Chaotic behavior learning of Chua's circuit*

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Least-square support vector machines (LS-SVM) are applied for learning the chaotic behavior of Chua's circuit. The system is divided into three multiple-input single-output (MISO) structures and the LS-SVM are trained individually. Comparing with classical approaches, the proposed one reduces the structural complexity and the selection of parameters is avoided. Some parameters of the attractor are used to compare the chaotic behavior of the reconstructed and the original systems for model validation. Results show that the LS-SVM combined with the MISO can be trained to identify the underlying link among Chua's circuit state variables, and exhibit the chaotic attractors under the autonomous working mode.

Keywords: chaotic behavior, Chua's circuit, support vector machines

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1. Introduction

Chua's circuit is a simple nonlinear electronic circuit exhibiting chaos behaviors, which has been verified by numerous laboratory experiments, numerically simulations, and rigorous mathematical analyses.[1] Chua's circuit has been used as a physical source of pseudo random signals and in secure communication systems and simulations of brain dynamics. Recently, it has also been exploited in avant-garde music compositions[2] and the evolution of natural languages.[3]

The key problem of the chaotic system is to construct a model for the unknown dynamical system that generates the time series. There are many traditional methods associated with the time series prediction such as linear regression and auto-regressive integrated moving average (ARIMA) models.[4,5] Nevertheless, in recent years, various innovative approaches, such as time-delay embedding,[6,7] wavelets,[8] and global dynamical models,[9] have been proposed to resolve the problems associated with the nonlinear time series. On the other hand, more recent studies place more emphasis on the general neural networks,[10−12] the radial basis function network,[13,14] the Gaussian process,[15] and the kennel methods such as support vector machines (SVM).[16−18] In addition, Grau-Sánchez *et al.* analyzed the chaotic behavior of a discrete dynamical system based on a uniparametric family of real functions.[19]

The chaotic phenomenon is a typical behavior in Chua's circuit and how to learn it is the object of many scientific research activities. Several approaches have been proposed, which can be grossly divided into two groups based on the structures of the data flows. The first regards the circuit as a multiple-input multiple-output system (MIMO) and then employs the black-box approaches, such as the neural networks, to capture the behavior of the circuit.[20−22] The second divides the system into multiple scalar time series and models them individually. Then the system model can be obtained by synthesizing the individual ones. The above-mentioned approaches can work well in most cases. However, some disadvantages still exist in those methods, e.g., high structural complexity and difficult training of the neural network for the MIMO system. In addition, embedding dimension and time delay need to be selected carefully for predicting the scalar time series.[7] These drawbacks may give rise to great difficulties when a more complex system is considered.

In this paper, Chua's circuit is modeled based on multiple-input single-output (MISO) structures and least-square support vector machines (LS-SVM).[17] The MISO structure and the LS-SVM allow solving

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high structural complexity and difficult training problems. In addition, the selection of embedding dimension and the time delay can be avoided.

2. Chua's circuit

Chua's circuit consists of a linear inductor L, two linear capacitors C_1, C_2, a linear resistor R and a voltage controlled resistor N_R. The electrical circuit is shown in Fig. 1. All three state variables are described by the following three ordinary differential equations:

$$\begin{aligned} C_1\frac{\mathrm{d}v_{C_1}}{\mathrm{d}t} &= \frac{v_{C_2}-v_{C_1}}{R} - g\left(v_{C_1}\right), \\ C_2\frac{\mathrm{d}v_{C_2}}{\mathrm{d}t} &= \frac{v_{C_1}-v_{C_2}}{R} + i_L, \\ L\frac{\mathrm{d}i_L}{\mathrm{d}t} &= -v_{C_2}, \end{aligned} \tag{1}$$

where v_{C_1} and v_{C_2} represent the voltages across capacitors C_1 and C_2, respectively, i_L is the current in inductor L and $g\left(v_{C_1}\right)$ is the current in the nonlinear resistor N_R. If these parameters are tuned to suitable values, then the chaotic behavior can be observed. When the nonlinear resistor N_R has Chua's diode characteristic (shown in Fig. 1(b)), the circuit generates the so-called double scroll attractor. The analytic expression of Chua's diode can be expressed as

$$\begin{aligned} g\left(v_{C_1}\right) = {} & G_{\mathrm{b}}v_{C_1} + \frac{1}{2}\left(G_{\mathrm{a}}-G_{\mathrm{b}}\right)\left|v_{C_1}+E\right| \\ & + \frac{1}{2}\left(G_{\mathrm{a}}-G_{\mathrm{b}}\right)\left|v_{C_1}-E\right|, \end{aligned} \tag{2}$$

where E is the breakpoint's abscissa.

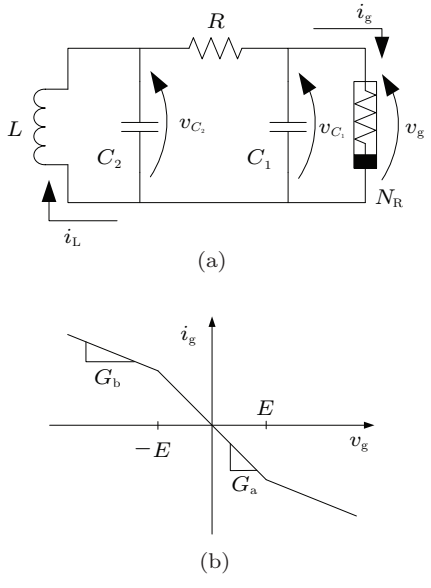


Fig. 1. (a) Chua's circuit; (b) Chua's diode characteristic.

3. Learning using an MISO structure and the LS-SVM

In this section, the system model and the LS-SVM foundations used in the presented study will be given.

3.1. LS-SVM regression

The LS-SVM is the least-square version of the support vector machines (SVMs), which are a set of related supervised learning methods and are used for classification and regression analyses.[17] In this version we find the solution by solving a set of linear equations instead of a convex quadratic programming (QP) problem.

Consider now the case in which there is noise in the description of the regression functions. Give a training set of size N defined as $D = \{(x_k, y_k) : x_k \in X, y_k \in Y; k = 1, 2, \ldots, N\}$, where

$$y_k = m\left(x_k\right) + e_k, \quad k = 1, \ldots, N, \tag{3}$$

and $E\left[e_k \left| X = x_k\right.\right] = 0$, $\mathrm{Var}[e_k] = \sigma^2 < \infty$, $m\left(x\right)$ is an unknown real-valued smooth function. Our goal is to find parameters w and b that minimize the empirical risk function

$$R_{\mathrm{emp}}\left(w, b\right) = \frac{1}{n}\sum_{k=1}^{N}\left(\left(w^{\mathrm{T}}\varphi\left(x_k\right)+b\right) - y_k\right)^2, \tag{4}$$

where φ is a nonlinear mapping in the future space. We can reduce the optimization problem of finding vector w and $b \in R$ to solving the following optimization problem

$$\min_{w,b,e} J\left(w, e\right) = \frac{1}{2}w^{\mathrm{T}}w + \frac{1}{2}\gamma\sum_{k=1}^{N}e_k^2, \tag{5}$$

such that

$$y_k = w^{\mathrm{T}}\varphi\left(x_k\right) + b + e_k, \quad k = 1, \ldots, N.$$

To solve the optimization problem, we define the Lagrangian function

$$\begin{aligned} & L\left(w, b, e; \alpha\right) \\ & = J\left(w, e\right) - \sum_{k=1}^{N}\alpha_k\left(w^{\mathrm{T}}\varphi\left(x_k\right) + b + e_k - y_k\right), \end{aligned} \tag{6}$$

where $\alpha_k \in R$ is the Lagrangian multiplier. The conditions for optimization are given by

$$\begin{aligned}
&\frac{\partial L}{\partial w} = 0 \to w = \sum_{k=1}^{N} \alpha_k \varphi(x_k), \\
&\frac{\partial L}{\partial b} = 0 \to \sum_{k=1}^{N} \alpha_k = 0, \\
&\frac{\partial L}{\partial e_k} = 0 \to \alpha_k = \gamma e_k, \quad k = 1, \ldots, N, \\
&\frac{\partial L}{\partial \alpha_k} = 0 \to w^{\mathrm{T}} \varphi(x_k) + b + e_k = y_k, \ k = 1, \ldots, N.
\end{aligned} \tag{7}$$

According to Mercer's theorem, the mapping function φ can be paraphrased by a kernel function, K, which means that

$$K(x_i, x_j) = \varphi(x_i)^{\mathrm{T}} \varphi(x_j). \tag{8}$$

In this paper, we will utilize the radial basis function (RBF) kernel defined as

$$K(x_i, x_j) = \exp\left(-\frac{\|x_i - x_j\|}{2\sigma^2}\right), \tag{9}$$

then the resulting LS-SVM model for function estimation becomes

$$\hat{m}(x) = \sum_{k=1}^{N} \hat{\alpha}_k K(x, x_k) + \hat{b}. \tag{10}$$

3.2. System model and learning procedure

Chua's circuit is a typical chaotic system when suitable parameters are selected. The chaotic system describes the time evolution of a system in some phase space $\Gamma \subset \boldsymbol{R}^m$. A state is specified by a vector $\boldsymbol{z} \in \boldsymbol{R}^m$. Then we can describe the dynamics by an explicit system of m first-order ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}(t) = f(t, \boldsymbol{z}(t)), \quad t \in \boldsymbol{R}. \tag{11}$$

This is the case in Eq. (1). When discrete time $t = n\Delta t$ is utilized, the above equation can be converted into

$$\boldsymbol{z}_{n+1} = \boldsymbol{F}(\boldsymbol{z}_n). \tag{12}$$

In the case of Chua's circuit, the components of $\boldsymbol{z}$ consist of v_{C_1}, v_{C_2}, and i_L. For a clear expression we denote v_{C_1}, v_{C_2}, and i_L as z_1, z_2, and z_3, respectively, namely vector $\boldsymbol{z} = (z_1, z_2, z_3)$. The goal of this study is to learn the character of function $\boldsymbol{F}$ in Eq. (12). As described in the introduction, some classical approaches regarding $\boldsymbol{F}$ as an MIMO system result in high structural complexity. Based on the reconstruction of a state space,[5] the other methods treating z_1, z_2, and z_3 individually will lose the coupling information between these quantities and suitable values of embedding dimension and time delay are also necessary. In this study, we model $\boldsymbol{F}$ as multiple MISO systems, and the details are shown in Fig. 2.

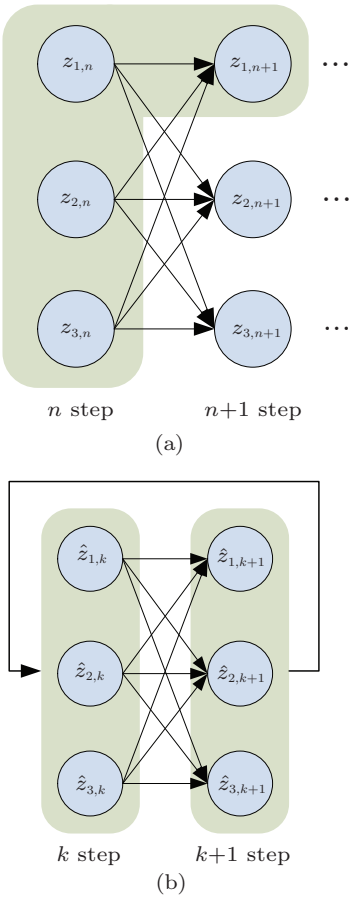


Fig. 2. (a) Training model and (b) reconstructed data generation.

Figure 2(a) shows the training model and the component in the shade shows an MISO system. The training model can be given as a multiple MISO model

$$z_{i,n+1} = F_i(\boldsymbol{z}_n), \quad i = 1, 2, 3; \ \ n = 1, \ldots, N, \tag{13}$$

where N is the number of state, F_i is learned by the LS-SVM and the training set is defined as $D_i = \{(\boldsymbol{z}_n, \boldsymbol{z}_{i,n+1}) : \boldsymbol{z}_n \in R^3, \boldsymbol{z}_{i,n+1} \in R; n = 1, \ldots, N\}$. So F_i is given by Eq. (10), and

$$\begin{aligned}
&\hat{F}_i(\boldsymbol{z}) = \sum_{n=1}^{N} \hat{\alpha}_{i,n} K(\boldsymbol{z}, \boldsymbol{z}_n) + \hat{b}_i, \\
&i = 1, 2, 3; \ \ n = 1, \ldots, N.
\end{aligned} \tag{14}$$

Based on the evaluated $\hat{F}_i$, the reconstructed data can be generated in an autonomous mode (i.e., output feedbacked as input), the structure is shown in Fig. 2(b). With an initial value z_0 (namely v_{C_1}, v_{C_2}, and i_L), the reconstructed data are generated as

$$\begin{aligned}
&\hat{z}_{i,1} = \hat{F}_i\left(\boldsymbol{z}_0\right), \\
&\vdots \\
&\hat{z}_{i,k+1} = \hat{F}_i\left(\hat{\boldsymbol{z}}_k\right), \quad i = 1,2,3, \\
&\vdots \\
&\hat{z}_{i,N'} = \hat{F}_i\left(\hat{\boldsymbol{z}}_{N'-1}\right),
\end{aligned} \tag{15}$$

where N' denotes the number of the reconstructed data. To evaluate the performance of learning, we will compare the parameters of attractor (a solution of Eq. (1) starting from an initial state called an orbit or attractor) of the original and the reconstructed data in a three-dimensional space.

We formulate our algorithm based on the MISO as follows.

(i) Generate the time series $\boldsymbol{z} = (z_1, z_2, z_3)$ (namely v_{C_1}, v_{C_2}, and i_L) based on Eq. (1).

(ii) Build the training data set $D_i = \left\{(\boldsymbol{z}_n, \boldsymbol{z}_{i,n+1}) : \boldsymbol{z}_n \in R^3, \boldsymbol{z}_{i,n+1} \in R; n = 1, \ldots, N\right\}$. The regression model is given by Eq. (13) and solved by the LS-SVM.

(iii) Given the initial value z_0, the new time series can then be generated by Eq. (15).

4. Numerical results

The main objective here is to verify the ability of the proposed approach to learn an autonomous model of Chua's circuit. The RBF kernel is used in the LS-SVM and parameters (γ, σ) are tuned by using the five-fold cross-validation method. Based on the Runge–Kutta integration method, the training data set is obtained by solving Eq. (1) in Matlab with a constant step length of 0.065. The parameters used are $1/C_1 = 9$, $1/C_2 = 1$, $1/L = 7$, $G = 0.7$, $G_{\rm a} = -0.8$, $G_{\rm b} = -0.5$, and $E = 1$, with the initial state being $z_0 = [0.9365 \quad -0.0610 \quad -0.1889]^{\rm T}$. A double-scroll attractor can be obtained with these parameters. The training set or the orbit to be tracked consists of the first 3000 data points (corresponding to samples in the time interval [0, 195]).

For judging the successfulness of the dynamic modeling, some rules such as correlation dimension and Lyapunov exponents are used, since they can describe the global properties of the attractor. The Lyapunov exponents measure the rate at which the nearby orbits converge or diverge. The number of Lyapunov exponents is equal to the dimensions in the state space of the system, and the largest is usually the most important one, since it indicates the chaotic degree of the dynamic system. Formally, the Lyapunov spectrum was defined by Wolf *et al.*[23] in an n-dimensional phase space, the long-term evolution of an infinitesimal sphere is monitored. As the sphere evolves, it will turn into an n ellipsoid.

The difference between strange attractor and purely stochastic (random) process is that the evolution of the points in the phase space of the strange attractor has a definite structure. The correlation integral provides a measure of the spatial organization of this structure. The correlation sum is the estimator of the correlation integral, which reflects the mean probability that the states at two different times are close

$$\begin{aligned}
C\left(r\right) = &\lim_{N\to\infty} \frac{1}{N\left(N-1\right)} \\
&\times \sum_{i\neq j} \Theta\left(r - \|\boldsymbol{s}\left(i\right) - \boldsymbol{s}\left(j\right)\|\right),
\end{aligned} \tag{16}$$

where Θ is the Heaviside function. Grassberger and Procaccia found that for a strange attractor, $C\left(r\right) \propto r^{\nu}$ holds for a limited range of r.[24] The power ν is called the correlation dimension of the attractor. Thus, we can plot the $\log C\left(r\right)$–$\log r$ graph to identify the attractor.

Besides the Lyapunov exponents and the correlation dimension, the power spectrum is a conventional method to analyze the time series and it can indicate if the dynamical system is periodic, quasiperiodic or chaotic.

Figures 3(a) and 3(b) show the attractors generated by the original and the reconstructed data, respectively. It can be seen that the relationship among the state variables reconstructed by the proposed approach mimics a double-scroll attractor and the behavior is in accord with the original ones. Thus, the proposed approach learns the underlying link among the state variables (i.e., a chaotic behavior).

Figure 4 shows the spectra of the original and the reconstructed Chua time series, which are obtained from the z_1 coordinate. Figures 4(a) and 4(b) denote the original and the reconstructed time series of the z_1 coordinate, respectively. Comparing Fig. 4(c) with Fig. 4(d), we can find that the spectrum of the reconstructed time series generated by our method follows very closely that of the original time series, only

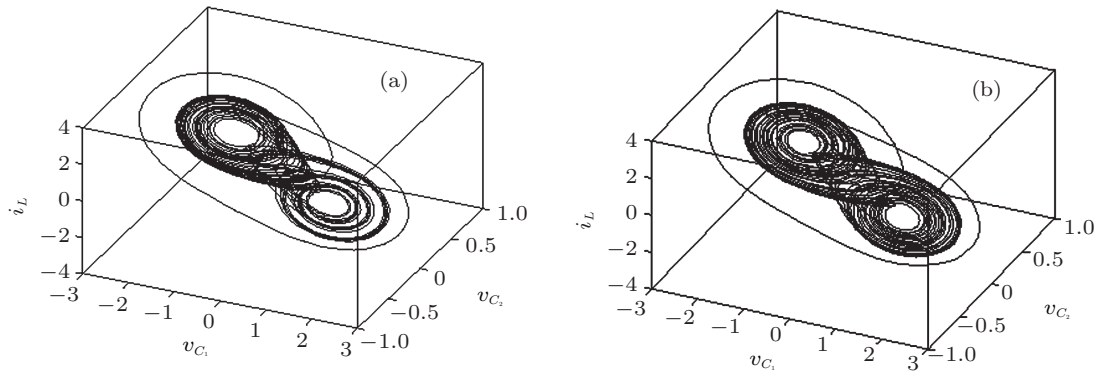


Fig. 3. Attractors of (a) the original system and (b) the reconstructed data.

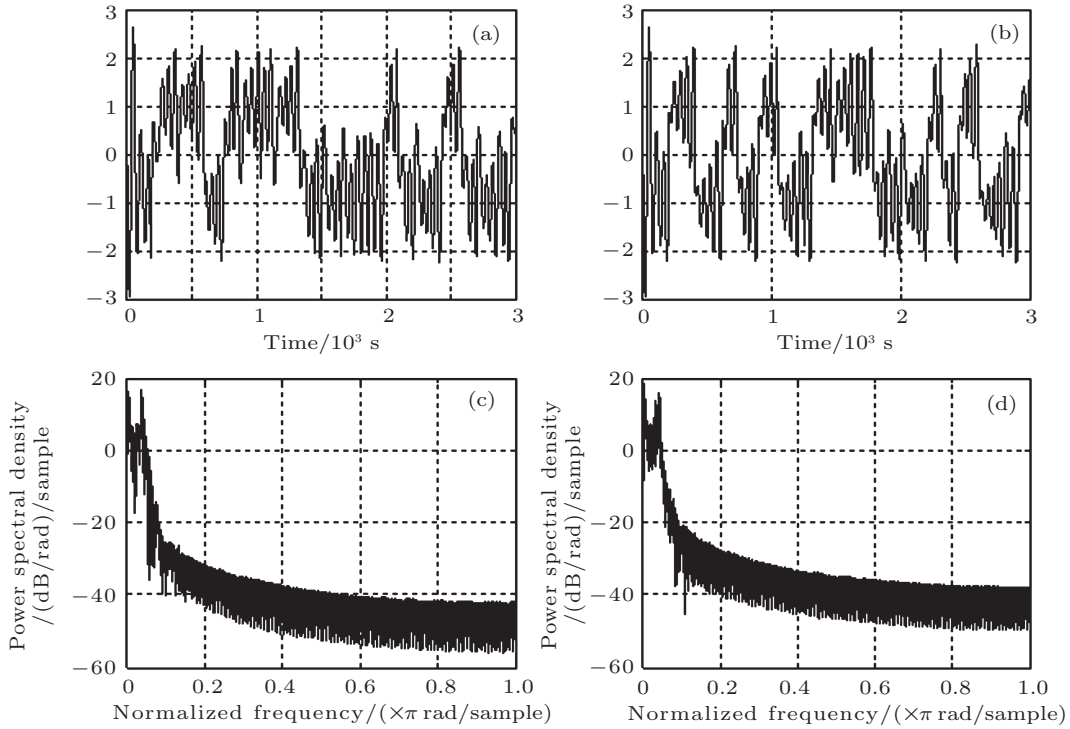


Fig. 4. Power spectra of the original and the reconstructed time series of Chua's circuit. Panels (a) and (b) show the original and the reconstructed time series. Panels (c) and (d) show the power spectra of the time series in panels (a) and (b), respectively.

with minor differences in the detail of the spectrum. Although these plots lead us to believe that the dynamics have been reasonably captured by using our algorithm, for the chaotic time series case, the correlation dimension and the largest Lyapunov exponent are necessary.

In Fig. 5 the correlation integral map (CIM) and its slope are depicted for the original and the reconstructed time series. For the strange attractor, $C(r) \propto r^{\nu}$ holds for a limited range of r. The power ν is called the correlation dimension of the attractor. Thus, the graph $\log C(r)$–$\log r$ can identify the attractor. The value of the correlation dimension is defined as the slope of the CIM curve for at least 3 consecutive embeddings. In this case the correlation dimension for the reconstructed time series is 1.9 in the embedding dimension region of 4–6 while that for the original time series is 1.8 in 5–7.

In Fig. 6, the largest Lyapunov exponent is also computed using Wolf's method. Comparing with 0.0511 of the original time series, the estimated value for the reconstructed time series by our method is 0.0385. Though a small difference of the largest Lyapunov exponent exists between the reconstructed and the original time series, the same order of magnitude is preserved and both estimates stabilize after 1500 steps. Obviously, our method has the ability to learn Chua's circuit.

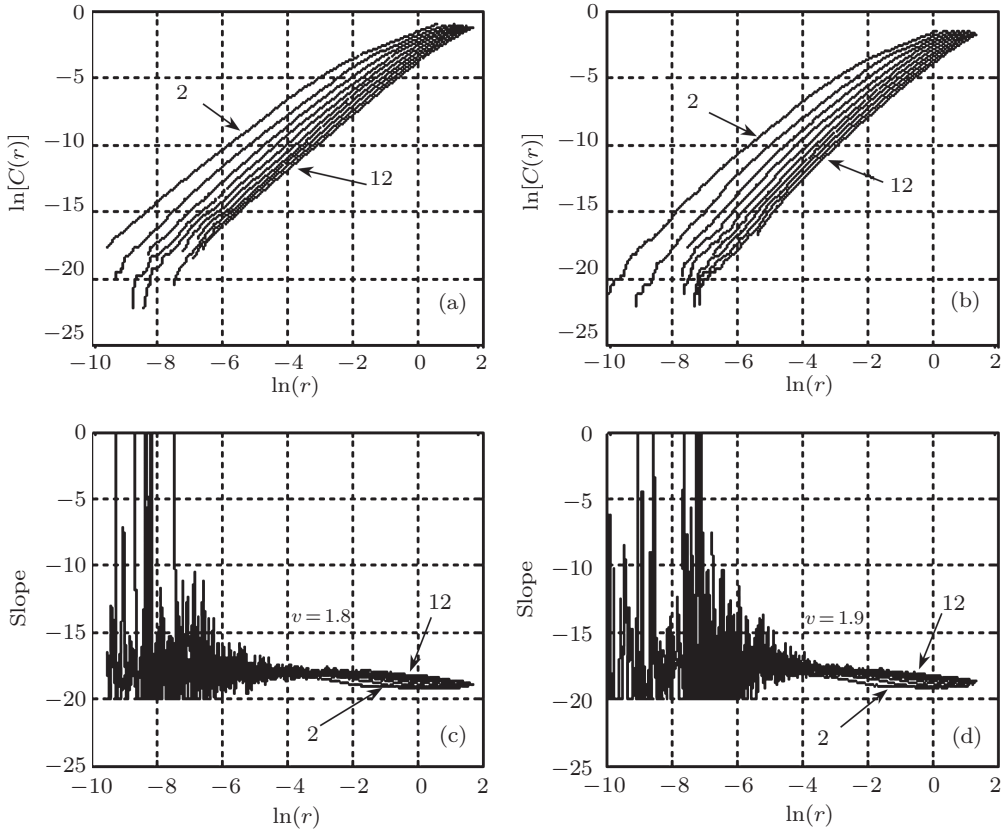


Fig. 5. Correlation dimension estimates for embedding dimension 2–12. Panels (a) and (b) show CIMs for the original and the reconstructed time series, respectively. Panels (c) and (d) show the slope (correlation dimension) estimates of CIM for the original and the reconstructed time series, respectively.

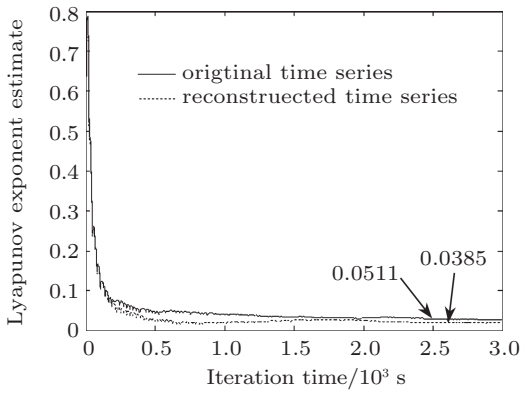


Fig. 6. Estimates of the largest Lyapunov exponent.

Analyzing the above-mentioned dynamical invariants and the state space trajectories, we can conclude that the dynamics of the reconstructed and the original time series are very similar. According to our definition, the dynamics of the system that produce the original time series have been reasonably well captured by our method. Notice that with the MISO and the LS-SVM we obtain an explicit representation of the dynamics.

5. Conclusion

In this paper, an approach has been proposed for learning the chaotic behaivor of Chua's circuit based on the LS-SVM combined with the MISO structure. The MISO structure allows the use of a simplified training structure and the complexity of the algorithm is reduced. In addition, the selection of an embedding dimension and the time delay are avoided. Comparing with the traditional black-box approach, e.g., neural networks, the LS-SVM has a global unique solution and a simple geometric interpretation. Simulation results show that the proposed method is more effective and can be used for analyzing and learning the chaotic behaivor of Chua's circuit.

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