Cosmological application on five-dimensional teleparallel theory equivalent to general relativity

Gamal G. L. Nashed[a)b)c)†]

[a)] *Mathematics Department, Faculty of Science, King Faisal University, P. O. Box* 380, *Al-Ahsaa* 31982, *the Kingdom of Saudi Arabia*
[b)] *Mathematics Department, Faculty of Science, Ain Shams University, Cairo* 11566, *Egypt*
[c)] *Center for Theoretical Physics, British University of Egypt, Sherouk City* 11837, *P. O. Box* 43, *Egypt*

(Received 8 February 2012; revised manuscript received 8 April 2012)

A theory of (4+1)-dimensional gravity has been developed on the basis of which equivalent to the theory of general relativity by teleparallel. The fundamental gravitational field variables are the 5-dimensional (5D) vector fields (pentad), defined globally on a manifold M, and gravity is attributed to the torsion. The Lagrangian density is quadratic in the torsion tensor. We then apply the field equations to two different homogenous and isotropic geometric structures which give the same line element, i.e., FRW in five dimensions. The cosmological parameters are calculated and some cosmological problems are discussed.

Keywords: 5D teleparallel equivalent of general relativity, 5D solutions, cosmological parameters, cosmological problems

PACS: 04.20.Jb, 04.50.–h, 04.50.Gh, 04.50.Kd **DOI:** 10.1088/1674-1056/21/10/100401

1. Introduction

In recent years an increasing studies have been directed to the understanding of black holes in higher dimensional gravity[1−8] for various reasons. Among them, the most important ones stem from the motivation of string and membrane theories, within the framework of string theory, which is believed to be a major candidate for the quantum theory of gravity and the unification of all interactions, black holes in higher dimensions are consistent with its mathematical requirement of higher dimensions. They may not only play a major role in the study of dynamics in higher dimensions but also help in understanding the compactification mechanisms. In particular, microscopic black holes may be useful to test some novel predictions of the string theory. On the other hand, in the scenario of the world brane, if assuming that there is the presence of extra dimensions in the universe holds true, it is possible to produce a higher dimensional small black holes in the future high energy colliders or observe them in the universe.[4,9,10]

In recent years increasing has been focused on the possibility that the universe may in fact be a higher-dimensional one than the usual 4-dimensional (4D) Einstein one. In the eighties, after the revival of Kaluza–Klein theory, some authors began to investigate the one-body problem in the context of five-dimensional $(4 + 1)$ general relativity.[11−13] An interesting extension of the $(3 + 1)$-dimensional Schwarzschild–de Sitter solution to five dimensions was then conducted by Wesson and coworkers.[14,15] In their analysis of the one-body problem they dealt with the issue of how an extra dimension would affect the classical tests of general relativity like the perihelion advance of Mercury, the gravitational red shift effect, and the deflection of light rays by the Sun. By comparing the geodesic motions in the corresponding $(4 + 1)$ and $(3 + 1)$-dimensional spacetimes they concluded that as far as the classical tests of general relativity are concerned there is no facility to differentiate between the 5-dimensional (5D) and 4D descriptions.[14,15] This conclusion, however, based on the assumption that the metric of the 5D embedding space has a certain (canonical) specific form, which allows the 5D and 4D geodesics to coincide.

Based on the above facts, it becomes clear that the study of finding exact solutions in higher dimensional gravity is of great importance. However, it is not an easy task to find solutions in higher dimensional general relativity, to be exact, compared with the

†Corresponding author. E-mail: nashed@bue.edu.eg, Egyptian Relativity Group (ERG) URL: http://www.erg.eg.net
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case of four dimensions, where several systematic techniques to generate stationary and axisymmetric solutions have been developed by some authors[16−18] and a lot of interesting solutions are presented, such as the unique rotating Kerr black hole for a given mass and angular momentum[19] and the charged Kerr–Newman black hole.[20] So far, there are still few known exact higher dimensional solutions in general relativity. The first exact rotating one of vacuum Einstein field equations was given by Myers and Perry (MP) in 1986.[1] This solution is a generalization of higher-dimensional Kerr black hole and has a spherical horizon topology. However, in the case of the 5D pure gravity, MP black hole is not unique. For a given mass and angular momentum, Emparan and Reall discovered a black ring solution in vacuum, whose horizon topology is $S_1 \times S_2$.[2] Then another general regular black ring solution with two angular momenta was also found.[21] Most recently, in Ref. [22], the authors presented a new 5D black hole solution with the horizon of distorted lens spaces.

Teleparallel theories are interesting for several reasons: first of all, general relativity (GR) can be viewed as a particular theory of teleparallelism and, therefore, teleparallelism can be considered, at the very least, as a different point of view that can lead to the same results.[23] Second, in this context, one can define an energy–momentum complex for the gravitational field which is a true tensor under general coordinate transformations, but ***not under local Lorentz transformation***. This is why teleparallelism was reconsidered by Möller when he was studying the problem of defining an energy–momentum tensor for the gravitational field.[24,25] The idea has been taken over by Pellegrini and Plebański who constructed the general Lagrangian for these theories.[26] The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full Poincaré group) and, therefore, they give an alternative explanation of GR.[27−34]

The objective of this work is to find a higher dimensional solution in teleparallel general relativity (TEGR). In Section 2, a brief review of teleparallel spacetime in five dimensions is offered. The Lagrangian and the field equations are given in Section 3. In Section 4, two different pentad fields having homogeneity and isotropy are given. An application of the pentads to the field equations is given. The cosmological models and cosmological parameters of the resulting models are achieved in Section 5. In Section 6, some cosmological problems are discussed and a comparison of the present results with observations is done. Final section is devoted to the main result.

2. Pentad, covariant deravative, and teleparallelism

The 5D spacetime M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ $(\mu,\nu = 0,1,2,3,4)$ related to the fields $\boldsymbol{h}^k = h^k{}_\mu \mathrm{d}x^\mu$, $(k = 0,1,2,3,4)$ through the relation $g_{\mu\nu} = \eta_{ab}h^a{}_\mu h^b{}_\nu$ with $\eta_{ij} = (-1,+1,+1,+1,+1)$. Here x^μ $\{\mu = 0,1,2,3,4\}$ is a local coordinate of the spacetime. The fields $\boldsymbol{h}_k = h^\mu{}_k \partial/\partial x^\mu$, which are the dual to $\boldsymbol{h}^k$, are the dreibein fields. The field strength of $h^k{}_\mu$ is given by[1)]

$$T^k{}_{\mu\nu} = \Gamma^k{}_{[\mu\nu]}, \quad \text{with} \quad \Gamma^k{}_{\mu\nu} = \partial_\mu h^k{}_\nu. \tag{1}$$

We define the covariant derivative of the Lorentzian vector field V^k by

$$\nabla_l V^k \overset{\text{def.}}{=\!=} h^\mu{}_l \partial_\mu V^k. \tag{2}$$

For the world vector fields $\boldsymbol{V} = V^\mu \partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma^\mu{}_{\nu\lambda}$ is defined by

$$\nabla_\nu V^\mu \overset{\text{def.}}{=\!=} \partial_\nu V^\mu + \Gamma^\mu{}_{\lambda\nu} V^\lambda. \tag{3}$$

The requirement

$$\nabla_l V^k = h^\nu{}_l h^k{}_\mu \nabla_\nu V^\mu \tag{4}$$

for $V^\mu = h^\mu{}_k V^k$ leads to

$$\Gamma^\mu{}_{\lambda\nu} \overset{\text{def.}}{=\!=} h^\mu{}_k \partial_\nu h^k{}_\lambda,$$

and hence we have the relation[35]

$$T^k{}_{\mu\nu} \overset{\text{def.}}{=\!=} h^k{}_\lambda T^\lambda{}_{\mu\nu},$$

$$R^\mu{}_{\nu\lambda\rho} \overset{\text{def.}}{=\!=} \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\tau\lambda}\Gamma^\tau{}_{\nu\rho} - \Gamma^\mu{}_{\tau\rho}\Gamma^\tau{}_{\nu\lambda} \equiv 0, \tag{5}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \equiv 0, \tag{6}$$

where $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}$. The components $T^\lambda{}_{\mu\nu}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion and of the curvature tensor, respectively. Equation (5) implies teleparallelism, and it together with Eq. (6) means that M

[1)] We will denote the symmetric part by ( ), for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$.

is the Weitzenböck spacetime. Also from Eq. (6) we obtain

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\} + K^{\lambda}{}_{\mu\nu} \tag{7}$$

with

$$\left\{{}^{\lambda}_{\mu\nu}\right\} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}\right),$$

where $K^{\lambda}{}_{\mu\nu}$ stands for the contortion tensor,

$$K^{\lambda}{}_{\mu\nu} = -\frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}\right) = -K_{\mu}{}^{\lambda}{}_{\nu}. \tag{8}$$

In what follows, the field components $h^k{}_\mu$ and $h^\mu{}_k$ are used to convert Latin and Greek indices, e.g., V^μ and $V^k(= h^k{}_\mu V^\mu)$. Also, rasing and lowering the indices $k, l, m, \ldots$ are accomplished with the aid of η^{ab}, $\eta_{ab} = (\eta^{ab})^{-1}$ respectively.

3. The Lagrangian density and the gravitation field equation

In this paper we follow the notions given in Refs. [36]–[41]. The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[2)]

$$\begin{aligned}\mathscr{L}_{\rm G} &= hL_{\rm G}\\ &= -\frac{h}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^aT_a\right) - L_{\rm m}\\ &= -\frac{h}{16\pi}\Sigma^{abc}T_{abc} - L_{\rm m},\end{aligned} \tag{9}$$

where $h = \det(h^a{}_\mu)$ and $L_{\rm m}$ is the matter Lagrangian. The tensor Σ^{abc} is defined by

$$\begin{aligned}\Sigma^{abc} \overset{\text{def.}}{=\!=} &\frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right)\\ &+ \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right).\end{aligned} \tag{10}$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$\begin{aligned}&T^a{}_{\mu\nu} \overset{\text{def.}}{=\!=} h^a{}_\lambda T^\lambda{}_{\mu\nu} = \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu,\\ &T^a{}_{bc} \overset{\text{def.}}{=\!=} h_b{}^\mu h_c{}^\nu T^a{}_{\mu\nu}, \qquad T^a \overset{\text{def.}}{=\!=} T^b{}_b{}^a.\end{aligned} \tag{11}$$

The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(h)$, except for a total divergence term.[42–48] $L_{\rm m}$ represents the Lagrangian density for matter fields.

The gravitational field equations for the system described by $L_{\rm G}$ are the following

$$\begin{aligned}&h_{a\lambda}h_{b\mu}\partial_\nu\left(h\Sigma^{b\lambda\nu}\right) - h\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}h_{a\mu}T_{bcd}\Sigma^{bcd}\right)\\ &= \frac{1}{2}\kappa h T_{a\mu},\end{aligned} \tag{12}$$

where

$$\frac{\delta L_{\rm m}}{\delta h^{a\mu}} \equiv hT_{a\mu}.$$

It is possible to prove by explicit calculations that the left-hand side of the symmetric field equations of Eq. (12) is exactly given by

$$\frac{h}{2}\left[R_{a\mu}(h) - \frac{1}{2}h_{a\mu}R(h)\right].$$

4. Homogenous and isotropic solution

Let us write the two pentad fields (written in the spherical coordinate, i.e., $x_0 = t$, $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, and $x_4 = \psi$) as

$$(h_i{}^\mu)_{\rm I} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & \dfrac{L_1\sin\theta\cos\phi}{R(t)} & \dfrac{L_1\cos\theta\cos\phi - r\sqrt{k}\sin\phi}{rR(t)} & -\dfrac{L_1\sin\phi + r\sqrt{k}\cos\theta\cos\phi}{rR(t)\sin\theta} & 0\\ 0 & \dfrac{L_1\sin\theta\sin\phi}{R(t)} & \dfrac{L_1\cos\theta\sin\phi + r\sqrt{k}\cos\phi}{rR(t)} & \dfrac{L_1\cos\phi - r\sqrt{k}\cos\theta\sin\phi}{rR(t)\sin\theta} & 0\\ 0 & \dfrac{L_1\cos\theta}{R(t)} & \dfrac{-L_1\sin\theta}{rR(t)} & \dfrac{\sqrt{k}}{R(t)} & 0\\ 0 & 0 & 0 & 0 & \dfrac{1}{L_1R(t)} \end{pmatrix}, \tag{13}$$

and

[2)] Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$.

$$(h_i{}^{\mu})_{\rm II} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{{L_1}^2 \sin\theta\cos\phi}{R(t)} & \dfrac{\cos\theta\cos\phi}{rR(t)} & -\dfrac{\sin\phi}{rR(t)\sin\theta} & \dfrac{r\sqrt{k}\sin\theta\cos\phi}{L_1 R(t)} \\ 0 & \dfrac{{L_1}^2 \sin\theta\sin\phi}{R(t)} & \dfrac{\cos\theta\sin\phi}{rR(t)} & \dfrac{\cos\phi}{rR(t)\sin\theta} & \dfrac{r\sqrt{k}\sin\theta\sin\phi}{L_1 R(t)} \\ 0 & \dfrac{{L_1}^2\cos\theta}{R(t)} & \dfrac{-\sin\theta}{rR(t)} & 0 & \dfrac{r\sqrt{k}\cos\theta}{L_1 R(t)} \\ 0 & \dfrac{r\sqrt{k}L_1}{R(t)} & 0 & 0 & \dfrac{1}{R(t)} \end{pmatrix}, \tag{14}$$

where L_1 is defined as

$$L_1 = \sqrt{1-kr^2} \tag{15}$$

with k being the curvature parameter ($k = -1, 0, +1$) corresponding to closed, flat, and open universes and $R(t)$ is the scale factor. The line element of the above two pentad fields have the same form and is given by

$$\begin{aligned} {\rm d}s^2 = &-{\rm d}t^2 + R^2(t)\Big[\frac{{\rm d}r^2}{1-kr^2} + r^2\left({\rm d}\theta^2 + \sin^2\theta{\rm d}\phi^2\right) \\ &+ (1-kr^2){\rm d}\psi^2\Big], \end{aligned} \tag{16}$$

which is the FRW line-element in five dimensions.[49]

Assuming that the matter filling the universe is in the form of a perfect fluid, then

$$T_0{}^0 = \rho, \qquad T_1{}^1 = T_2{}^2 = T_3{}^3 = T_4{}^4 = -P, \tag{17}$$

where $\rho = \rho(t)$ is the density and $P = P(t)$ is the pressure. Applying Eq. (13) or Eq. (14) to the field equation (12) we obtain

$$8\pi\rho = 6\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right], \tag{18}$$

$$8\pi P = -3\left[\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2}\right] \tag{19}$$

with $\dot{R} = \partial R(t)/\partial t$.

Now we have two differential equations in three unknown functions $R(t)$, $\rho(t)$, and $P(t)$. Therefore we need an extra condition to be able to solve Eqs. (18) and (19) for each value of k. This extra condition is usually provided by an equation of state (EoS) characterizing the fluid filling the model. Here we consider the EoS has the form

$$P = (\omega - 1)\rho, \tag{20}$$

where $\omega \in [0, 2]$. When $\omega = 0$, then $P = -\rho$ (dark energy case), $\omega = 1$, then $P = 0$ (dust case), $\omega = 4/3$ then $P = \rho/3$ (radiation case), and when $\omega = 2$ then $P = \rho$ (stiff matter case). The conservation law (continuity equation) has the form $T_{\mu}{}^{\nu}{}_{;\,\nu} = 0$. This will give rise to the relation

$$\dot{\rho} + 4\frac{\dot{R}(t)}{R(t)}(\rho + P) = 0. \tag{21}$$

Equation (18), (19), and (21) are three nonlinear partial differential equations in three unknowns. Solving Eq. (21), we obtain

$$\rho(t) = \frac{c_1}{R(t)^{4\omega}}, \tag{22}$$

where the EoS has been used and c_1 is a constant of integration. The constant c_1 can be determined by using $\omega = \omega_0$ and $\rho = \rho_c$ (critical density) at time $t = t_0$, where the subscript 0 refers to the present value. We will discard the case $\omega = 0$ since this will make the density parameter, given by Eq. (22), always constant, in spite of the expansion of the universe. This needs an extensive discussion which will be done in the future work.

Now we are going to give the definitions of some cosmological parameters

$$\begin{aligned} &\text{The Hubble's parameter } H_0 = \frac{\dot{R}(t_0)}{R(t_0)} = \frac{\dot{R}_0}{R_0}; \quad \text{The Hubble's time } \tau_0 \overset{\text{def.}}{=\!=} {H^{-1}}_0; \\ &\text{The deceleration parameter } q_0 \overset{\text{def.}}{=\!=} -\frac{\ddot{R}_0}{R_0}{H^{-2}}_0 = -1 - \frac{\dot{H}_0}{{H_0}^2}; \quad \text{The density parameter } \sigma_0 \overset{\text{def.}}{=\!=} \frac{\rho_0}{\rho_{\rm c}}, \\ &\text{where } \rho_{\rm c} \text{ is the critical density defined by } \rho_{\rm c} \overset{\text{def.}}{=\!=} \frac{3{H^2}_0}{8\pi G}; \\ &\text{The state-finder pair is defined by}[50]\ r \overset{\text{def.}}{=\!=} \frac{\dddot{a}}{aH^3},\ s = \frac{r-1}{3(q-1/2)}. \end{aligned} \tag{23}$$

5. Models resulting from the above pentad fields

We summarize the solutions of equations (18) and (19) using equation (20) into the following table (Table 1).

Table 1. Comparison between the results of scale factors $R(t)$ for $\omega = 1$, or $4/3$, or 2 and for different choices of k.

Curvature constant	Dust case $\omega = 1$	Radiation case $\omega = 4/3$	Stiff matter case $\omega = 2$
$k = -1$	$R(t) = \sqrt{t^2 - 2c_2 t + 2c_3}$	$\int_0^{R(t)} \frac{x^{10/3}}{\sqrt{x^{10/3}\left(x^{10/3} + c_4\right)}} \mathrm{d}x - t - c_5$	$\int_0^{R(t)} \frac{x^3}{\sqrt{x^6 + c_6}} \mathrm{d}x - t - c_7$
$k = 0$	$R(t) = \sqrt{2tc_8 + 2c_9}$	$R(t) = \frac{\sqrt{3}(216tc_{10} + 216c_{11})^{3/8}}{9}$	$R(t) = (4tc_{12} + 4c_{13})^{1/4}$
$k = +1$	$R(t) = \sqrt{2c_{14} - t^2 - 2tc_{15}}$	$\int_0^{R(t)} \frac{x^{10/3}}{\sqrt{x^{10/3}\left(c_{16} - x^{10/3}\right)}} \mathrm{d}x - t - c_{17}$	$\int_0^{R(t)} \frac{x^3}{\sqrt{c_{18} - x^6}} \mathrm{d}x - t - c_{19}$

The asymptotic values of the scale factor given in Table 1 is plotted in Figs. 1, 2, and 3.

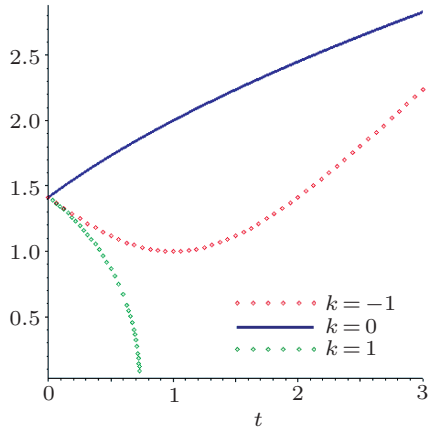


Fig. 1. (colour online) The scale factor $R(t)$ of dust case for different values of k when $c_2 = 1$, $c_3 = 1$, $c_8 = 1$, $c_9 = 1$, $c_{14} = 1$, and $c_{15} = 1$.

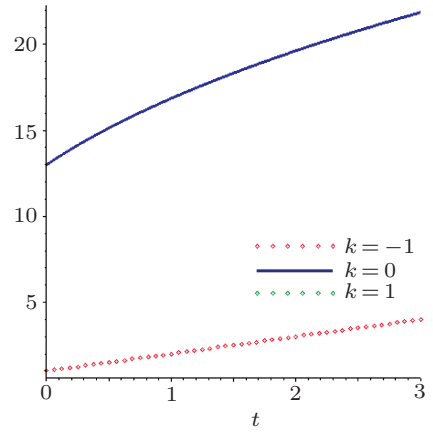


Fig. 2. (colour online) The scale factor $R(t)$ of radiation case for different values of k when $c_4 = 1$, $c_5 = 1$, $c_{10} = 1$, $c_{11} = 1$, $c_{16} = 1$, and $c_{17} = 1$.

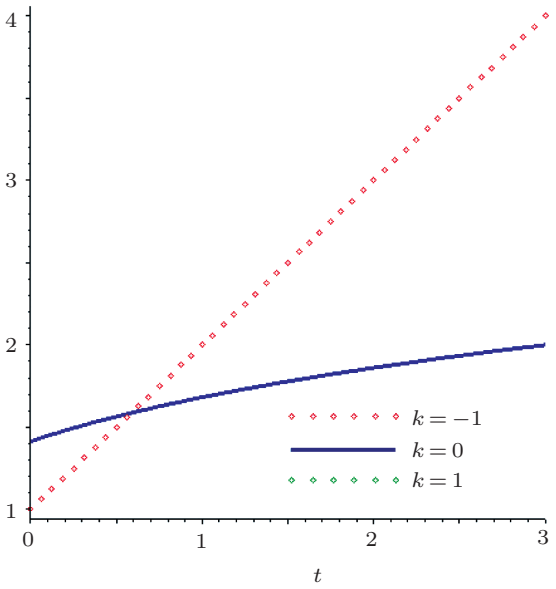


Fig. 3. (colour online) The scale factor $R(t)$ of stiff matter case for different values of k when $c_6 = 1$, $c_7 = 1$, $c_{12} = 1$, $c_{13} = 1$, $c_{18} = 1$, and $c_{19} = 1$.

It is of interest to note that the behavior of the scale factor of the radiation and stiff cases when $k = \pm$ are drawn for the asymptotic value up to order $O(1/t)$.

The calculated values of Hubble parameter are given in the following table (Table 2).

The asymptotic behavior of the dust case is given in Fig. 4. Using the values of Hubble parameter given in Table 2 for the dust case, the asymptote behavior of the deceleration parameter are shown in Fig. 5.

Table 2. Comparison between the results of Hubble parameters $H(t)$ for $\omega = 1$ or 4/3 or 2 and for different choices of k.

Curvature constant	Dust case $\omega = 1$	Radiation case $\omega = 4/3$	Stiff matter case $\omega = 2$
$k = -1$	$H(t) = \dfrac{t - c_2}{t^2 - 2tc_2 - 2c_3}$	$H(t) = \dfrac{\dot{R}(t)R(t)^{5/3} - \sqrt{R(t)^{10/3} + c_{12}}}{\sqrt{R(t)^{10/3} + c_{12}}\left(\displaystyle\int_0^{R(t)} \frac{x^{5/3}}{x^{5/3} + c_{12}}\,\mathrm{d}x - t - c_{13}\right)}$	$H(t) = \dfrac{\dot{R}(t)R(t)^3 - \sqrt{R(t)^6 + c_{14}}}{\sqrt{R(t)^6 + c_{14}}\left(\displaystyle\int_0^{R(t)} \frac{x^3}{x^6 + c_{12}}\,\mathrm{d}x - t - c_{13}\right)}$
$k = 0$	$H(t) = \dfrac{c_4}{2(tc_4 + c_5)}$	$H(t) = \dfrac{3c_16}{8(tc_{16} + c_{17})}$	$H(t) = \dfrac{c_{18}}{4(tc_{18} + c_{19})}$
$k = +1$	$H(t) = \dfrac{t + c_8}{(t^2 + 2tc_8 - 2c_9)}$	$H(t) = \dfrac{\dot{R}(t)R(t)^{5/3} - \sqrt{c_{20} - R(t)^{10/3}}}{\sqrt{c_{20} - R(t)^{10/3}}\left(\displaystyle\int_0^{R(t)} \frac{x^{5/3}}{\sqrt{c_{20} - R(t)^{10/3}}}\,\mathrm{d}x - t - c_{13}\right)}$	$H(t) = \dfrac{\dot{R}(t)R(t)^3 - \sqrt{c_{22} - R(t)^6}}{\sqrt{c_{22} - R(t)^6}\left(\displaystyle\int_0^{R(t)} \frac{x^{5/3}}{\sqrt{c_{22} - R(t)^6}}\,\mathrm{d}x - t - c_{23}\right)}$

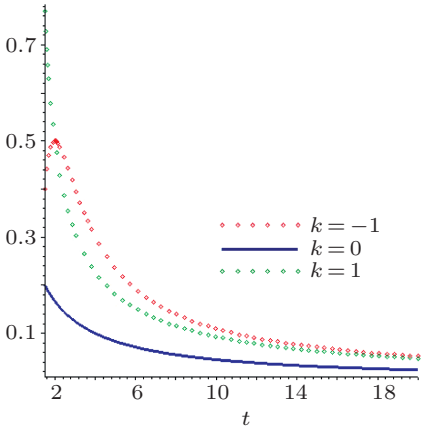


Fig. 4. (colour online) The Hubble parameter $H(t)$ of dust case for different values of k when $c_2 = 1$, $c_3 = 1$, $c_4 = 1$, $c_5 = 1$, $c_8 = 1$, and $c_9 = 1$.

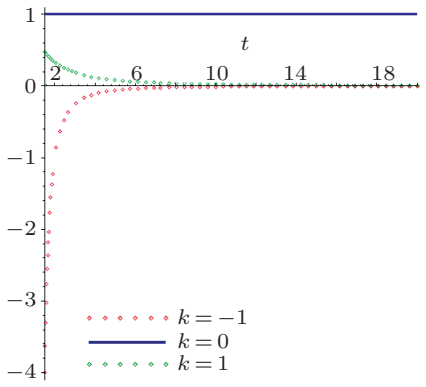


Fig. 5. (colour online) The deceleration parameter $q(t)$ of dust case for different values of k when $c_2 = 1$, $c_3 = 1$, $c_4 = 1$, $c_5 = 1$, $c_8 = 1$, $c_9 = 1$.

6. Physical properties of the derived models

(i) Red shift

Consider an observer O, located at the origin of space coordinate and a galaxy S at an arbitrary point with spatial coordinates (r, θ, ϕ, ψ). According to Eq. (16), the velocity of a radial ray of light $(\mathrm{d}s = 0, \mathrm{d}\theta = 0, \mathrm{d}\phi = 0, \mathrm{d}\psi = 0)$ is given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\frac{\sqrt{1 - kr^2}}{R(t)}. \tag{24}$$

Hence if a light pulse is emitted by the galaxy S at the instant t_1, is received by O at the instant t_0 $(t_0 > t_1)$, we obtain from Eq. (24)

$$f(r_1) = -\int_{t_1}^{t_0} \frac{\mathrm{d}t}{R(t)} = \begin{cases} \sin^{-1} r_1, & k = +1, \\ r_1, & k = 0, \\ \sinh^{-1} r_1, & k = -1. \end{cases} \tag{25}$$

The value of the red-shift Z in all obtained models can be proved to have the form

$$Z = \frac{\delta\lambda}{\lambda} = \frac{R_0}{R_1} - 1. \tag{26}$$

(ii) Particle horizons

One of the important questions in cosmological theory is to find out, according to a certain model, the answer of the following questions: How much of our universe can be observed at a given instant? To answer this question we consider, as mentioned above, a galaxy S at the point (r, θ, ϕ, ψ) and an observer O, located at the origin space coordinates. Then we consider a light signal emitted by S at the instant t_1 and received by O at the instant t_0. Light signal moves along null geodesic whose equation gives rise to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\sqrt{1-kr^2}}{R(t)} \Rightarrow \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = \frac{\mathrm{d}t}{R(t)} = f(r_1). \tag{27}$$

We have to distinguish between two different distances, the radial coordinate distance defined by

$$r \overset{\text{def.}}{=\!=} \int_{t_1}^{t_0} \frac{\mathrm{d}t}{R(t)}, \tag{28}$$

and the proper distance s defined by

$$s \overset{\text{def.}}{=\!=} R(t_0)r. \tag{29}$$

Recalling the red-shift as given by Eq. (26), one can always express the proper distance s in terms of the red-shift Z. Then if $s \to \infty$ as $Z \to \infty$, we can say that the model is free from particle horizons, i.e., isotropic observers can communicate by sending and receiving signals. But if s attains a finite value as $Z \to \infty$ we say that the model contains particle horizons.

For example we are going to explore the existence of particle horizons in one of the models discussed above. For the dust model ($\omega = 1, k = 0$), given in Table 1, we have

$$\begin{aligned} f(r_1) = r_1 &= \int_{t_1}^{t_0} \frac{\mathrm{d}t}{R(t)} = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{\sqrt{2tc_8 + 2c_9}} \\ &= \frac{ZR_1(t)}{c_8}. \end{aligned} \tag{30}$$

Then from Eq. (30) we have

$$s = R_0(t)r_1 = \frac{ZR_0(t)R_1(t)}{c_8}. \tag{31}$$

From Eq. (31) we see that there is no particle horizons since when $Z \to \infty$ the proper distance s tends to infinity as well.

7. Main results and discussion

We construct a five-dimensional pentadic theory of gravitation. Basic entity of this theory is the pentad field $h_a{}^\mu$. A Lagrangian that is quadratic in torsion in 5D spacetime is constructed. By making a variational principles to the Lagrangian we have obtained the field equation.

We have assumed the existence of an additional dimensional and solved the TEGR field equations in this context. We consider cosmological application in the domain of 5D spacetime using two different pentad fields. In spite that the torsion components of these pentads are different as indicated in the appendix below, the field equations did not give any difference. The difference may appear in other higher dimensional theories, this study will be in details in the future work.

We have found new homogenous and isotropic solutions to the field equations of TEGR. Most of the models generating from the studied three cases, i.e., dust, radiation, and stiff matter have no singularity. Also most of our solutions are in agreement with the current observation of our universe, for example, the Hubble parameter decreases and the deceleration parameter related to most of our solutions is negative, as shown in Figs. 4 and 5, that in agreement with current accelerated universe. Another interesting result is that some of the resulting models are free from horizons.

Appendix A

Here we are going to write the non-vanishing components of the torsion $T^\mu{}_{\nu\sigma}$ and the non-vanishing vector T^μ needed for the calculations.

The non-vanishing components of the torsion of pentad (13) have the form

$$\begin{aligned} &T^1{}_{10} = T^2{}_{20} = T^3{}_{30} = T^4{}_{40} = \frac{\dot{R}(t)}{R(t)}, \\ &T^1{}_{23} = -2\sqrt{k}r^2 \sin\theta\sqrt{1-kr^2}, \\ &T^2{}_{13} = \frac{2\sqrt{k}\sin\theta}{\sqrt{1-kr^2}}, \quad T^3{}_{12} = \frac{-2\sqrt{k}}{\sqrt{1-kr^2}}, \\ &T^4{}_{14} = \frac{kr}{1-kr^2}, \quad T^4{}_{24} = \cot\theta. \end{aligned} \tag{A1}$$

The non-vanishing components of the vector T^μ of pentad (13) have the form

$$T_0 = \frac{-4\dot{R}(t)}{R(t)}, \quad T_1 = \frac{kr}{1-kr^2}. \tag{A2}$$

The non-vanishing components of the torsion of pentad (14) have the form

$$T^1{}_{10} = T^2{}_{20} = T^3{}_{30} = T^4{}_{40} = \frac{\dot{R}(t)}{R(t)},$$
$$T^1{}_{14} = T^2{}_{24} = T^3{}_{34} = \sqrt{k}\sqrt{1-kr^2},$$
$$T^4{}_{14} = \frac{\sqrt{k}r}{\sqrt{1-kr^2}}. \quad \text{(A3)}$$

The non-vanishing components of the vector T^μ of pentad (14) have the form

$$T_0 = \frac{-4\dot{R}(t)}{R(t)}, \quad T_1 = \frac{kr}{1-kr^2},$$
$$T_4 = -3\sqrt{k}\sqrt{1-kr^2}. \quad \text{(A4)}$$

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