Iterative quantum algorithm for distributed clock synchronization*

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Clock synchronization is a well-studied problem with many practical and scientific applications. We propose an arbitrary accuracy iterative quantum algorithm for distributed clock synchronization using only three qubits. The n bits of the time difference Δ between two spatially separated clocks can be deterministically extracted by communicating only $O(n)$ messages and executing the quantum iteration process n times based on the classical feedback and measurement operations. Finally, we also give the algorithm using only two qubits and discuss the success probability of the algorithm.

Keywords: clock synchronization, iterative algorithm

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Clock synchronization, which is the synchronization of a pair of spatially separated clocks, A and B, that are at rest in a common inertial frame, is an important technology with many practical and scientific applications, and has occupied a prominent position at the frontier of technology for a long time.[1,2] The basic problem of clock synchronization is to precisely determine the time difference Δ between two spatially separated clocks using minimum communication resources. Accurate timekeeping lies at the heart of many projects, such as telecommunication, long baseline interferometry, the global positioning system (GPS), electric power generation, and tests of the general theory of relativity. There are two standard methods for clock synchronization in the special theory of relativity, the Einstein clock synchronization[3] and the Eddington's slow clock transport,[4] both of which can be accomplished using only classical resources. However, both methods have certain limitations and difficulties in high accuracy nonlocal synchronization, in which the relativistic effects, for example, the rotating disk problem, have to be taken into consideration.

Over the last decade, several new quantum protocols, which apply quantum mechanical means to this classical problem and show that the quantum clock synchronization could gain significant improvement compared to its classical counterpart, have been proposed to synchronize remote clocks in both theory and experiment.[5−15] Two of the most famous quantum protocols are the shared-prior-entanglement (SPE) based protocol[5] and the ticking-qubit-handshake (TQH) based protocol (also called the quantum-phase-estimation (QPE) based protocol),[6] which are proposed by Jozsa *et al.* and Chuang, respectively. The SPE protocol utilizes the shared prior quantum entanglement between the synchronizing parties and assumes that the remote entanglement distribution of pure singlets could be accomplished without any relative phase error. However, the validity of this assumption has been discussed and questioned by some authors in Refs. [16]–[18]. To determine the time difference Δ between clocks A and B by monitoring the oscillation of function $f(\Delta) \sim \cos(\omega\Delta)$, $O(2^{2n})$, shared singlets are needed in the SPE protocol when the entanglement distribution is efficiently achieved. In the TQH protocol, it uses the quantum phase estimation method to obtain Δ to n bits of accuracy by communicating only $O(n)$ qubits and using an $O(2^n)$ range of frequencies. The TQH protocol obtains an exponential improvement over the classical algorithms which require $O(2^{2n})$ message exchanges.

In this paper, we first propose an alternative distributed clock synchronization algorithm using only three qubits based on the quantum iteration method.

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Two spatially separated clocks can be synchronized to an arbitrary given accuracy by performing classical feedback and measurement operations continuously without applying any discrete quantum Fourier transform, which is a key ingredient in the TQH protocol, to the qubits. The present algorithm has the same success probability as the TQH protocol.[6] We also give a quantum circuit to achieve this algorithm by using only two qubits. By increasing the number of the times of qubit T being transmitted between two parities Alice and Bob, fewer quantum operations and measurements are needed.

To determine the time difference Δ accurately, the ticking qubit in the TQH protocol must be transmitted twice between Alice and Bob, the transmission route is Alice–Bob–Alice. While in our protocol, the ticking qubit needs to be transmitted only once before performing the classical feedback and measurement operations, and the transmission route can be either Alice–Bob or Bob–Alice. For the purpose of distinguishing with the TQH method in Chuang's protocol[6] and for the convenience of presentation, we here call it the one-way ticking qubit handshake OTQH(ω, $|\psi\rangle$). (i) At time t_1^a (t_1^b), where t_1^a (t_1^b) is the local time on Alice's (Bob's) clock, Alice (Bob) sends $(t_1^a, |\psi\rangle, \omega)$ $\big((t_1^b, |\psi\rangle, \omega)\big)$ to Bob (Alice), with ω being the tick rate of the qubit $|\psi\rangle$. (ii) Bob (Alice) receives $(t_1^a, \mathrm{e}^{\mathrm{i}\omega t_{12}^a Z}|\psi\rangle, \omega)$ $\big((t_1^b, \mathrm{e}^{\mathrm{i}\omega t_{12}^b Z}|\psi\rangle, \omega)\big)$ at time t_2^b (t_2^a), where t_{12}^a (t_{12}^b) is the time the qubit spent in transit from Alice to Bob (Bob to Alice). (iii) Bob (Alice) applies the operation $C_{12}^a = \mathrm{e}^{-\mathrm{i}\omega(t_2^b - t_1^a)Z}$ $\big(C_{12}^b = \mathrm{e}^{-\mathrm{i}\omega(t_2^a - t_1^b)Z}\big)$ to the qubit and obtains $\mathrm{e}^{-\mathrm{i}\omega Z\Delta}|\psi\rangle$. The OTQH works in the same way as the TQH method, and Chuang has illustrated how to ensure the same time difference Δ in each TQH process, read Ref. [6] for details. Furthermore, for the ticking qubit transmitted between Alice and Bob, the same discussion as that in Ref. [6] is valid, which is not repeated here. In what follows, we describe in detail our algorithm to determine Δ to n bits of accuracy using the iterative method.

To derive the time difference Δ accurately, we assume that the value $\omega\Delta$ has a binary expansion with no more than n bits, $\omega\Delta = \varphi = (0.\varphi_1\varphi_2\varphi_3\cdots\varphi_n000\cdots)$. The quantum circuit implementation for the distributed clock synchronization in our algorithm is shown in Fig. 1. At the beginning, Alice and Bob start with three qubits initialized to $|\phi_0\rangle = |0\rangle_A|0\rangle_T|0\rangle_B$, where qubit T is the ticking qubit, which is transmitted between Alice and Bob back and forth, and qubits A and B are the ancillary qubits held by Alice and Bob, respectively. Alice applies a Hadamard (H) gate transform to qubit A, then she performs a controlled-NOT (CNOT) gate operation on qubits A and T with A being the control qubit and T being the target qubit, obtaining

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AT} + |11\rangle_{AT})|0\rangle_B. \tag{1}$$

Alice sends the qubit T to Bob. Now executing the OTQH($\pi 2^{n-1}\omega$,$|\psi\rangle$) gives

$$\begin{aligned}|\phi_2\rangle = \frac{1}{\sqrt{2}}(&\mathrm{e}^{-\mathrm{i}\pi 2^{n-1}\omega\Delta}|00\rangle_{AT} \\ &+ \mathrm{e}^{\mathrm{i}\pi 2^{n-1}\omega\Delta}|11\rangle_{AT})|0\rangle_B.\end{aligned} \tag{2}$$

Substituting $\omega\Delta = \varphi = (0.\varphi_1\varphi_2\varphi_3\cdots\varphi_n000\cdots)$ into Eq. (2) yields

$$|\phi_2'\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AT} + \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n}|11\rangle_{AT})|0\rangle_B, \tag{3}$$

where the common phase factor $\mathrm{e}^{-\mathrm{i}\pi 2^{n-1}\omega\Delta}$ has been discarded. Bob first applies a CNOT gate operation to qubits T and B with the control qubit being T and the target qubit being B. Then Bob performs a second CNOT operation on qubits T and B with the control qubit being B and the target qubit being T, as shown in Fig. 1, giving

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n}|11\rangle_{AB})|0\rangle_T. \tag{4}$$

After that, Alice and Bob apply the H gate transform to qubits A and B, respectively, obtaining

$$\begin{aligned}|\phi_4\rangle = \frac{1}{2\sqrt{2}}\big\{&[(1 + \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n})|0\rangle_A] \\ &+ (1 - \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n})|1\rangle_A]|0\rangle_B \\ &+ [(1 - \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n})|0\rangle_A] \\ &+ (1 + \mathrm{e}^{\mathrm{i}2\pi 0.\varphi_n})|1\rangle_A]|1\rangle_B\big\}.\end{aligned} \tag{5}$$

Alice and Bob make measurements on qubits A and B, respectively. The probability to obtain $|0\rangle_A|0\rangle_B$ and $|1\rangle_A|1\rangle_B$ is $P_a = \cos^2[\pi(0.\varphi_n00\cdots)]$, and that to obtain $|0\rangle_A|1\rangle_B$ and $|1\rangle_A|0\rangle_B$ is $P_b = \sin^2[\pi(0.\varphi_n00\cdots)]$, which correspond to $P_a = 0, P_b = 1$ for $\varphi_n = 1$ and $P_a = 1, P_b = 0$ for $\varphi_n = 0$. Therefore, the n-th bit φ_n of φ is extracted deterministically. This is the first iteration. In the second iteration, the qubit T is sent from Bob to Alice and executed with OTQH($\pi 2^{n-2}\omega$,$|\psi\rangle$). The phase of the qubit state $|11\rangle_{AB}$ in Eq. (4) before the rotation operation $R_z(\Delta_\alpha)$ shown in Fig. 2 is $2\pi(0.\varphi_{n-1}\varphi_n00\cdots)$. After the rotation operation $R_z(\Delta_n)$ with rotation angle $\Delta_n = -2\pi(0.0\varphi_n)$, the probabilities become $P_a =$

$\cos^2[\pi(0.\varphi_{n-1}00\cdots)]$ and $P_b = \sin^2[\pi(0.\varphi_{n-1}00\cdots)]$. Thus the $(n-1)$-th bit φ_{n-1} of φ is also extracted deterministically based on the feedback operation. Here the feedback angle Δ_α ($\alpha = n, n-1, \ldots, 2$) is defined as $\Delta_\alpha = -2\pi(0.0\varphi_\alpha\varphi_{\alpha+1}\ldots\varphi_n)$, and

$$R_z(\Delta_\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\Delta_\alpha} \end{pmatrix}, \tag{6}$$

which is a one-bit rotation gate expressed in the basis $\{|0\rangle, |1\rangle\}$. Therefore, repeating the iteration n times, all n bits of $\omega\Delta$ can be extracted deterministically and hence Δ.

In this algorithm, we use Dobšíček *et al.*'s iteration method[19] as the subroutine to determine the time difference Δ to an arbitrary accuracy. We assume that the first n bits of the binary expansion of the time difference φ ($\varphi = \omega\Delta$) is denoted as $\widetilde{\varphi} = 0.\varphi_1\varphi_2\ldots\varphi_n$, and all the later bits are 0. Generally, as illustrated in Ref. [19], however, there will be a remainder $0 \le \gamma < 1$, making $\varphi - \widetilde{\varphi} = \gamma 2^{-n}$. In this case, the conditional probability for each bit to be measured correctly is $P_m = \cos^2(\pi 2^{m-n-1}\gamma)$, and thus the total success probability of our algorithm to extract $\widetilde{\varphi}$ accurately is

$$P(\gamma) = \prod_{m=1}^{n} P_m = \frac{\sin^2(\pi\gamma)}{2^{2n}\sin^2(\pi 2^{-n}\gamma)}, \tag{7}$$

which equals the outcome probability given in Ref. [19]. The total success probability is plotted in Fig. 3 as a function of γ and n. We can see from Fig. 3

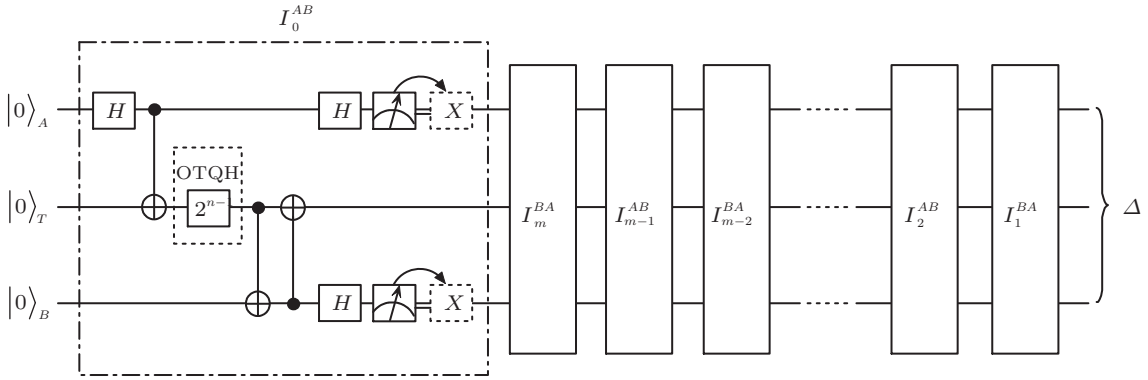


Fig. 1. Quantum circuit for implementing the iterative quantum clock synchronization algorithm. Time goes from left to right. Here n is the number of iteration, $m = n-1$, the boxed 2^l gates denote calls to OTQH($\pi 2^l\omega$, $|\psi\rangle$), meters denote projective measurements, X is a NOT gate operation performed on the qubit if and only if the measurement result is 1, and I_α^{BA} and $I_{\alpha-1}^{AB}$ represent quantum iterations, which will be shown in Fig. 2.

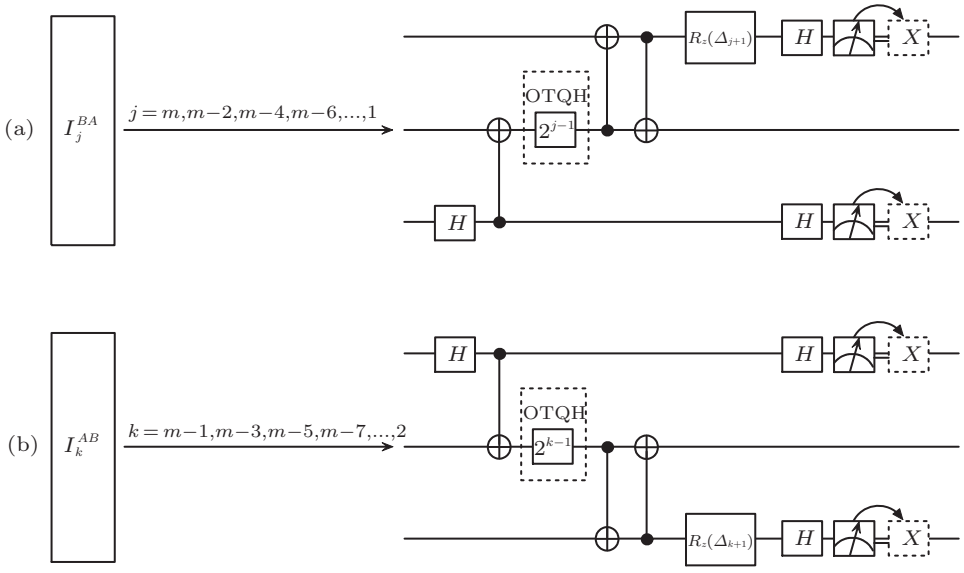


Fig. 2. Iterative quantum circuit for (a) I_α^{BA} and (b) $I_{\alpha-1}^{AB}$ shown in Fig. 1, where $R_z(\Delta_\beta)$ is the classical feedback rotation operation given in Eq. (6).

that the success probability is only relevant to the remainder γ (independent of n). According to the binomial distribution, therefore, the effective success probability increases exponentially with the increasing number of repetitions based on the fact that γ will decrease exponentially following the increment of the number of iterations. The present algorithm determines the time difference Δ with accuracy $1/2^n$ and with a success probability of $P_s > 8/\pi^2$.

Furthermore, the iterative process can also be achieved based on the feedback operation by using only two qubits if we use the TQH method proposed by Chuang,[6] as shown in Fig. 4. In the two-qubit iterative algorithm, the number of the times that qubit T is transmitted between Alice and Bob is twice that of in the three-qubit algorithm, but fewer quantum measurements and quantum operations, including one- and two-qubit operations, are used.

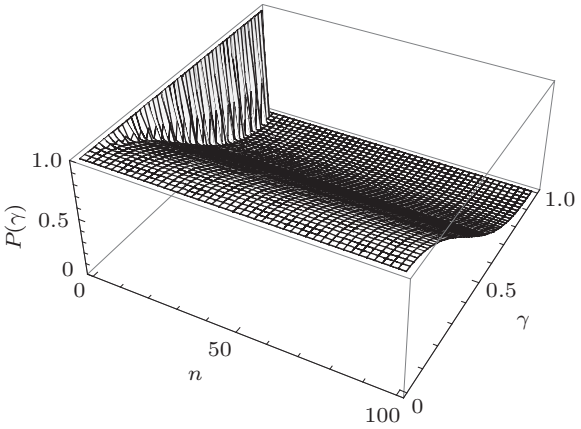


Fig. 3. The overall success probability for the proposed algorithm as a function of γ and n.

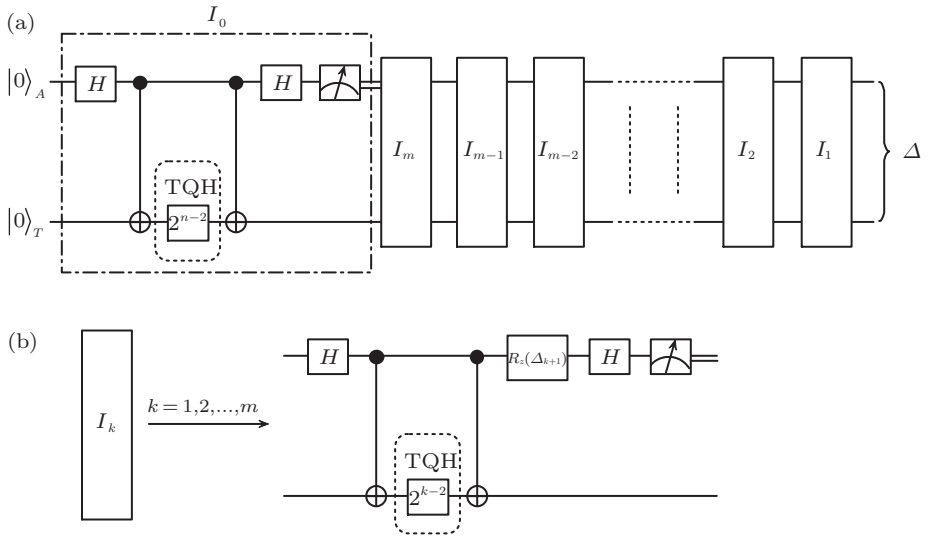


Fig. 4. Quantum circuit of the two-qubit iterative algorithm for clock synchronization, where $m = n - 1$, the boxed 2^j gates denote calls to TQH($\pi 2^j\omega$,$|\psi\rangle$), which has been given in Ref. [6].

In conclusion, we have proposed a distributed clock synchronization algorithm to determine the time difference Δ to an arbitrary given accuracy with only three qubits or two qubits based on the quantum iteration. In our protocol, the discrete quantum Fourier transform, which is a very important tool for clock synchronization in the TQH protocol[6] but cannot be effectively realized for multi-qubit in the present experiment, is replaced by the classical feedback and measurement operations. The proposed algorithm has the same success probability as the TQH protocol, and in the meantime, fewer qubits are used, which greatly reduces the quantum resources required.

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