &= -\frac{\left\|X\left(t\right)\right\|_1^{-c}\mathrm{d}\left\|X\left(t\right)\right\|_1}{m\left\|X\left(t\right)\right\|_1^{1-c}+n} \\ &= -\frac{1}{1-c}\frac{\mathrm{d}\left\|X\left(t\right)\right\|_1^{1-c}}{m\left\|X\left(t\right)\right\|_1^{1-c}+n}. \end{aligned} \tag{19}$$

Taking the integral of both sides of expression (19) from 0 to T and letting $x_1(t_1) = x_2(t_2) = 0$ and $X(T) = 0$ ($T = \max\{t_1, t_2\}$), one has

$$\begin{aligned} T &\leq -\frac{1}{1-c}\int_{X(0)}^{X(T)}\frac{\mathrm{d}\left\|X\left(t\right)\right\|_1^{1-c}}{m\left\|X\left(t\right)\right\|_1^{1-c}+n} \\ &= \frac{-1}{m\left(1-c\right)}\ln\left(m\left\|X\left(t\right)\right\|_1^{1-c}+n\right)\Bigg|_{X(0)}^{X(T)} \\ &= \frac{1}{m\left(1-c\right)}\ln\left(\frac{m\left\|X\left(0\right)\right\|_1^{1-c}+n}{n}\right). \end{aligned} \tag{20}$$

Therefore, the chaotic motions of uncertain non-autonomous FOMEMRS (10) will be suppressed in the finite time

$$T \leq \frac{1}{m\left(1-c\right)}\ln\left(\frac{m\left\|X\left(0\right)\right\|_1^{1-c}+n}{n}\right).$$

This completes the proof.

Remark 1 It is worth noting that, as reported in Ref. [32], non-smooth Lyapunov functions can be used to prove the finite-time convergence of a system. Therefore, choosing a non-smooth Lyapunov function as $V(t) = \|X(t)\|_1$ to show the finite-stability of the system is practical and common in the literature.

Remark 2 In system (1), if we assume $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \omega t$, the non-autonomous system (1) is transformed into the following form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\mu x_2 - \alpha x_1 - \beta x_1^3 + \gamma\left[\dfrac{1}{(1-x_1)^2} - \dfrac{1}{(1+x_1)^2}\right] + \dfrac{A}{(1-x_1)^2}\sin x_3, \\ \dot{x}_3 = \omega. \end{cases} \tag{21}$$

Based on the definition of the fractional order differential equations and using Eq. (21), the following fractional order model for the autonomous MEMS resonator is given

$$\begin{cases} D^q x_1 = x_2, \\ D^q x_2 = -\mu x_2 - \alpha x_1 - \beta x_1^3 + \gamma\left[\dfrac{1}{(1-x_1)^2} - \dfrac{1}{(1+x_1)^2}\right] + \dfrac{A}{(1-x_1)^2}\sin x_3 + \Delta f\left(X\right) + d\left(t\right) + u(t), \\ D^q x_3 = \omega. \end{cases} \tag{22}$$

Theorem 3 Consider autonomous uncertain chaotic FOMEMRS (22). If this system is controlled by control law (23), then its state trajectories will be converged to zero in a finite time

$$\begin{aligned} u(t) = &-f(x,t) - D^{q-1}[(a_1 + l_1\left|x_1\right|^c + a_2 \\ &+ l_2\left|x_2\right|^c + l_3\left|x_3\right|^c + \mathrm{sign}(x_1)D^{1-q}(x_2) \\ &+ \mathrm{sign}(x_3)D^{1-q}(\omega) + k_1\left|x_1\right| \\ &+ k_3 x_3)\,\mathrm{sign}(x_2) - k_2 x_2], \end{aligned} \tag{23}$$

where

$$\begin{aligned} f\left(x,t\right) = &-\mu x_2 - \alpha x_1 - \beta x_1^3 \\ &+ \gamma\left[\frac{1}{\left(1-x_1\right)^2} - \frac{1}{\left(1+x_1\right)^2}\right] \\ &+ \frac{A}{\left(1-x_1\right)^2}\sin\omega, \end{aligned}$$

l_1, l_2, l_3, k_1, k_2, and k_3 are positive constants and $0 < c < 1$ is a real number.

Proof Choosing a Lyapunov function candidate in the form of $V(t) = \|X(t)\|_1$, where $X(t) = [x_1, x_2, x_3]^{\mathrm{T}}$ and taking its time derivative, one obtains

$$\begin{aligned} \dot{V}\left(t\right) = &\ \mathrm{sign}\left(x_1\right)\dot{x}_1 + \mathrm{sign}\left(x_2\right)\dot{x}_2 + \mathrm{sign}\left(x_3\right)\dot{x}_3 \\ = &\ \mathrm{sign}\left(x_1\right)D^{1-q}\left(D^q x_1\right) \\ &+ \mathrm{sign}\left(x_2\right)D^{1-q}\left(D^q x_2\right) \\ &+ \mathrm{sign}\left(x_3\right)D^{1-q}\left(D^q x_3\right). \end{aligned} \tag{24}$$

Substituting $D^q x_1$, $D^q x_2$, and $D^q x_3$ from system (22)

into Eq. (24), we have

$$\begin{aligned}\dot{V}(t) &= \operatorname{sign}(x_1) D^{1-q}(x_2) \\ &\quad + \operatorname{sign}(x_2) D^{1-q}\Big[-\mu x_2 - \alpha x_1 - \beta x_1^3 \\ &\quad + \gamma\left(\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2}\right) \\ &\quad + \frac{A}{(1-x_1)^2}\sin x_3 + \Delta f(x) + d(t) + u(t) \\ &\quad + \operatorname{sign}(x_3) D^{1-q}(\omega)\Big]. \end{aligned} \tag{25}$$

Using Assumption 1, one can obtain

$$\begin{aligned}\dot{V}(t) &\le \operatorname{sign}(x_1) D^{1-q}(x_2) + \operatorname{sign}(x_2) D^{1-q} \\ &\quad \times \Big[-\mu x_2 - \alpha x_1 - \beta x_1^3 \\ &\quad + \gamma\left(\frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2}\right) \\ &\quad + \frac{A}{(1-x_1)^2}\sin x_3 + u(t) \\ &\quad + \operatorname{sign}(x_3) D^{1-q}(\omega)\Big] + a_1 + a_2. \end{aligned} \tag{26}$$

Introducing control law (23) into the right-hand side of Eq. (25), one has

$$\begin{aligned}\dot{V}(t) \le & -\big(k_1|x_1| + k_2|x_2| + k_3|x_3| \\ & + l_1|x_1|^c + l_2|x_2|^c + l_3|x_3|^c\big). \end{aligned} \tag{27}$$

After some simple mathematical manipulations, we have

$$\begin{aligned}\dot{V}(t) &\le -\left(M(|x_1| + |x_2|) + N(|x_1| + |x_2|)^c\right) \\ &= -M\|X(t)\|_1 - N\|X(t)\|_1^c \\ &\le -M\|X(t)\|_1, \end{aligned} \tag{28}$$

where $M = \min\{k_1, k_2, k_3\}$ and $N = \min\{l_1, l_2, l_3\}$ are two positive constants.

Therefore, according to Theorem 1, the state trajectories of uncertain autonomous FOMEMRS (22) will converge to zero asymptotically. Using a similar approach in Theorem 2, we can show that the convergence takes place in the following finite time

$$T \le \frac{1}{M(1-c)} \ln\left(\frac{M\|X(0)\|_1^{1-c} + N}{N}\right).$$

Thus the proof is achieved completely

3.2. Validation of the proposed finite time fractional controller in the chaos suppression of an FOMEMRS

In order to validate the efficiency and applicability of proposed finite-time controller (12) in suppressing the chaotic behavior of the uncertain FOMEMRS in a given finite time, an illustrative example is presented. We use the numerical approach introduced in Ref. [33] to solve the nonlinear fractional-order equations of the system. We assume that the fractional order of system (10) is equal to 0.54, where the system has an MLE of 0.17. Moreover, the parameters of system (10) are chosen as $\alpha = 1$, $\beta = 12$, $\gamma = 0.338$, $\mu = 0.01$, $V_{\rm b} = 3.8$, $\omega = 0.5$, and $V_{\rm AC} = 0.4$. Subsequently, the following model uncertainty and external disturbance are considered in the simulation

$$\Delta f(x) = 0.1\sin(t)\, x_2(t), \quad d(t) = -0.1\cos(t). \tag{29}$$

The parameters of controller (12) are selected as $a_1 = a_2 = 0.3$, $l_1 = l_2 = k_1 = k_2 = 0.1$, and $c = 0.98$. The initial conditions of the system are set to $x_1(0) = x_2(0) = 0.5$. The state trajectories of the uncontrolled and controlled uncertain FOMEMRS (10) are shown in Figs. 7 and 8, respectively. It is obvious that the chaotic motions of the non-autonomous uncertain chaotic FOMEMRS system are suppressed quickly in a finite time. The time history of the applied control input is depicted in Fig. 9. It is seen that the control input converges to zero.

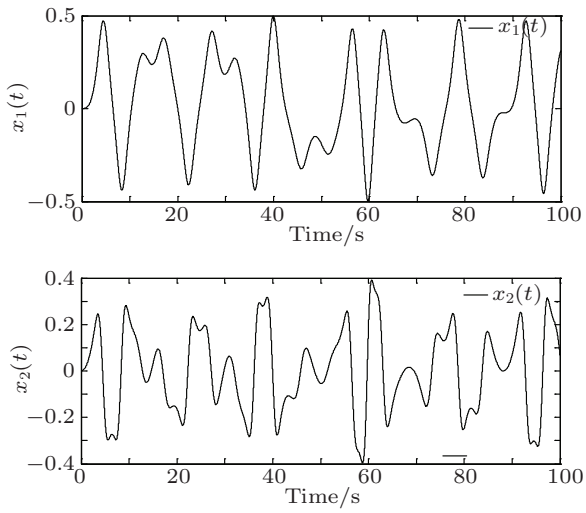


Fig. 7. The state trajectories of uncontrolled FOMEMRS (10).

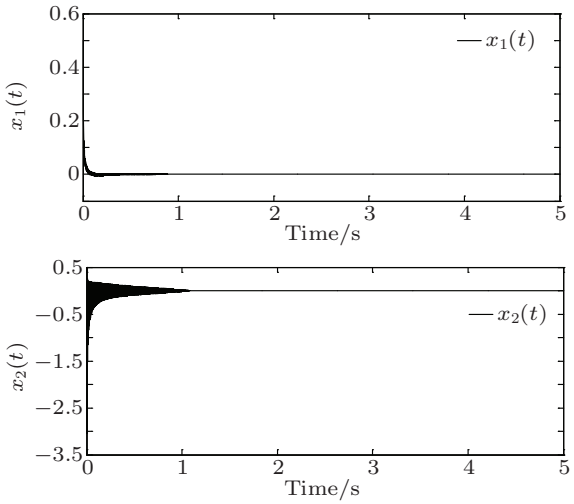


Fig. 8. The state trajectories of controlled uncertain FOMEMRS (10).

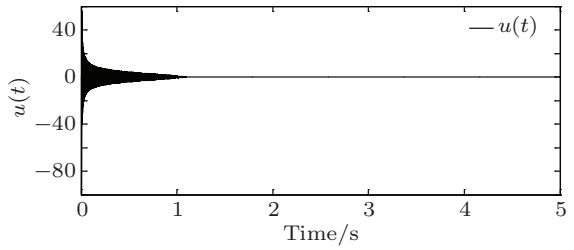


Fig. 9. The time histories of applied control input (12).

4. Conclusion

This paper studies the existence of chaos for a non-autonomous fractional-order micro-electromechanical resonator system (FOMEMRS). Using the maximal Lyapunov exponent criterion, it is shown that the FOMEMRS exhibits chaotic behavior. A robust finite-time controller is then designed to attenuate the chaotic behavior of the system in the presence of both model uncertainty and external disturbance. Using the fractional Lyapunov stability theory, the finite-time stability and robustness of the proposed scheme is mathematically proved. A numerical simulation illustrates the usefulness and effectiveness of the proposed technique.

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