The Fresnel–Weyl complementary transformation*

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Based on the newly developed coherent-entangled state representation, we propose the so-called Fresnel–Weyl complementary transformation operator. The new operator plays the roles of both Fresnel transformation (for $(a_1 - a_2)/\sqrt{2}$) and the Weyl transformation (for $(a_1 + a_2)/\sqrt{2}$). Physically, $(a_1 - a_2)/\sqrt{2}$ and $(a_1 + a_2)/\sqrt{2}$ could be a symmetric beamsplitter's two output fields for the incoming fields a_1 and a_2. We show that the two transformations are concisely expressed in the coherent-entangled state representation as a projective operator in the integration form.

Keywords: coherent-entangled state representation, Fresnel–Weyl complementary transformation, beamsplitter

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1. Introduction

Dirac[1] has pointed out that the transformation theory is the basic ingredient of theoretical physics. In quantum mechanics, the transformation theory based on the representation theory is the core for solving dynamic problems,[2−5] and is the theoretical basis of quantum control and quantum information. In this work, we propose a new transformation, the Fresnel–Weyl complementary transformation. In Refs. [6], [7], and [8], we have found the unitary operator

$$F(r,s) = \sqrt{s}\int \frac{\mathrm{d}^2 z}{\pi} \left|sz - rz^*\right\rangle \left\langle z\right|. \tag{1}$$

By using the integration within an ordered product of operators (IWOP) [9−13] technique, we obtain

$$F = \exp\left[\left(-\frac{r}{2s^*}\right)a^{\dagger 2}\right] \exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\frac{1}{s^*}\right] \times \exp\left[\left(\frac{r^*}{2s^*}\right)a^2\right], \tag{2}$$

which corresponds to the optical Fresnel transformation, since the matrix element in the coordinate representation

$$\left\langle x_2\right| F(r,s) \left|x_1\right\rangle = \frac{1}{\sqrt{2\pi \mathrm{i} B}} \exp\left\{\frac{\mathrm{i}}{2B}\left(Ax_1^2 - 2x_1x_2 + Dx_2^2\right)\right\}, \tag{3}$$

is just the the Fresnel integration kernel in classical optics. On the other hand, in quantum statistics theory,[14,15] we often use the Wigner function to describe quantum states (pure or mixed states), the Wigner function is the natural outcome when we perform the so-called Weyl transformation to density operator ρ. In the one-dimensional case, the Weyl transformation[16] is

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \left\langle x + \frac{v}{2}\right| \rho \left|x - \frac{v}{2}\right\rangle \times \mathrm{e}^{-\mathrm{i}pv/\hbar}\mathrm{d}v. \tag{4}$$

Correspondingly, the single-mode Wigner operator with coordinate and momentum variables is

$$\Delta(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \left|x - \frac{v}{2}\right\rangle \left\langle x + \frac{v}{2}\right| \mathrm{e}^{-\mathrm{i}pv/\hbar}\mathrm{d}v, \tag{5}$$

where $|x\rangle$ is the eigenvector of the coordinate operator $X = (a^\dagger + a)/\sqrt{2}$ (or named the quadrature operator in quantum optics theory), $\left[a, a^\dagger\right] = 1$. In the following, we combine the above two different transformations (Eqs. (2) and (4)) in a unified way in the coherent-entangled state representation, which will be named as the Fresnel–Weyl complementary transformation. This paper is arranged as follows. In Section 2, we give a brief review of the coherent-entangled state representation. In Section 3, we construct the

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Fresnel–Weyl complementary transformation. Finally, we give some discussions.

2. Review of the coherent-entangled state representation

By observing the commutator $[a_1 - a_2, X_1 + X_2] = 0$, where $X_i = (a_i + a_i^\dagger)/\sqrt{2}, i = 1, 2$, is the quadrature operator, we have constructed in Ref. [17] the coherent-entangled state, which is the common eigenvector of $(a_1 - a_2)$ and $(X_1 + X_2)$,

$$|z, x\rangle = \frac{1}{\sqrt{2\sqrt{\pi}}} \exp\left[-\frac{|z|^2}{4} - \frac{x^2}{2} - \frac{\left(a_1^\dagger + a_2^\dagger\right)^2}{4} + \left(x + \frac{z}{2}\right) a_1^\dagger + \left(x - \frac{z}{2}\right) a_2^\dagger \right] |00\rangle, \quad (6)$$

where z is a complex number, and x is real. In fact, due to

$$a_1 |z, x\rangle = \left[\frac{1}{2}\left(z - a_1^\dagger - a_2^\dagger\right) + x \right] |z, x\rangle,$$
$$a_2 |z, x\rangle = \left[-\frac{1}{2}\left(a_2^\dagger + a_1^\dagger + z\right) + x \right] |z, x\rangle, \quad (7)$$

we have

$$(a_1 - a_2) |z, x\rangle = z |z, x\rangle,$$
$$\frac{X_1 + X_2}{2} |z, x\rangle = \frac{1}{\sqrt{2}} x |z, x\rangle. \quad (8)$$

Equation (8) embodies the characteristics of both coherent and entangled states. Using the IWOP technique, we can prove the completeness of $|z, x\rangle$,

$$\int_{-\infty}^{\infty} \mathrm{d}x \int \frac{\mathrm{d}^2 z}{\pi} |z, x\rangle \langle z, x| = 1. \quad (9)$$

3. The Fresnel–Weyl complementary operator

Enlightened by the constructions of the Fresnel and the Weyl transformations, we construct the Fresnel–Weyl complementary transformation as

$$U = \sqrt{s} \int \frac{\mathrm{d}^2 z}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{\mathrm{i}pu} \times \left| sz - rz^*, q + \frac{u}{2} \right\rangle \left\langle z, q - \frac{u}{2} \right|. \quad (10)$$

Substituting Eq. (6) into Eq. (10), using the IWOP technique and the two-mode vacuum projector's normally ordered form $|00\rangle\langle 00| =: \exp\left[-a_1^\dagger a_1 - a_2^\dagger a_2\right] :$, and separating the integration variances, we obtain

$$\begin{aligned} U &= \sqrt{s} \int \frac{\mathrm{d}^2 z}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{\mathrm{i}pu} \\ &\quad \times \left| sz - rz^*, q + \frac{u}{2} \right\rangle \left\langle z, q - \frac{u}{2} \right| \\ &= \int \frac{\mathrm{d}^2 z}{\pi} :\ \exp\left[-\frac{|sz - rz^*|^2}{4} - \frac{|z|^2}{4} \right. \\ &\quad \left. + \frac{sz - rz^*}{2}\left(a_1^\dagger - a_2^\dagger\right) + \frac{z^*}{2}(a_1 - a_2) \right] \\ &\quad \cdot \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{\mathrm{i}pu} \exp\left[-\frac{(q + u/2)^2}{2} - \frac{(q - u/2)^2}{2} \right. \\ &\quad \left. + \left(q + \frac{u}{2}\right)\left(a_1^\dagger + a_2^\dagger\right) + \left(q - \frac{u}{2}\right)(a_1 + a_2) \right] \\ &\quad \cdot \exp\left[-\frac{\left(a_1^\dagger + a_2^\dagger\right)^2}{4} - \frac{(a_1 + a_2)^2}{4} \right. \\ &\quad \left. - a_1^\dagger a_1 - a_2^\dagger a_2 \right] :. \end{aligned} \quad (11)$$

Calculating the above two integrations in Eq. (11) yield

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{\mathrm{i}pu} \exp\left[-\frac{(q + u/2)^2}{2} - \frac{(q - u/2)^2}{2} \right. \\ &\quad \left. + \left(q + \frac{u}{2}\right)\left(a_1^\dagger + a_2^\dagger\right) + \left(q - \frac{u}{2}\right)(a_1 + a_2) \right] \\ &= \frac{1}{\sqrt{\pi}} \exp\left[\frac{\left(a_1^\dagger + a_2^\dagger - a_1 - a_2 + 2\mathrm{i}p\right)^2}{4} - q^2 \right. \\ &\quad \left. + q\left(a_1^\dagger + a_2^\dagger + a_1 + a_2\right) \right], \end{aligned} \quad (12)$$

and

$$\begin{aligned} &\int \frac{\mathrm{d}^2 z}{\pi} \exp\left[-\frac{|sz - rz^*|^2}{4} - \frac{|z|^2}{4} \right. \\ &\quad \left. + \frac{sz - rz^*}{2}\left(a_1^\dagger - a_2^\dagger\right) + \frac{z^*}{2}(a_1 - a_2) \right] \\ &= \frac{2}{\sqrt{|s|^2}} \exp\left[-\frac{s\left(a_1^\dagger - a_2^\dagger\right)}{2}(ra_1^\dagger - ra_2^\dagger \right. \\ &\quad \left. - a_1 + a_2) + \frac{r^*}{4s^*}\left(ra_1^\dagger - ra_2^\dagger - a_1 + a_2\right)^2 \right. \\ &\quad \left. + \frac{rs}{4}\left(a_1^\dagger - a_2^\dagger\right)^2 \right], \end{aligned} \quad (13)$$

where we have use the formula[18]

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\{\zeta |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}\}$$

$$= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left\{\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right\}. \quad (14)$$

Combing Eqs. (12) and (13), we rewrite Eq. (11) as

$$\begin{aligned} U = {} & \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{s^*}} {:}\ \exp\left\{-\frac{1}{2}\frac{r}{s^*}\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)^2\right. \\ & + \left(\frac{1}{s^*} - 1\right)\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)\left(\frac{a_1 - a_2}{\sqrt{2}}\right) \\ & + \frac{1}{2}\frac{r^*}{s^*}\left(\frac{a_1 - a_2}{\sqrt{2}}\right)^2 \\ & - \left(p - \frac{a_1 - a_1^\dagger + a_2 - a_2^\dagger}{2\mathrm{i}}\right)^2 \\ & \left. - \left(q - \frac{a_1^\dagger + a_1 + a_2^\dagger + a_2}{2}\right)^2\right\} {:}\,. \end{aligned} \quad (15)$$

After some rearrangement, we finally obtain

$$\begin{aligned} U = {} & \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{s^*}} {:}\ \exp\left\{-\frac{1}{2}\frac{r}{s^*}\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)^2\right. \\ & + \left(\frac{1}{s^*} - 1\right)\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)\left(\frac{a_1 - a_2}{\sqrt{2}}\right) \\ & + \frac{1}{2}\frac{r^*}{s^*}\left(\frac{a_1 - a_2}{\sqrt{2}}\right)^2 - \left(p - \frac{P_1 + P_2}{\sqrt{2}}\right)^2 \\ & \left. - \left(q - \frac{X_1 + X_2}{\sqrt{2}}\right)^2\right\} {:}\,. \end{aligned} \quad (16)$$

As

$$\left[\frac{a_1 - a_2}{\sqrt{2}}, \frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}\right] = 0,$$

and $X_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/\sqrt{2}\mathrm{i}$, $(i = 1, 2)$, we can rewrite Eq. (16) as

$$U = WF, \quad (17)$$

where

$$\begin{aligned} W = {} & \frac{2}{\sqrt{\pi}} {:}\ \exp\left[-\left(p - \frac{P_1 + P_2}{\sqrt{2}}\right)^2\right. \\ & \left. - \left(q - \frac{X_1 + X_2}{\sqrt{2}}\right)^2\right] {:}\,, \end{aligned} \quad (18)$$

and operator F is expressed as

$$\begin{aligned} F = {} & \exp\left[-\frac{r}{2s^*}\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)^2\right] \\ & \times \exp\left\{\left[\left(\frac{a_1^\dagger - a_2^\dagger}{\sqrt{2}}\right)\left(\frac{a_1 - a_2}{\sqrt{2}}\right) + \frac{1}{2}\right]\ln\frac{1}{s^*}\right\} \\ & \times \exp\left[\frac{r^*}{2s^*}\left(\frac{a_1 - a_2}{\sqrt{2}}\right)^2\right]. \end{aligned} \quad (19)$$

The W is named the Weyl transformation for mode $(a_1 + a_2)/\sqrt{2}$, F is named the Fresnel transformation for mode $(a_1 - a_2)/\sqrt{2}$. We can see that the Fresnel–Weyl complementary operator can play both the roles of Fresnel transformation for $(a_1 - a_2)/\sqrt{2}$ and Weyl transformation for $(a_1 + a_2)/\sqrt{2}$. Physically, $(a_1 - a_2)/\sqrt{2}$ and $(a_1 + a_2)/\sqrt{2}$ can be two output fields of a beamsplitter for the incoming fields a_1 and a_2, and they are independent of each other.

In summary, by virtue of the IWOP technique, we have introduced the Fresnel–Weyl complementary transformation operator. This operator can complementarily play the role of two transformations, one is the Fresnel transformation for mode $(a_1 - a_2)/\sqrt{2}$, the other is the Weyl transformation for $(a_1 + a_2)/\sqrt{2}$. We have shown that the transformation can be concisely expressed in the coherent-entangled state representation as a projective operator in the integration form. We also find that physically, for the Fresnel–Weyl complementary transformation, $(a_1 - a_2)/\sqrt{2}$ and $(a_1 + a_2)/\sqrt{2}$ can be two output fields of a beamsplitter. Experimentally, the Fresnel–Weyl complementary transformation can be designed as an optical device, which can be directly applied to these two output fields of the beamsplitter.

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