# Improved frequency standard via weighted graph states*

Xue Peng(薛 鹏)†

*Department of Physics, Southeast University, Nanjing* 211189, *China*

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We study the spin squeezing property of weighted graph states, which can be used to improve sensitivity in interferometry. We study the time evolution of spin squeezing under local decoherence acting independently on each qubit. Based on the analysis, the spin squeezing of the weighted graph states is somehow robust in the presence of decoherence and the decoherence limit in the improvement of the interferometric sensitivity is still achievable. Furthermore, one can obtain the optimal improvement of sensitivity by tuning the weighted of each edges of the weighted graph state.

**Keywords:** spin squeezing, weighted graph states, interferometric sensitivity

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## 1. Introduction

Quantum correlations have attracted much interest as intriguing manifestations of nonclassical phenomena and they have also found many promising applications such as achieving interferometric[1−4] and spectroscopic[5] sensitivities beyond the standard quantum noise limit using squeezed states and entangled states. The spin squeezed states are quantum correlated states[6] with reduced fluctuations in one of the collective spin components, with possible applications in atomic interferometers and high-precision atomic clocks. Entanglement is based on the superposition principle combined with the Hilbert space structure, while squeezing is originated from another fundamental principle of quantum mechanics, i.e., the uncertainty principle. Therefore, it is interesting and important to study their relationship. It is found that spin squeezing is closely related to and implies quantum entanglement. In this paper we will study their different robustness to the quantum noise and decoherence.

In the absence of decoherence, the squeezed uncertainty of a transverse spin component directly improves the sensitivity in interferometry,[3] enabling the improvement factors of up to $N^{1/3}$ for one-axis twisting and $N^{1/2}$ for two-axis twisting with $N$ particles.[7] Maximally entangled states can also improve interferometric sensitivity by a factor of $N^{1/2}$ treating $N$ particles as a single quantum object. The phase evolution of an object consisting of $N$ particles is $N$ times as fast as that of a particle itself, or an equivalently deBroglie wavelength is shortened by a factor of $N$, giving the standard quantum limit for an object $\sim 1$.[8]

In the presence of decoherence, a Greenberger–Horne–Zeilinger (GHZ) type maximally entangled state with zero spin squeezing parameter does not provide higher resolution.[4,9] Whereas a partially entangled state with a high symmetry, which is actually a spin squeezed state, gives an improved sensitivity. Thus in the presence of decoherence the spin squeezing results in the improvement of interferometry sensitivity. The aim of this paper is to find the spin squeezed states which are robust to decoherence. Graph states proved to be very robust against many sources of noise.[10−12] In this paper we will study a special kind of graph state and combine the robustness of the spin squeezing property of these states with a scheme capable of improving interferometric sensitivity.

## 2. Spin squeezing and pairwise entanglement

We consider an ensemble of $N$ two-level particles with a lower (upper) state $|\downarrow\rangle$ ($|\uparrow\rangle$). Adopting the nomenclature of spin-1/2 particles, we introduce the

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†Corresponding author. E-mail: gnep.eux@gmail.com

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total angular momentum (i.e., Bloch vector)

$$\boldsymbol{J} = \sum_{j=1}^{N} \boldsymbol{S}_j, \tag{1}$$

where

$$\boldsymbol{S}_z^j = \frac{1}{2}\hat{\sigma}_z^j = \frac{1}{2}\left(|\uparrow\rangle_j \langle\uparrow| - |\downarrow\rangle_j \langle\downarrow|\right). \tag{2}$$

At this point, it is convenient to introduce the following two kinds of spin squeezing parameters[5,13]

$$\xi_1^2 = \frac{4\left(\Delta J_{\boldsymbol{n}_\perp}\right)_{\min}^2}{N}, \qquad \xi_2^2 = \frac{N^2}{4\langle \boldsymbol{J}\rangle^2}\xi_1^2. \tag{3}$$

Here, the minimization in the first equation is over all directions denoted by $\boldsymbol{n}_\perp$, perpendicular to the mean spin direction $\boldsymbol{n} = \langle \boldsymbol{J}\rangle/|\langle \boldsymbol{J}\rangle|$. If $\xi_2^2 < 1$ is satisfied, the spin squeezing occurs and the $N$-qubit state is entangled.

Although we cannot say that the squeezed state via the parameters $\xi_1^2$ is entangled, it is indeed closely related to quantum entanglement. For an arbitrary symmetric multi-qubit state, spin squeezing implies pairwise entanglement. Due to the exchange symmetry,[14] the expectation value $\langle J_{\boldsymbol{n}_\perp}^2\rangle$ can be written as

$$\langle J_{\boldsymbol{n}_\perp}^2\rangle = N + N(N-1)\langle S_{\boldsymbol{n}_\perp}^j \otimes S_{\boldsymbol{n}_\perp}^k\rangle. \tag{4}$$

The spin squeezing parameter $\xi_1$ can be rewritten as

$$\xi_1^2 = 4 + 4(N-1)\langle S_{\boldsymbol{n}_\perp}^j \otimes S_{\boldsymbol{n}_\perp}^k\rangle. \tag{5}$$

The above equation shows that spin squeezing is equivalent to the negative pairwise correlation $\langle S_{\boldsymbol{n}_\perp}^j \otimes S_{\boldsymbol{n}_\perp}^k\rangle < 0$.

It is known that the spin squeezing manifested by $\xi_2^2 < 1$ leads to an improvement in frequency measurement.[4,9] Generally speaking, an interferometer is quantum mechanically described as a collective, linear, rotation of an input state by an angle $\varphi$:

$$\rho_{\rm out}(\varphi)\ \mathrm{e}^{\mathrm{i}\varphi \boldsymbol{J}_{\boldsymbol{n}}}\rho_{\rm in}\,\mathrm{e}^{-\mathrm{i}\varphi \boldsymbol{J}_{\boldsymbol{n}}}. \tag{6}$$

The goal is to estimate $\varphi$ with a sensitivity overcoming the shot-noise limit $\delta\varphi = 1/\sqrt{N}$.[15]

Spin squeezed states are many-body states showing pairwise entanglement and reduced uncertainty in the collective spin moment in one direction. This reduction in the measurement uncertainty, achieved without violating the minimum uncertainty principle by a redistribution of the quantum fluctuations between noncommuting variables, can be exploited to perform metrology beyond the Heisenberg limit. The degree of squeezing of a spin ensemble is evaluated by the squeezing parameter $\xi_2^2$. When spin squeezing is instead used in the context of quantum limited metrology, the squeezing parameter should measure the improvement in signal-to-noise ratio for the measured quantity $\varphi$ and the phase sensitivity is

$$\delta\varphi = \frac{\xi_2}{\sqrt{N}}. \tag{7}$$

This definition is associated to Ramsey-type experiments,[16] in which an external magnetic field is measured via the detection of the accumulated phase $\varphi$ due to the Zeeman interaction and the phase uncertainty for a product state is $\sim\sqrt{N}$, and for a GHZ-type maximally entangled state is $\sim N$ in the absence of decoherence. In the limit of large spin numbers, using the one- and two-axis squeezing operators, the optimal squeezing parameters are $\xi_1^2 \sim 3^{2/3}/(2N^{2/3})$ (Ref. [7]) and $\xi_2^2 \sim (1+2\sqrt{3})/(2N)$,[17,18] respectively.

## 3. Spin squeezing property of weighted graph states

We consider an $N$ spin-1/2 system with pairwise interactions described by an Ising-type Hamiltonian

$$\hat{H} = \sum_{j,k}\frac{1}{4}f(j,k)\left(\mathbb{I} - \hat{\sigma}_z^i\right)\otimes\left(\mathbb{I} - \hat{\sigma}_z^j\right). \tag{8}$$

The weighted graph state[10,11] is obtained by the evolution of the above Hamiltonian

$$\begin{aligned}
|\psi_t\rangle &= \exp\left(-\mathrm{i}\hat{H}t\right)|+\rangle^{\otimes N} \\
&= \prod_{j,k}\exp\left[-\frac{1}{4}\mathrm{i}f(j,k)t\right] \\
&\quad\times\exp\left[\frac{1}{4}\mathrm{i}f(j,k)t\left(\hat{\sigma}_z^i + \hat{\sigma}_z^j\right)\right] \\
&\quad\times\exp\left[-\frac{1}{4}\mathrm{i}f(j,k)t\hat{\sigma}_z^i\hat{\sigma}_z^j\right]|+\rangle^{\otimes N} \\
&= \prod_{j,k}\exp\left[-\frac{1}{4}\mathrm{i}f(j,k)t\right] \\
&\quad\times\prod_{i,j}\exp\left[\frac{1}{4}\mathrm{i}f(j,k)t\left(\hat{\sigma}_z^i + \hat{\sigma}_z^j\right)\right] \\
&\quad\times\prod_{j,k}\exp\left[-\frac{1}{4}\mathrm{i}f(j,k)t\hat{\sigma}_z^i\hat{\sigma}_z^j\right]|+\rangle^{\otimes N} \\
&= U'\prod_{j,k}\exp\left[-\frac{1}{4}\mathrm{i}f(j,k)t\hat{\sigma}_z^i\hat{\sigma}_z^j\right]|+\rangle^{\otimes N},
\end{aligned} \tag{9}$$

where $|+\rangle = \left(|\uparrow\rangle + |\downarrow\rangle\right)/\sqrt{2}$. If we choose the evolution time to satisfy $f(j,k)t = m\pi$ with $m$ being an integer, the state $|\psi_t\rangle$ is a product state. If

$f(j,k)t = (2m+1)\,\pi/2$, $|\psi_t\rangle$ becomes a graph state. And for $0 < f(j,k)t < \pi/2$, $|\psi_t\rangle$ is a weighted graph state with spin squeezing parameter $\xi_2 < 1$.

In this paper we only consider two kinds of the simplest weighted graph states: weighted cluster states and weighted full-connected graph states. For the weighted cluster state we have the form

$$|\psi_t\rangle = \prod_{j=1}^{N-1} \exp\left[-\frac{\mathrm{i}}{4} f_j t \left(\mathbb{I} - \hat{\sigma}_z^j\right) \otimes \left(\mathbb{I} - \hat{\sigma}_z^{j+1}\right)\right] |+\rangle^{\otimes N}. \quad (10)$$

If we make our case further simpler by assuming $f_j \equiv f$ for each qubit $j$ of the weighted cluster state and $ft \equiv \alpha$, the spin squeezing parameter takes this form

$$\xi_2^2 = \frac{1 + \cos^2\phi \sin^2(\alpha/2) + \sin 2\phi \sin\alpha}{\cos^4(\alpha/2)} \quad (11)$$

with the mean spin direction

$$\boldsymbol{n} = (\cos\alpha, \sin(-\alpha), 0), \quad (12)$$

the orthogonal direction

$$\boldsymbol{n}_\perp = (-\cos\phi \sin(-\alpha), \cos\phi\cos\alpha, \sin\phi), \quad (13)$$

and $\phi \in [0, 2\pi]$. The spin squeezing parameter is independent of the number of the qubits $N$ and can be optimized to be $\xi_{\min} = 0.7312$ with respect to $\alpha$ and $\phi$ shown in Fig. 1. It is clear that with the proper choice of $\alpha$ and $\phi$ the weighted cluster state can be used to improve the phase sensitivity.

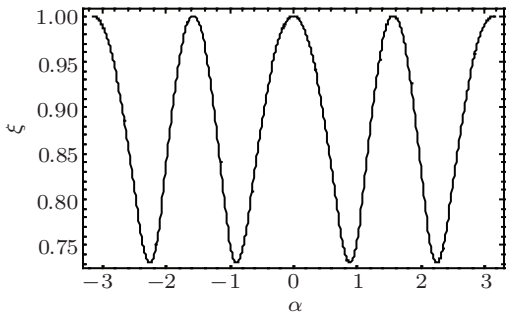



**Fig. 1.** Simulation of spin squeezing parameter for weighted cluster state $\xi_2$ as a function of $\alpha$.

For the weighted fully-connected graph state, we have the form

$$|\psi_t\rangle = \prod_{j\neq k} \exp\left[-\frac{\mathrm{i}}{4}\alpha \left(\mathbb{I} - \hat{\sigma}_z^j\right) \otimes \left(\mathbb{I} - \hat{\sigma}_z^k\right)\right] |+\rangle^{\otimes N}. \quad (14)$$

The spin squeezing parameter of the weighted fully-connected graph state with the same weight $\alpha$ takes this form

$$\xi_2^2 = \frac{1 - (N-1)\left[\sqrt{A^2 + B^2} - A\right]/4}{\cos^{2N-2}\alpha}, \quad (15)$$

where

$$A = 1 - \cos^{N-2}(2\alpha), \qquad B = 4\sin\alpha\cos^{N-2}\alpha. \quad (16)$$

The mean spin direction for the weighted fully-connected graph state is

$$\boldsymbol{n} = (\cos(N\alpha), \sin(-N\alpha), 0), \quad (17)$$

and the orthogonal direction is then

$$\boldsymbol{n}_\perp = (-\cos\phi\sin(-N\alpha), \cos\phi\cos(N\alpha), \sin\phi). \quad (18)$$

The minimum spin squeezing parameter with respect to $\alpha$ is obtained $\xi_2 \propto 1/N^{1/3}$ as shown in Fig. 2.

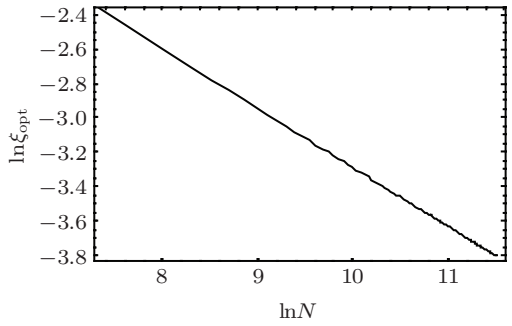



**Fig. 2.** The ln–ln plot of the spin squeezing parameter for weighted fully connected graph state $\xi_2$ versus the number of the qubits $N$ optimized with respect to $\alpha$.

# 4. Evolution of spin squeezing of weighted graph states in the presence of decoherence

For a system consisting of $N$ two-level particles, we would be interested in the effect of decoherence on the spin squeezing properties of this system. We consider as a decoherence model individual coupling of each of the qubits to a thermal bath, where the evolution of the $k$-th qubit is described by the map $\mathscr{E}_k$ with Pauli operators $\hat{\sigma}_j$ $(j = 0, 1, 2, 3)$ acting on qubit $k$. We would be interested in the evolution of a given weighted graph state $\psi$ of $N$ qubit under this decoherence model. That is, the initial state $\psi$ suffers from decoherence and evolves in time to a mixed state $\rho(t)$ given by

$$\rho(t) = \mathscr{E}_1\mathscr{E}_2\cdots\mathscr{E}_N |\psi_t\rangle\langle\psi_t|. \quad (19)$$

Disregarding a physical description in terms of an underlying interaction between the system and its environment, we will in the following also consider decoherence due to individual noise processes of the particles described by some Pauli channel:

$$\mathscr{E}\chi = \sum_{j=0}^{3} p_j(t)\hat{\sigma}_j\chi\hat{\sigma}_j \tag{20}$$

with $\chi$ being a density matrix for a single-qubit state and $\sum_{j=0}^{3} p_j(t) = 1$. These noise channels are particles of interest in quantum information theory, especially in the study of fault-tolerance of quantum computation. This class contains for example: (i) for $p_0 = (1+3p)/4$ and $p_1 = p_2 = p_3 = (1-p)/4$ the above depolarizing channel; (ii) for $p_0 = (1+p)/2$, $p_1 = p_2 = 0$, and $p_3 = (1-p)/2$ the above dephasing channel; (iii) for $p_0 = (1+p)/2$, $p_2 = p_3 = 0$, and $p_1 = (1-p)/2$ the bit-flip channel.[11]

Entangled states are known to decay more rapidly due to decoherence than product states, so that in practice the improvement in sensitivity is often counterbalanced by the need to reduce the interrogation time. In particular, if the system is prepared in the GHZ-type maximally entangled state, the sensitivity improvement is completely lost in the presence of some classes of decoherence such as dephasing, as the system is $N$-times more sensitive both to the signal and the noise. A partially entangled state with spin squeezing robust to decoherence, such as a weighted graph state, can instead provide an advantage over product states or GHZ-type maximally entangled states.

Let us now examine the same situation in a realistic experimental scenario,[9] where decoherence effects are inevitably present. The main type of decoherence in a spin ensemble is dephasing due to processes that cause random phase changes while preserving the population in the spin states. Important mechanisms that result in dephasing effects are collisions, stray fields, and laser instabilities. We model the time evolution of the density operator for the spin ensemble $\rho(t)$ in the individual dephasing channel by the following master equation

$$\frac{\partial\rho(t)}{\partial t} = -\mathrm{i}\left[\hat{H}_{\mathrm{r}}, \rho(t)\right] + \frac{\gamma}{2}\sum_{j}^{N}\left[\hat{\sigma}_z^j\rho(t)\hat{\sigma}_z^j - \rho(t)\right], \tag{21}$$

where the reference system is the standard system used in phase sensitivity and it is defined as

$$\hat{H}_{\mathrm{r}} = \frac{\Delta}{2}\sum_{j=1}^{N}\hat{\sigma}_z^j. \tag{22}$$

For the weighted cluster state, we can easily evaluate the modified main spin direction in the presence of dephasing as

$$\boldsymbol{n}' = (\cos(\Delta t - \alpha), \sin(\Delta t - \alpha), 0), \tag{23}$$

and the modified orthogonal direction is

$$\boldsymbol{n}'_{\perp} = (-\cos\phi\sin(\Delta t - \alpha), \cos\phi\cos(\Delta t - \alpha), \sin\phi). \tag{24}$$

Then the spin squeezing parameter of the weighted cluster state under the dephasing evolves

$$\xi_2^2(t) = \frac{1 + \mathrm{e}^{-2\gamma t}\cos^2\phi\sin^2(\alpha/2) + \mathrm{e}^{-\gamma t}\sin(2\phi)\sin\alpha}{\mathrm{e}^{-2\gamma t}\cos^4(\alpha/2)}. \tag{25}$$

In Fig. 3, with dephasing the spin squeezing parameter of the weighted cluster state still holds ($< 1$) for small $2\gamma t < 0.42$.

For the weighted fully-connected graph state, we consider the same dephasing channel. The modified mean spin direction and the orthogonal direction are calculated as

$$\begin{aligned}\boldsymbol{n}' &= (\cos(\Delta t - N\alpha), \sin(\Delta t - N\alpha), 0),\\ \boldsymbol{n}'_{\perp} &= (-\cos\phi\sin(\Delta t - N\alpha), \cos\phi\cos(\Delta t - N\alpha), \sin\phi).\end{aligned} \tag{26}$$

Then the spin squeezing parameter of the weighted fully-connected graph state evolves to

$$\xi_2^2(t) = \frac{1 + \frac{1}{4}\mathrm{e}^{-2\gamma t}(N-1)\left(A - \frac{A^2}{\sqrt{A^2+B^2}}\right) - \frac{1}{4}\mathrm{e}^{-\gamma t}(N-1)\frac{B^2}{\sqrt{A^2+B^2}}}{\mathrm{e}^{-2\gamma t}\cos^{2N-2}\alpha}. \tag{27}$$

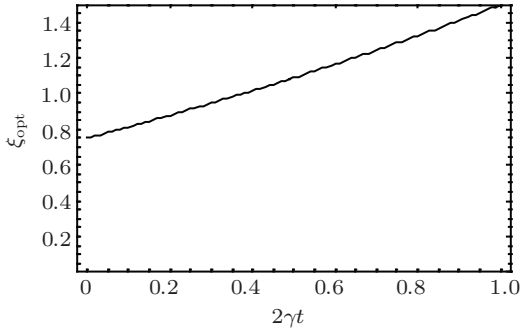



**Fig. 3.** The spin squeezing parameter of the weighted cluster state $\xi_2$ as a function of $2\gamma t$ optimized with respect to $\alpha$.

As we mentioned in Section 2, the spin squeezed state can be used to improve the frequency standards. In the absence of decoherence, using the product state with $N$ qubits, the phase sensitivity $\delta\varphi \propto 1/\sqrt{N}$, and using the $N$-qubit maximally entangled state such as GHZ state the phase sensitivity can be improved to be $\delta\varphi \propto 1/N$. In the presence of decoherence (for example dephasing), one will obtain the same accuracy for frequency standards for either product states and GHZ states as $\delta\varphi \propto 1/\sqrt{N}$. That means there is no improvement even with the maximally entangled multi-qubit states. However with one of the weighted graph states, i.e., weighted fully-connected graph states, the sensitivity is reduced in the presence of decoherence but our scheme remains superior over several reference schemes with states such as GHZ states and product states because of the robustness of the spin squeezing property. For example, by optimizing with respect to the weights, the spin squeezing parameter of the weighted fully-connected graph state is obtained numerically $\xi_2^2 \propto 1/N^{0.38}$ with the dephasing rate satisfying $2\gamma t = 1$ as shown in Figs. 4 and 5. Thus the phase sensitivity $\delta\varphi \propto 1/N^{0.69}$ follows a power law on the number of the particles not a constant.

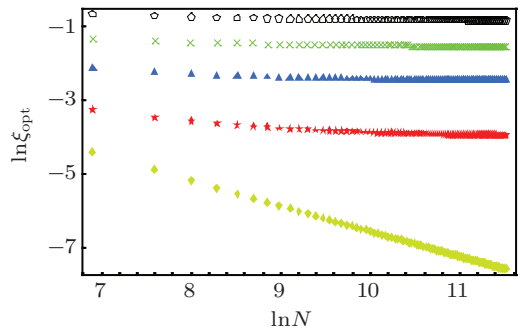



**Fig. 4.** (colour online) The ln–ln plots of the spin squeezing parameters $\xi_2$ of the weighted fully-connected graph state versus the number of the qubits $N$ with different decoherence rate $\gamma$: $2\gamma t = 0$ (yellow diamonds), $2\gamma t = 0.25$ (red stars), $2\gamma t = 0.5$ (blue triangles), $2\gamma t = 0.75$ (green crosses), $2\gamma t = 1$ (black circles).

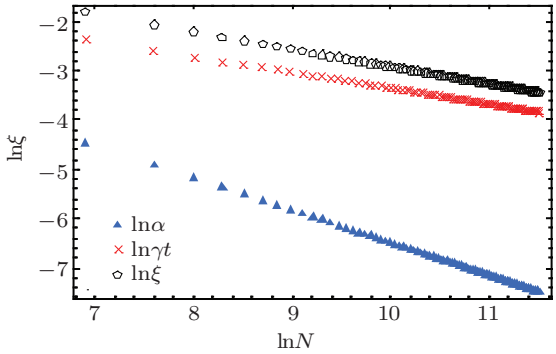



**Fig. 5.** (colour online) The ln–ln plot of the spin squeezing parameters $\xi_2$ of the weighted fully-connected graph state versus the number of the qubits $N$ and ln–ln plots of the corresponding parameters $\alpha$ and $\gamma t$.

## 5. Conclusion

We have studied the spin squeezing property of weighted graph states and shown that the spin squeezing can be provide a significant gain in the frequency standard. In the presence of decoherence the ability of improving sensitivity is due to the robustness of spin squeezing against decoherence. Based on the analysis, in the presence of decoherence, when $N$ is large enough, the setup with the maximally entangled states and product states all collapse, while the scheme with weighted graph states still survive without any measurement optimization. Furthermore, one can obtain the optimal improvement of sensitivity via the joint measurement.

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