Dynamical behaviors of a system with switches between the Rössler oscillator and Chua circuits*

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The behaviors of a system that alternates between the Rössler oscillator and Chua's circuit is investigated to explore the influence of the switches on the dynamical evolution. Switches related to the state variables are introduced, upon which a typical switching dynamical model is established. Bifurcation sets of the subsystems are derived via analysis of the related equilibrium points, which divide the parameters into several regions corresponding to different types of attractors. The dynamics behave typically in period orbits with the variation of the parameters. The focus/cycle periodic switching phenomenon is explored in detail to present the mechanism of the movement. The period-doubling bifurcation to chaos can be observed via the doubling increase of the turning points related to the switches. Furthermore, period-decreasing sequences have been obtained, which can be explained by the variation of the eigenvalues associated with the equilibrium points of the subsystems.

Keywords: switching dynamical system, bifurcation mechanism, focus/cycle switching, chaotic oscillation

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1. Introduction

Many practical systems, such as electrical circuits,[1] chemical processing,[2] and communication networks,[3] involving switches, can be modeled by a family of continuous-time subsystems with certain switching rules. Generally, two types of switches exist, one corresponds to certain critical conditions associated with the values of the state variables, while the other is related to the fixed time for the occurrence of alteration.[4] Due to the introduction of switching rules, the vector fields that determine the behaviors of the systems may change between two types of flows described by smooth subsystems,[5] leading to some special phenomena, such as grazing bifurcation and multiple-crossing bifurcation.[6] Because of the wide existence of switches, the behaviors of switching systems have received much attention during the last decades, and a lot of results have been reported.[7–9] Bhattacharyya and Mukhopadhyay presented the condition of global stability for an eco-epidemiological model with switches.[10] Li *et al.* dealt with liable stabilization and control scheme problems of a class of switched Lipschitz system.[11] Xiang *et al.* investigated the robust reliable control of the switching neutral system.[12] Sharan and Banerjee derived the nature of the switching map for the case of grazing orbits of power electrical circuits.[13]

Up to now, most of the related works focus on the stability as well as the control schemes of the switching system.[14] Many problems remain unsolved, such as the dynamical evolution with the variation of the parameters, the difference between the behaviors caused by the two types of switches, the bifurcations associated with the switches, and the mechanism of the complexity.

Since critical changes may occur at the switches, implying non-smooth vector field evolution in the switching system though the trajectory remains continuous, the traditional nonlinear theory cannot be directly used to explore the bifurcation behaviors of the switched system, which leads to investigation difficulties. Furthermore, great differences may exist between the dynamics of systems with the two types of switches.

To explore the dynamical behaviors and the mechanism of the switching systems, here we study a switching system alternating between two subsystems described by the Rössler oscillator and Chua's circuit,

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respectively. The switches are defined with the conditions related to the state variables. Some interesting phenomena, such as periodic switching and period-adding sequences, are obtained. Based on the bifurcation analysis of the two subsystems and the critical behaviors at the switches, the mechanism related to the special phenomena observed in the switching system is presented to account for the evolutions of the trajectories.

2. Mathematical model and bifurcations

2.1. Mathematical model

The switching system often behaves in the way that the vector fields alternate between two types of flows distinguished by the special switching rule. Here we consider two oscillators with the special switching rule described by

$$\{\mathrm{d}\boldsymbol{X}/\mathrm{d}t = f_\tau(\boldsymbol{X}), \quad \boldsymbol{X} \in D_\tau, \tau \in \{1,2\}\}, \qquad (1)$$

where $\boldsymbol{X} = (x,y,z)^{\mathrm{T}}$, $f_1(\boldsymbol{X}) = (-y-z, x+\mu y, xz-\sigma z+\eta)^{\mathrm{T}}$, $f_2(\boldsymbol{X}) = (\alpha(y-g(x)),\ x-y+z, -\beta y)^{\mathrm{T}}$, $g(x) = bx + (a-b)(|x+c| - |x-c|)/2$, $D_1 = \{(x,y,z) \in R^3 \,|y > y_{\mathrm{ref}}\}$, $D_2 = \{(x,\ y,z) \in R^3 \,|x < x_{\mathrm{ref}}\}$.

The trajectory of the system starting from the vector field $f_1(\boldsymbol{X})$, i.e., $\tau = 1$, may turn to the flow $f_2(\boldsymbol{X})$ once the first switching condition $S_1 = \{(x,y,z) \in R^3 \,|y = y_{\mathrm{ref}}\}$ is satisfied, i.e., τ changes from 1 to 2, while the trajectory moving according to $f_2(\boldsymbol{X})$ may change back to the vector field $f_1(\boldsymbol{X})$ when the second switching condition $S_2 = \{(x,y,z) \in R^3 \,|x = x_{\mathrm{ref}}\}$ is met, i.e., τ changes back from 2 to 1, which forms the whole switching system.

Obviously, the two subsystems correspond to the typical forms of Rössler and Chua's oscillators, respectively, implying that the vector field of the switching system may alternate between two famous nonlinear oscillators, while the switches are determined by the special conditions described above.

Note that both subsystems may undergo different types of bifurcations, such as Hopf bifurcations and period-doubling bifurcations, with the variations of the parameters,[15] which leads to increased complexity of the subsystems. The combination of the behaviors of the two subsystems may appear in the dynamics of the switching system because of the alternations determined by the switches. Therefore, not only the bifurcation details of the two subsystems but also the dynamics exactly at the switches may influence the dynamical evolution of the switched system. In the following, we will investigate the nonlinear phenomena as well as the bifurcation mechanism of full system (1) with switches.

2.2. Bifurcation analysis of subsystems

For $\eta = 4.0$, the bifurcation sets of the Rössler system are presented in Fig. 1, dividing the parameter plane (μ,σ) into three regions, where in region I, no equilibrium point can be observed. Two equilibrium points denoted by

$$E_{R\pm} = \left(\frac{1}{2}(\sigma \pm \sqrt{\sigma^2 - 16\mu}),\ \frac{1}{2\mu}(\sigma \pm \sqrt{\sigma^2 - 16\mu}),\ \frac{1}{2\mu}(\sigma \pm \sqrt{\sigma^2 - 16\mu})\right) \qquad (2)$$

exist in regions II and III, and they may meet with each other at the set B_{F}, causing the fold bifurcation. The E_{R+} is always unstable, which can be checked by using the characteristic polynomial. While stable E_{R-} may lose its stability at the set B_{H}, leading to a stable periodic oscillation in region III via the Hopf bifurcation.

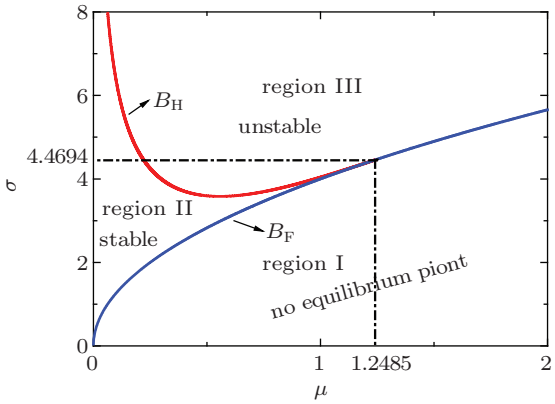


Fig. 1. (colour online) Bifurcation sets on parameter plane (μ,σ).

Now we turn to the local analysis of Chua's oscillator. For $a = 0.3$, $b = -0.15$, $c = 1.0$, two unstable equilibrium points $E_{C\pm} = (\pm 3, 0, \mp 3)$ can be obtained, while the unstable equilibrium point $E_{C0} = (0,0,0)$ in region I shown in Fig. 2 may turn into the stable focus in region II via a simple bifurcation at the set B_{S}.

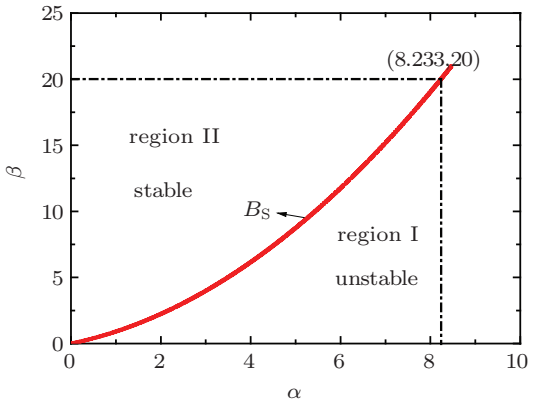


Fig. 2. (colour online) Bifurcation set on parameter plane (α, β).

Because the system may change between two types of vector fields with the switches associated with the values of certain variables, complicated behaviors may occur, and the dynamics may alternate between the states determined by the two oscillators.

Obviously, since the behaviors of the single oscillator may be determined by the parameters taken, different phenomena can be observed for different fixed parameters, while at the switches connecting the two vector fields, qualitative changes may take place on the phase trajectory. To reveal the influence of the switches on the system, we fix the parameters at $\beta = 20.0$, $a = 0.3$, $b = -0.15$, $c = 1.0$, $\mu = 0.398$, $\sigma = 4.0$, $\eta = 4.0$, and take parameter α as the bifurcation parameter to investigate the dynamical evolution of the oscillator. It can be checked that a periodic oscillation can be observed for the Rössler system, while a stable focus exists for Chua's oscillator with the above parameters when $\alpha < 8.233$.

3. Focus/cycle periodic switching behaviors and the bifurcation mechanism

Once the switches described above with $x_{\rm ref} = 0$ and $y_{\rm ref} = -1$ are introduced, not only periodic movements but also chaotic oscillations can be found. Both switching conditions are related to the fixed values of some state variables, which can be easily realized with a certain controlling scheme. Furthermore, in order to cause the switching phenomenon, the switching conditions should be defined that the trajectories of the two subsystems may pass across the switching conditions S_1 and S_2, respectively. Because the initial conditions may influence the structures of the attractors of the switching system, here we fix the initial condition at $(-2.2778,\ 2.6964,\ 0.3049)^{\rm T}$. The bifurcation diagram with the variation of α is plotted in Fig. 3. Note that when α is relatively small, the periodic oscillation can be observed. The regular oscillation and the chaotic movements may alternate with the increase of parameter α. To reveal the details of the dynamical evolution of system, here we focus on several typical regions of parameter α to investigate the behaviors of the switching system.

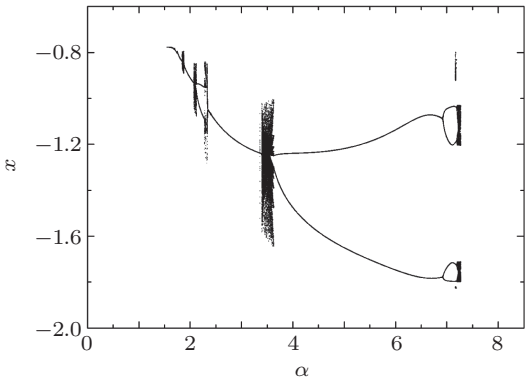


Fig. 3. Bifurcation diagram of the switching system with the variation of parameter α.

3.1. Focus/cycle periodic switching

When $\alpha = 1.6$, periodic oscillation can be observed, as shown in Fig. 4(a). Meanwhile, the overlap of the phase portraits related to the oscillation with the attractors of the two subsystems in the $(x,\ y)$ plane is presented in Fig. 4(b).

The trajectory of the periodic oscillation in Fig. 4(a) alternates between two behaviors associated with the stable focus and the stable limit cycle, which can be demonstrated by the overlap plotted in Fig. 4(b). Two turning points, denoted by A and B, can be observed, which divide the trajectory of the periodic oscillation into two parts. The C is assumed to be the initial point, as shown in Fig. 4(b), the trajectory of the switching system governed by the Rössler may move along CB to point B. If no switch is applied to the system, the trajectory may settle down at the stable limit cycle LC_R, the detail of which is shown by $CBDE$ in Fig. 4(b). However, because the switching condition S_1 is satisfied at point B, the vector field will turn to Chua's oscillator. A qualitative change occurs at point B, causing the trajectory moves along BF instead of settling down at the limit cycle LC_R, which agrees with the phase trajectory determined by Chua's circuit with the initial condition taken at point

B. Another switch can be observed at A, as the condition S_2 is satisfied, which causes the interruption of the trajectory settling down to the stable focus E_{C0}. The switching system then evolves according to the dynamics governed by the Rössler oscillator, implying that the trajectory moves asymptotically to the limit cycle LC_R across point C until point B, at which the switch to Chua's oscillator can be observed. The alternation between two states forms the period oscillation of the switching system, in which two types of switches between stable focus and stable limit cycle can be observed, leading to focus/cycle periodic oscillation with two turning points called 2T-focus/cycle periodic switching.

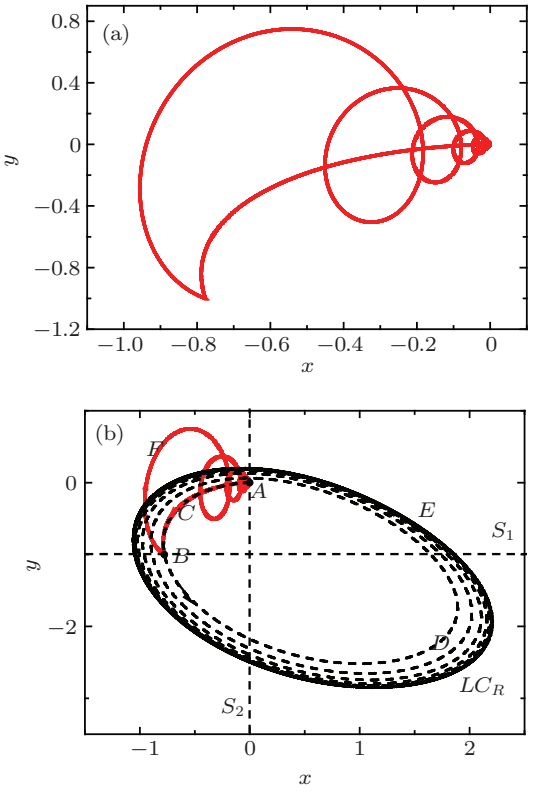


Fig. 4. (colour online) 2T-focus/cycle periodic switching when $\alpha = 1.6$. Panels (a) and (b) show the phase portrait and the overlap of the phase portrait with the attractors of the subsystems, respectively.

3.2. Period-decreasing cascades

With the increasing of parameter α, the distribution of the eigenvalues related to the stable focus determined by Chua's oscillator will change, which leads to the period-decreasing phenomenon in the 2T-focus/cycle periodic switching. The projections of phase portraits for $\alpha = 1.55$ and $\alpha = 1.70$ in the (x, y) plane are presented in Fig. 5, from which we find that the structure of the periodic switching may change slightly, though the switching attractors still alternate between stable focus and stable limit cycle.

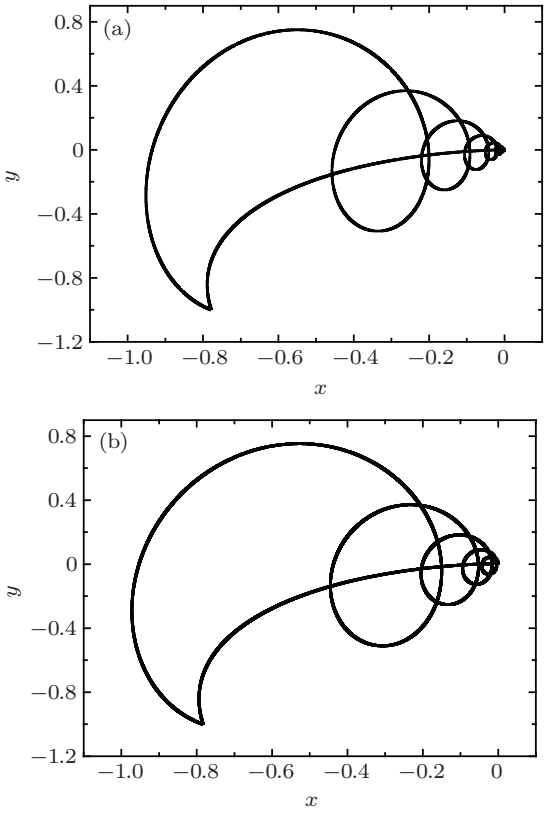


Fig. 5. Phase portraits of periodic switching for (a) $\alpha = 1.55$ and (b) $\alpha = 1.70$.

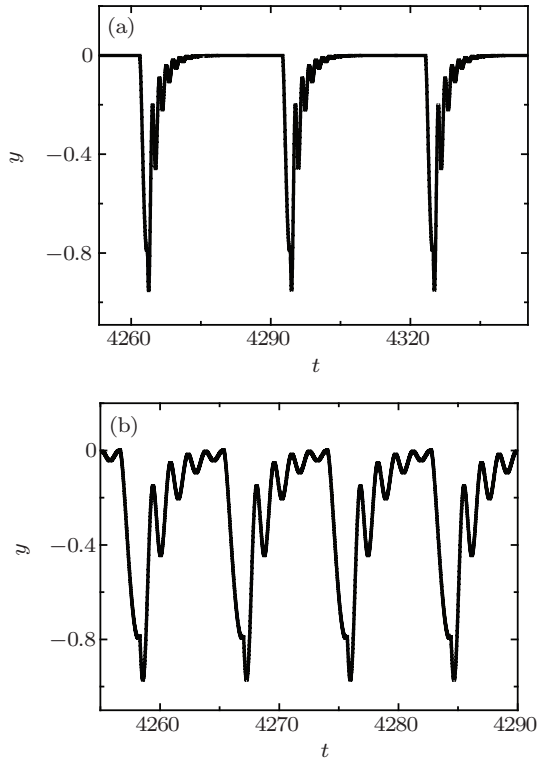


Fig. 6. Time histories for (a) $\alpha = 1.55$ and (b) $\alpha = 1.70$.

Further investigation reveals that the period of the attractor may decrease with the increase of α, see the related time histories for different α in Fig. 6. The period for $\alpha = 1.55$ can be computed to be 30.86, while that for $\alpha = 1.70$ is approximately 9.69, implying that the rapid decrease of the period of the switching attractor may occur with the increase of α.

The phenomenon can be understood by analyzing the eigenvalues associated with the stable focus defined by Chua's system. For $\alpha = 1.55$, the eigenvalues of the stable focus in Chua's oscillator can be easily computed to be $\lambda_1 = -0.5046$, $\lambda_{2,3} = -0.4802 \pm 4.2661\mathrm{i}$, which may change to be $\lambda_1 = -0.5223$, $\lambda_{2,3} = -0.4788 \pm 4.2603\mathrm{i}$ at $\alpha = 1.6$ and $\lambda_1 = -0.5580$, $\lambda_{2,3} = -0.5116 \pm 4.2506\mathrm{i}$ at $\alpha = 1.70$. Note that a relatively rapid increase of the absolute value of the real eigenvalue associated with the focus takes place, for example, α from -0.5046 to -0.5580 with the increase of α from 1.55 to 1.70. This implies that when the system switches to Chua's oscillator, the trajectory may settle down much quickly to the focus with a larger α, leading to the decrease of the time of oscillation in Chua's oscillator. Then the period-decreasing cascades appear in the periodic movement of the system.

3.3. Periodic-doubling bifurcation to chaos

Further increasing parameter α may lead to complicated switching phenomena in the switching system, such as chaos via cascades of period-doubling bifurcations, as presented in Fig. 7. The number of turning points in the orbit of the period switching attractor changes from two to four at $\alpha = 3.654$, and the period-doubling bifurcation occurs, implying that the trajectory alternates between the two subsystems twice during one period of oscillation.

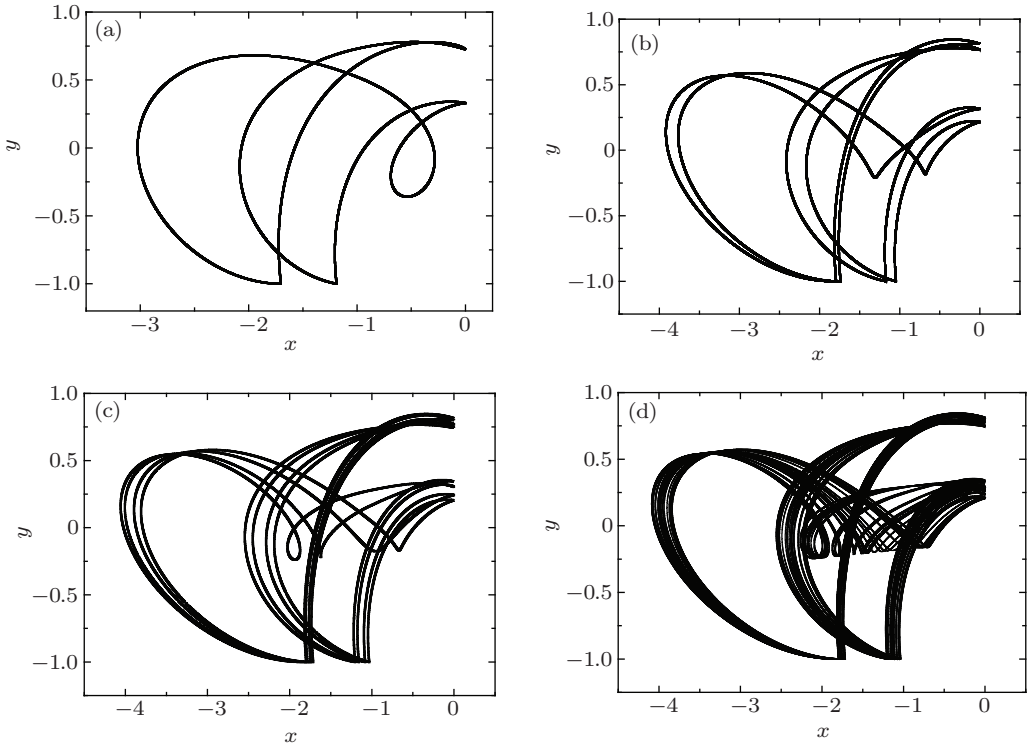


Fig. 7. Periodic switching attractors for (a) $\alpha = 5.5$; (b) $\alpha = 7.0$; (c) $\alpha = 7.21$; (d) $\alpha = 7.25$.

The number of the switching points continues to be doubled with the increase of α, implying that the vector field alternates doubling times between the two subsystems, which leads to the cascades of period-doubling bifurcation and finally to the chaotic oscillation in the switching system.

We now focus on the periodic oscillation with four turning points shown in Fig. 8(a) to explore the mechanism of the movement. Assume that the trajectory of the solution starts from point A in the Rössler system, as presented in Fig. 8(b) for $\alpha = 5.0$. Because of the sole stable limit cycle for the subsystem with the fixed parameters, it may move asymptotically to the limit cycle LC_R. However, the switching condition

S_1 is met during the procedure, leading to switching point A_1, at which the trajectory if further governed by Chua's oscillator, implying that the trajectory may settle down to the stable focus determined by Chua's oscillator with A_1 as the initial point and along A_1B_1 until the other switching condition S_2 is satisfied. Condition S_2 may cause another turning point in the orbit, denoted as B_1. Obviously, the trajectory may try to settle down at the stable limit cycle asymptotically until condition S_1 is satisfied. Note that both switching conditions S_1 and S_2 are only transversal planes in the cubic space, which may cause different locations of the switching points. Instead of the same turning points between the two subsystems, when $\alpha = [3.654,\ 6.976]$, another switching point A_2 from the Rössler system to Chua's oscillator occurs. Though the trajectory still asymptotically approaches the stable focus, it moves along with another path, i.e., A_2B_2, which is still governed by Chua's oscillator. The difference between A_1B_1 and A_2B_2 lies in the different initial points A_1 and A_2 in Chua's oscillator. A new switching point B_2 appears when S_2 is satisfied again, at which the trajectory turns to be governed by the Rössler system along the path B_2A to the starting point A, forming the period-two orbit with four turning points.

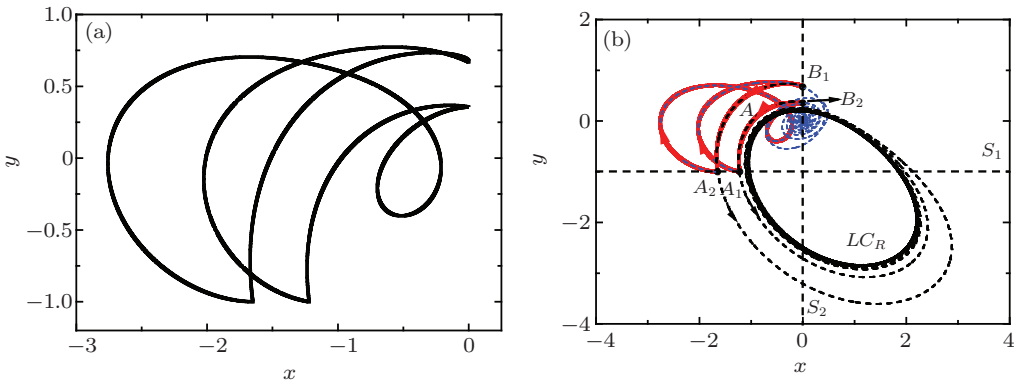


Fig. 8. (colour online) 2T-focus/cycle periodic switching for $\alpha = 5.0$. Panels (a) and (b) show the phase portrait and the overlap of the phase portrait with the attractors of the subsystems, respectively.

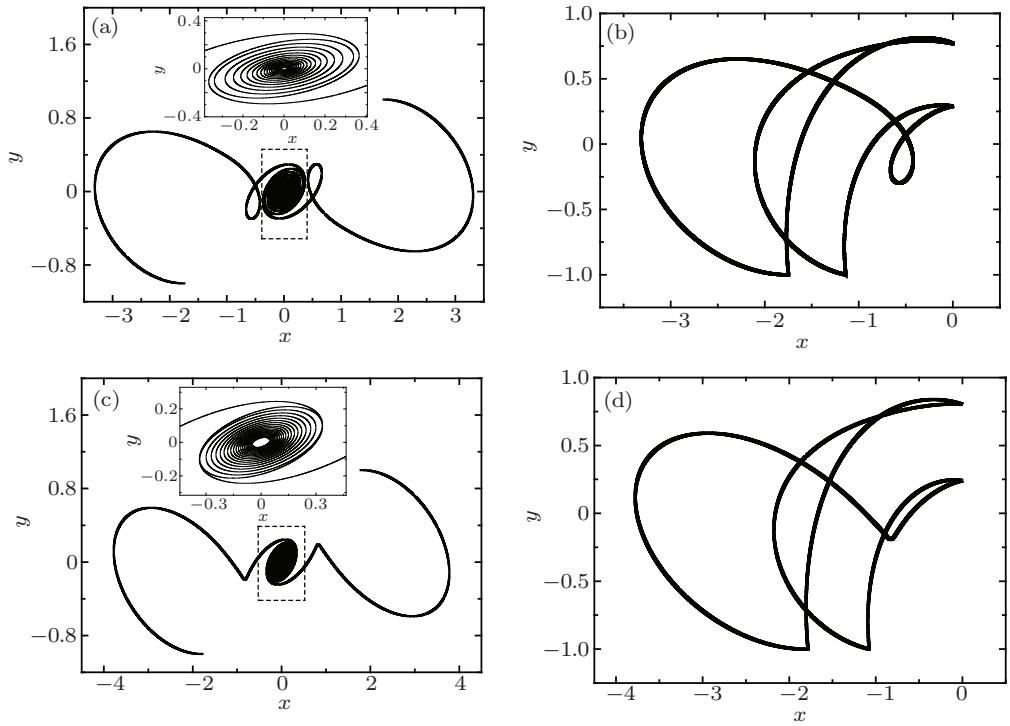


Fig. 9. The 4T-focus/cycle periodic switching attractors, where panels (a) and (b) correspond to $\alpha = 6.0$, and panels (c) and (d) are related to $\alpha = 6.8$. Panels (a) and (c) show the overlap of the phase portrait with the trajectories to the attractors of the subsystems phase portrait, while panels (b) and (d) present the phase portraits of the oscillations.

It is easy to find that the numbers of the turning points determined by S_1 and S_2 are the same, because each path between the two turning points defined by S_1 and S_2 forms a connection between the two subsystems. Furthermore, when the number of the turning points determined by S_1 or S_2 with doubling cascades tends to infinity, the chaotic switching phenomenon takes place, though only regular stable attractors can be observed in the two subsystems, and the trajectory of the chaotic oscillation moves alternatively between the two types of paths.

Since the parameter may influence the dynamics of the subsystems, the format of the switching attractor may change with the variation of the parameter. For the two qualitatively same switching attractor governed by Chua's oscillator, because there exist two unstable focuses $E_{C\pm} = (\pm 3, 0, \mp 3)$, the trajectory may scroll down to the stable focus with the special path corresponding to the starting point at the switching attractor. For example, for $\alpha = 6.0$, the trajectory related to the equilibrium point plotted in Fig. 9(a) may lead to the periodic switching attractor in Fig. 9(b), while for $\alpha = 6.8$, the trajectories in Fig. 9(c) may result in the periodic oscillation in Fig. 9(d).

4. Conclusion

Switches between different subsystems may lead to combined movements, the trajectory of which can be divided into parts determined by the subsystems respectively, while at the turning points related to the switches, obvious non-smooth behaviors can be observed, resulting in the qualitatively change of the orbit. For the fixed parameters, the attractors of the subsystems can be obtained accordingly, the transient process may influence the phase portraits of the whole system associated with the switching rule introduced. With the variation of the parameters, the doubling increase of the number of the turning points on the trajectory can be found, implying the alternating times between the two subsystems may be doubled during one period of oscillation, which may cause the sequence of period-doubling bifurcation to chaos in the switching system. Furthermore, the distribution of eigenvelues related to the attractors of the subsystems may be affected by the parameters taken, which may lead to the period-adding or period-decreasing cascades in the dynamical evolution of the switching system.

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