Analysis of the equal width wave equation with the mesh-free reproducing kernel particle Ritz method*

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In this paper, we analyse the equal width (EW) wave equation by using the mesh-free reproducing kernel particle Ritz (kp-Ritz) method. The mesh-free kernel particle estimate is employed to approximate the displacement field. A system of discrete equations is obtained through the application of the Ritz minimization procedure to the energy expressions. The effectiveness of the kp-Ritz method for the EW wave equation is investigated by numerical examples in this paper.

Keywords: meshless method, mesh-free kp-Ritz method, equal width (EW) wave equation, solitary wave

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1. Introduction

The study of numerical methods of solving nonlinear partial differential equations has enjoyed an intense period of activity over the last 40 years from both theoretical and practical points of view. The finite element method (FEM) and boundary element method (BEM) are well established by numerical techniques, which have been used to obtain numerical solutions for various problems in the field of engineering and science.[1−3] Even though these methods are very effective for solving various kinds of partial differential equations, they also have their own limits. For the FEM, the need to produce a body-fitted mesh in two- and three-dimensional problems makes this method time-consuming and difficult to use. As for the BEM, for an inhomogeneous equation, it requires a domain node distribution in addition to a boundary mesh and depends on the fundamental solutions. To avoid these problems, in recent years meshless techniques have attracted the attention of researchers. In a meshless (mesh-free) method, a set of scattered nodes is used instead of meshing the domain of the problem. For many engineering problems, such as large deformation and crack growth, it is necessary to deal with extremely large deformations or fractures of the mesh with the re-meshing technique. The meshless method is a new and interesting numerical technique that can solve many engineering problems that are not suited to conventional numerical methods with a minimum of meshing or no meshing at all.[4−27] Some meshless methods have been developed, such as smooth particle hydrodynamics (SPH) methods,[28] the radial basis function (RBF),[29] the element-free Galerkin (EFG) method,[30] the reproducing kernel particle method (RKPM),[31,32] the meshless local Petrov–Galerkin (MLPG) method,[33] the finite point method (FPM),[34] and so on.

The equal width (EW) wave equation was suggested by Morrison and Meiss[35] to be used as a model partial differential equation for the simulation of one-dimensional wave propagation in a nonlinear medium with a dispersion process. The EW wave equation is an alternative description of the nonlinear dispersive waves to the more usual Korteweg de Vries (KdV) equation. It has been shown to have solitary wave solutions and to govern a large number of important physical phenomena such as shallow water waves and plasma waves. It is difficult to find analytical solutions for such equations, so the only choice left is an approximate numerical solution. A large number of numerical techniques have been developed to solve the equation, such as the Adomain decomposition method,[36] the tanh

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method,[37] the extended tanh-function method,[38] the modified extended tanh-function method,[39] the variational iteration method,[40] the generalized hyperbolic-function,[41] the separation of variables method,[42] and the sine–cosine method.[43]

The Ritz[44] approximation approach, developed almost a century ago, is the generalization of the Rayleigh[45] method. It is based on the principle that a resonant vibrating system completely interchanges its kinetic and potential energy form. In the Rayleigh method, a single trial function for the mode shape satisfying at least the geometric boundary conditions, is employed, and then by equating the maximum kinetic and potential energies, an upper bound frequency solution is obtained accordingly. The Ritz or Rayleigh–Ritz method is a proven approximation technique in computational mechanics; notable studies have been conducted by Kitipornchai *et al.*,[46] Liew *et al.*,[47,48] Cheung and Zhou,[49,50] Zeng and Bert,[51] Su and Xiang,[52] Lim and Liew,[53] and Liew and Feng.[54]

In the present paper, the reproducing kernel particle estimation is employed to study the EW wave equation. In this study, the displacement field is approximated as a kernel function, and an energy formulation is formulated and a system of nonlinear discrete equations is obtained by the Ritz minimization procedure. In the conventional Ritz method, it is difficult to choose appropriate functions to satisfy some complicated boundary conditions, and the eigen-equations need to be re-evaluated for different boundary conditions. In the present kp-Ritz method, a standard weight function is employed to express the interior field, and the boundary conditions are enforced by the penalty method, thereby avoiding disadvantages of the conventional Ritz method. Comprehensive convergence studies are conducted to validate the accuracy as well as the stability of the present approach. The present results are compared with the results available from the existingliterature.

2. Kernel particle Ritz method for EW wave equation

2.1. Energy formulation

Consider the following EW wave equation

$$u_t + \varepsilon u u_x - \mu u_{xxt} = 0, \quad (a < x < b), \tag{1a}$$

with boundary conditions

$$u(a,t) = \alpha,\ u(b,t) = \beta, \tag{1b}$$

and initial condition

$$u(x,0) = f(x), \tag{1c}$$

where μ is the positive parameter and subscripts x and t denote differentiation.

The weighted integral form of Eq. (1a) is obtained as follows:

$$\int_\Omega w \cdot (u_t + \varepsilon u u_x - \mu u_{xxt}) \mathrm{d}\Omega = 0. \tag{2}$$

The weak form of Eq. (2) is

$$\int_\Gamma w^{\mathrm{T}} u_t \mathrm{d}\Gamma + \varepsilon \int_\Gamma w^{\mathrm{T}} u u_x \mathrm{d}\Gamma + \mu \int_\Gamma w_x^{\mathrm{T}} u_{xt} \mathrm{d}\Gamma = 0. \tag{3}$$

The energy functional $\Pi(u)$ can be written as

$$\Pi(u) = \int_\Gamma u^{\mathrm{T}} u_t \mathrm{d}\Gamma + \varepsilon \int_\Gamma u^{\mathrm{T}} u u_x \mathrm{d}\Gamma + \mu \int_\Gamma u_x^{\mathrm{T}} u_{xt} \mathrm{d}\Gamma. \tag{4}$$

2.2. Kernel particle shape functions

Approximation of the displacement field can be expressed as

$$u(x,t) = \sum_{I=1}^{n} \Phi_I(\boldsymbol{x}) \cdot U_I(t) = \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{U}, \tag{5}$$

where n is the total number of particles, $\Phi_I(\boldsymbol{x})$ is the shape function, and $U_I(t)$ is the unknown nodal value of u at a sampling point I. Based on the reproducing kernel particle estimate, the shape function is given by

$$\Phi_I(\boldsymbol{x}) = C(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}_I) w_r(\boldsymbol{x} - \boldsymbol{x}_I), \tag{6}$$

where $C(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}_I)$ is the correcting function, and $w_r(\boldsymbol{x} - \boldsymbol{x}_I)$ is the kernel function. The kernel function is expressed as

$$w_r(\boldsymbol{x} - \boldsymbol{x}_I) = \frac{1}{r} \phi\left(\frac{\boldsymbol{x} - \boldsymbol{x}_I}{r}\right), \tag{7}$$

and the correction function is written as

$$c(\boldsymbol{x}; \boldsymbol{x} - \boldsymbol{x}') = \sum_{i=1}^{m} p_i(\boldsymbol{x} - \boldsymbol{x}') b_i(\boldsymbol{x}) = \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}') \boldsymbol{b}(\boldsymbol{x}), \quad (x \in \Omega), \tag{8}$$

where

$$\boldsymbol{P} = \begin{bmatrix} p_1(\boldsymbol{x} - \boldsymbol{x}_1) & p_1(\boldsymbol{x} - \boldsymbol{x}_2) & \cdots & p_1(\boldsymbol{x} - \boldsymbol{x}_n) \\ p_2(\boldsymbol{x} - \boldsymbol{x}_1) & p_2(\boldsymbol{x} - \boldsymbol{x}_2) & \cdots & p_2(\boldsymbol{x} - \boldsymbol{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ p_m(\boldsymbol{x} - \boldsymbol{x}_1) & p_m(\boldsymbol{x} - \boldsymbol{x}_2) & \cdots & p_m(\boldsymbol{x} - \boldsymbol{x}_n) \end{bmatrix}, \tag{9}$$

$$\boldsymbol{b}^{\mathrm{T}}(\boldsymbol{x}) = (b_1(\boldsymbol{x}), b_2(\boldsymbol{x}), \cdots, b_m(\boldsymbol{x})), \tag{10}$$

where m is the number of terms in the basis, $p_i(\boldsymbol{x}-\boldsymbol{x}')$ is the monomial basis function, and $b_i(\boldsymbol{x})$ is the coefficients of monomial basis functions, which are to be determined. Generally, the basis can be as follows:

(i) for linear basis ($p = 1$),

$$\boldsymbol{p}^{\mathrm{T}} = (1, x_1 - x_1'), \quad (1\mathrm{D}), \tag{11}$$

$$\boldsymbol{p}^{\mathrm{T}} = (1, x_1 - x_1', x_2 - x_2'), \quad (2\mathrm{D}); \tag{12}$$

(ii) for quadratic basis ($p = 2$),

$$\boldsymbol{p}^{\mathrm{T}} = \left(1, x_1 - x_1', (x_1 - x_1')^2\right), \quad (1\mathrm{D}), \tag{13}$$

$$\boldsymbol{p}^{\mathrm{T}} = (1, x_1 - x_1', x_2 - x_2', (x_1 - x_1')^2, (x_1 - x_1')(x_2 - x_2'), (x_2 - x_2')^2), \quad (2\mathrm{D}). \tag{14}$$

Thus the shape function can be written as

$$\Phi_I(\boldsymbol{x}) = b^{\mathrm{T}}(\boldsymbol{x})P(\boldsymbol{x} - \boldsymbol{x}_I)w_r(\boldsymbol{x} - \boldsymbol{x}_I). \tag{15}$$

Equation (15) can be rewritten as

$$\Phi_I(\boldsymbol{x}) = b^{\mathrm{T}}(\boldsymbol{x})B_I(\boldsymbol{x} - \boldsymbol{x}_I), \tag{16}$$

where

$$B_I(\boldsymbol{x} - \boldsymbol{x}_I) = P(\boldsymbol{x} - \boldsymbol{x}_I)\phi_a(\boldsymbol{x} - \boldsymbol{x}_I), \tag{17}$$

$$\boldsymbol{b}(\boldsymbol{x}) = M^{-1}(\boldsymbol{x})H, \tag{18}$$

$$H = (1, 0, \cdots, 0)^{\mathrm{T}}. \tag{19}$$

The explicit expression for $M(\boldsymbol{x})$ is given by

$$M(\boldsymbol{x}) = \sum_{I=1}^{n} \boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}_I)\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_I)w_r(\boldsymbol{x} - \boldsymbol{x}_I). \tag{20}$$

Therefore, the shape functions can be expressed as

$$\Phi_I(\boldsymbol{x}) = H^{\mathrm{T}}M^{-1}(\boldsymbol{x})\boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}_I)w_r(\boldsymbol{x} - \boldsymbol{x}_I). \tag{21}$$

For the sine-Gordon equation, the first derivatives of shape functions need to be determined. Equation (18) can be rewritten as

$$M(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) = H. \tag{22}$$

Vector $\boldsymbol{b}(\boldsymbol{x})$ can be determined using the LU decomposition of matrix $M(\boldsymbol{x})$, followed by back substitution. Derivatives of $\boldsymbol{b}(\boldsymbol{x})$ can be obtained similarly. Taking derivatives of Eq. (22), we can obtain

$$M_x(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}) + M(\boldsymbol{x}) \cdot \boldsymbol{b}_x(\boldsymbol{x}) = P_x(0), \tag{23}$$

which can be rearranged as

$$M(\boldsymbol{x}) \cdot \boldsymbol{b}_x(\boldsymbol{x}) = P_x(0) - M_x(\boldsymbol{x}) \cdot \boldsymbol{b}(\boldsymbol{x}). \tag{24}$$

Taking derivatives of Eq. (16), we can obtain the first derivatives of shape function as

$$\Phi_{I,x}(\boldsymbol{x}) = b_x^{\mathrm{T}}(\boldsymbol{x})B_I(\boldsymbol{x} - \boldsymbol{x}_I) + b^{\mathrm{T}}(\boldsymbol{x})B_{I,x}(\boldsymbol{x} - \boldsymbol{x}_I). \tag{25}$$

The cubic spline weight function used in the present analysis can be written as a function of normalized radius, and expressed as

$$\begin{aligned} w_i &= w(\boldsymbol{x} - \boldsymbol{x}_i) \\ &\equiv w(r) \\ &= \begin{cases} \dfrac{2}{3} - 4r^2 + 4r^3, & r \le \dfrac{1}{2}, \\ \dfrac{4}{3} - 4r + 4r^2 - \dfrac{4}{3}r^3, & \dfrac{1}{2} < r \le 1, \\ 0, & r > 1, \end{cases} \end{aligned} \tag{26}$$

with

$$r = \frac{d_i}{d_{m_i}}, \quad d_i = \|x - x_i\|, \quad d_{m_i} = d_{\max}c_i, \tag{27}$$

where $d_{\max}$ is a scaling parameter and c_i is the distance that is chosen such that matrix $M(x)$ is not singular. In order to compute the derivatives of shape functions, it is necessary to compute derivatives of weight functions. The first and second order derivatives of the weight functions can be easily obtained by chain rule

$$\begin{aligned} &\frac{\mathrm{d}w_i}{\mathrm{d}x} \\ &= \frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}x} \\ &= \begin{cases} (-8r + 12r^2)\dfrac{\mathrm{sign}(x - x_i)}{d_{m_i}}, & r \le \dfrac{1}{2}, \\ (-4 + 8r - 4r^2)\dfrac{\mathrm{sign}(x - x_i)}{d_{m_i}}, & \dfrac{1}{2} < r \le 1, \\ 0, & r > 1, \end{cases} \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\mathrm{d}^2w}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right) \\ &= \begin{cases} (-8 + 24r), & r \le \dfrac{1}{2}, \\ (8 - 8r), & \dfrac{1}{2} < r \le 1, \\ 0, & r > 1. \end{cases} \end{aligned} \tag{29}$$

2.3. Enforcement of boundary conditions: a penalty approach

There are several approaches to enforcing essential boundary conditions in meshless methods, such

as the use of penalty methods, Lagrange multipliers, modified variational principles, etc. In the present work, the penalty method is used to implement essential boundary conditions.

2.3.1. Simply-supported boundary conditions

For the domain bounded by l_u, the displacement boundary condition is

$$u = \bar{u} \quad \text{on} \quad l_u, \tag{30}$$

where $\bar{u}$ is the prescribed displacement on displacement boundary l_u. Condition (30) is treated as a constraint and is introduced into the formulation using the penalty method. The variational form of the penalty function is given by

$$\Gamma_u = \frac{\alpha}{2}\int_S (u-\bar{u})^2 \,\mathrm{d}S, \tag{31}$$

where α is the penalty parameter, which can be chosen to be $10^3 \sim 10^7$.

2.3.2. Clamped boundary conditions

In the clamped case, for the domain bounded by l_u, besides the boundary condition described by Eq. (30), the rotation boundary condition is also included as

$$\beta = \bar{\beta} \quad \text{on} \quad l_u, \tag{32}$$

where

$$\beta = \frac{\partial u}{\partial n}, \tag{33}$$

and $\bar{\beta}$ is the prescribed rotation on the boundary. The variational form due to the rotational constraint is given by

$$\Gamma_\beta = \frac{\alpha}{2}\int_S (\beta-\bar{\beta})^2 \,\mathrm{d}S. \tag{34}$$

Although, in general, the penalty parameter for each constraint can be taken to be different, here the same penalty parameter is used for both boundary constraints.

2.4. Ritz minimization procedure

The variational form due to the boundary conditions can be expressed as

$$\Gamma_\mathrm{B} = \Gamma_u + \Gamma_\beta, \tag{35}$$

and the total energy functional for this problem thus becomes

$$\Pi^*(u) = \Pi(u) + \Gamma_\mathrm{B}. \tag{36}$$

Equation (36) can also be written as

$$\Pi^*(u) = \int_\Gamma u^\mathrm{T} u_t \,\mathrm{d}\Gamma + \varepsilon\int_\Gamma u^\mathrm{T} u u_x \,\mathrm{d}\Gamma + \mu\int_\Gamma u_x^\mathrm{T} u_{xt}\,\mathrm{d}\Gamma + \frac{\alpha}{2}(u-\bar{u})^2|_{\Gamma_u}. \tag{37}$$

By Eq. (5), we can obtain the approximation of the temperature function as

$$u^h(x,t) = \sum_{I=1}^n \Phi_I(\boldsymbol{x})\cdot\boldsymbol{u}_I(t) = \boldsymbol{\Phi}(x)\cdot\boldsymbol{U}(t), \tag{38}$$

$$\frac{\partial u^h(x,t)}{\partial t} = \frac{\partial}{\partial t}\sum_{I=1}^n \Phi_I(\boldsymbol{x})\cdot\boldsymbol{u}_I(t) = \sum_{I=1}^n \Phi_I(\boldsymbol{x})\cdot\frac{\partial U_I(t)}{\partial t} = \boldsymbol{\Phi}(x)\dot{\boldsymbol{U}}(t), \tag{39}$$

$$\frac{\partial u^h(x,t)}{\partial x} = \frac{\partial}{\partial x}\sum_{I=1}^n \Phi_I(\boldsymbol{x})\cdot u_I(t) = \sum_{I=1}^n \Phi_{I,x}(x)\cdot u_I(t) = \boldsymbol{\Phi}_x(x)\boldsymbol{U}(t), \tag{40}$$

$$\frac{\partial u^h(x,t)}{\partial x\partial t} = \frac{\partial}{\partial x\partial t}\sum_{I=1}^n \Phi_I(x)\cdot u_I(t) = \sum_{I=1}^n \Phi_{I,x}(x)\cdot\frac{\partial U_I(t)}{\partial t} = \boldsymbol{\Phi}_x(x)\dot{\boldsymbol{U}}(t), \tag{41}$$

where

$$\boldsymbol{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)), \tag{42}$$

$$\boldsymbol{U}(t) = (u_1(t), u_2(t), \ldots, u_n(t))^\mathrm{T}, \tag{43}$$

$$\dot{\boldsymbol{U}}(t) = \left(\frac{\partial u_1(t)}{\partial t}, \frac{\partial u_2(t)}{\partial t}, \ldots, \frac{\partial u_n(t)}{\partial t}\right)^\mathrm{T}, \tag{44}$$

$$\boldsymbol{\Phi}_x(x) = (\Phi_{1,x}(x), \Phi_{2,x}(x), \ldots, \Phi_{n,x}(x)). \tag{45}$$

Substituting Eqs. (38)–(41) into Eq. (37) and applying the Ritz minimization procedure to the maximum energy function Γ_max, we obtain

$$\frac{\partial\Gamma_\mathrm{max}}{\partial\Delta} = 0,\ \Delta = u_I(t), \frac{\partial u_I(t)}{\partial t},\ I = 1,2,\ldots,n. \tag{46}$$

In the matrix form, the results can be expressed as

$$(K+G)\dot{U}(t) + HU(t) + \int_\Gamma \varepsilon\Phi(uu_x)\,\mathrm{d}\Gamma = \boldsymbol{Q}(t), \tag{47}$$

where

$$K_{IJ} = \int_\Gamma \Phi_I(\boldsymbol{x})\Phi_J(x)\,\mathrm{d}\Gamma, \tag{48}$$

$$G_{IJ} = \mu\int_\Gamma \Phi_{I,x}(x)\Phi_{J,x}(x)\,\mathrm{d}\Gamma, \tag{49}$$

$$H_{IJ} = \alpha(\Phi_I\Phi_J|_{x=a} + \Phi_I\Phi_J|_{x=b}), \quad (50)$$

$$Q_I = \alpha(\Phi_I\bar{u}|_{x=a} + \Phi_I\bar{u}|_{x=b}). \quad (51)$$

Forming the time discretization of Eq. (47) with the central difference method, we obtain

$$\begin{aligned}&(\boldsymbol{K}+\boldsymbol{G})\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t} + \boldsymbol{H}\frac{\boldsymbol{U}^{n+1}+\boldsymbol{U}^n}{2}\\ &+\int_\Gamma \varepsilon\Phi\frac{(uu_x)^{n+1}+(uu_x)^n}{2}\,\mathrm{d}\Gamma\\ &=\frac{\boldsymbol{Q}^{n+1}+\boldsymbol{Q}^n}{2},\end{aligned} \quad (52)$$

where Δt is the time of step and

$$\begin{aligned}U^n &= U(n\Delta t)\\ &= (u_1(n\Delta t), u_2(n\Delta t), \ldots, u_n(n\Delta t)).\end{aligned} \quad (53)$$

The nonlinear term $(uu_x)^{n+1}$ in the above Eq. (52) is linearized by using the following term

$$(uv)^{n+1} = u^{n+1}v^n + u^nv^{n+1} - u^nv^n. \quad (54)$$

So we obtain

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u_x^nu^{n+1} - u^nu_x^n. \quad (55)$$

Substituting Eq. (55) into Eq. (52) yields

$$\begin{aligned}&(\boldsymbol{K}+\boldsymbol{G})\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t} + \boldsymbol{H}\frac{\boldsymbol{U}^{n+1}+\boldsymbol{U}^n}{2}\\ &+\left(\int_\Gamma \varepsilon\Phi\frac{\boldsymbol{U}^n(\Phi_x\Phi^{\mathrm{T}}+\Phi\Phi_x^{\mathrm{T}})}{2}\,\mathrm{d}\Gamma\right)\cdot\boldsymbol{U}^{n+1}\\ &=\frac{\boldsymbol{Q}^{n+1}+\boldsymbol{Q}^n}{2}.\end{aligned} \quad (56)$$

By solving the above iteration equation, we can obtain the numerical solution of EW equation.

3. Numerical examples

The proposed kp-Ritz method is applied to the numerical solution of EW equation. A regular arrangement of nodes and the background mesh of cells are used for numerical integrations to calculate the system equation. The linear basis and Gauss weight function are used in the kp-Ritz method.

3.1. Example 1: Motion of a single solitary wave

Consider the following problem

$$u_t + \varepsilon uu_x - \mu u_{xxt} = 0, \quad (0 < x < 30), \quad (57)$$

with initial condition

$$u(x,0) = 3c\,\mathrm{sech}^2(k(x-x_0)) \quad (58)$$

and boundary conditions

$$u(0,t) = 3c\,\mathrm{sech}^2(k(-x_0-ct)), \quad (59)$$

$$u(30,t) = 3c\,\mathrm{sech}^2(k(30-x_0-ct)), \quad (60)$$

then the exact solution will be obtained as

$$u(x,t) = 3c\,\mathrm{sech}^2(k(x-x_0-ct)), \quad (61)$$

where $\mu = 1$, $\varepsilon = 1$, $c = 0.3$, $k = 1/\sqrt{4\mu}$, and $x_0 = 10$.

The kp-Ritz method is applied to the above EW wave equation with $\varepsilon = 1$, $\mu = 1$, $c = 0.3$, $k = 1/\sqrt{4\mu}$, $x_0 = 10$, the penalty factor $\alpha = 10^5$ and scaling parameter $d_{\max} = 2.15$, and the time-step length $\Delta t = 0.1$. The weight function is chosen to be a Gauss function and the bases are chosen to be linear. In Fig. 1, the exact solutions are plotted in comparison with numerical results when $t = 1$, 3, 5, 10, and 20, respectively.

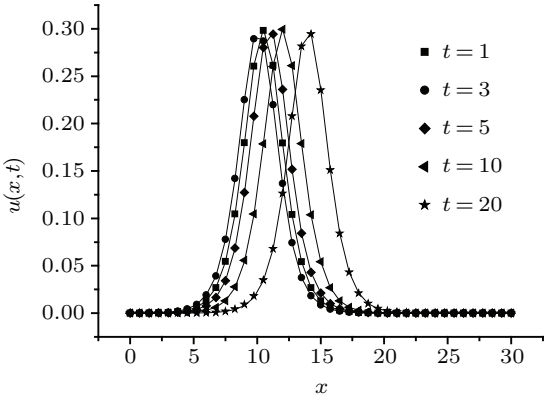


Fig. 1. Numerical solutions and exact solutions of $u(x,t)$ when $t = 1$, 3, 5, 10, 20 (example 1).

3.2. Example 2: Interaction of two solitary waves[56]

Consider the following problem

$$u_t + uu_x - u_{xxt} = 0 \quad (0 < x < 80) \quad (62)$$

with initial condition

$$\begin{aligned}u(x,0) &= 3A_1\,\mathrm{sech}^2(k_1(x-x_1-A_1))\\ &\quad + 3A_2\,\mathrm{sech}^2(k_2(x-x_2-A_2))\end{aligned} \quad (63)$$

and boundary conditions

$$u(0,t) = u(80,t) = 0. \quad (64)$$

Parameters $k_1 = k_2 = 0.5$, $x_1 = 10$, $x_2 = 25$, $A_1 = 1.5$, $A_2 = 0.75$ are chosen to coincide with those used by Esen.[57]

We use the kp-Ritz method to solve the above equations. The penalty factor $\alpha = 10^5$ and scaling parameter $d_{\max} = 2.15$, and the time-step length $\Delta t = 0.1$. The weight function is chosen as a Gauss function and the bases are chosen to be linear. In Fig. 2, two solitary wave solutions are plotted with the initial condition and given parameters at times $t = 0$, 15, 30.

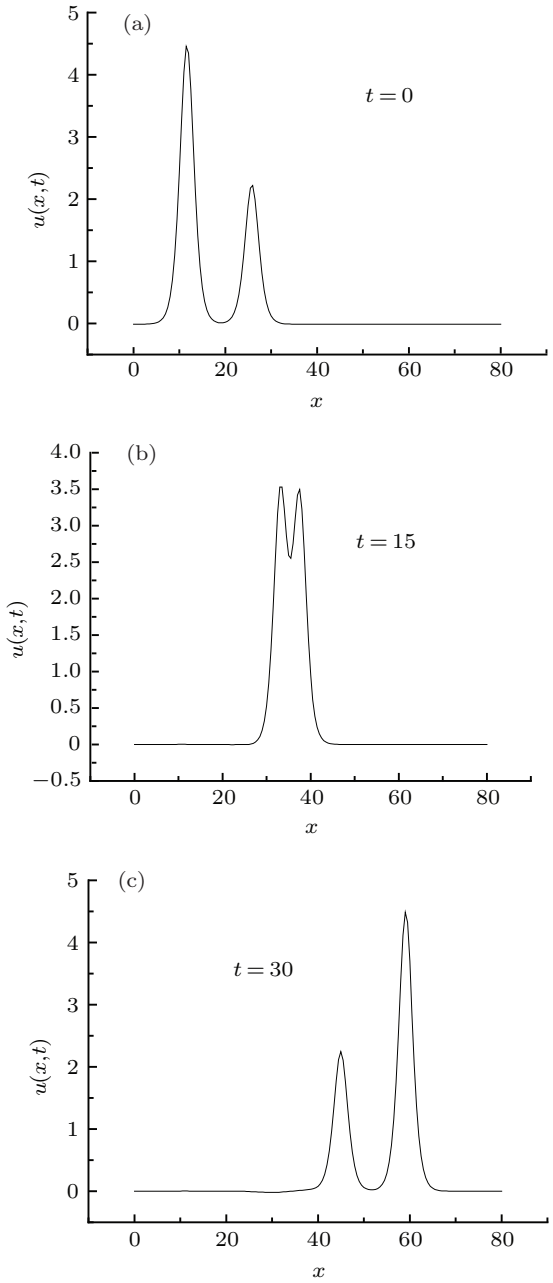


Fig. 2. Two solitary wave solutions when $t = 0$ (a), 15 (b), 30(c) (example 2).

3.3. Example 3: The undular bore

We finally consider Eq. (1a) with the initial condition

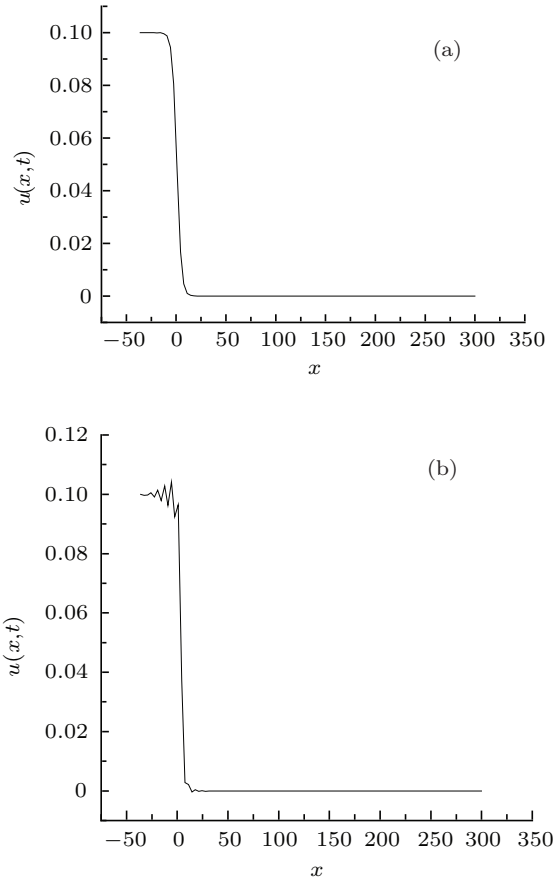
$$u(x,0) = 0.5U_0 \left[1 - \tanh\left(\frac{x - x_c}{d}\right)\right] \tag{65}$$

and the boundary condition

$$u(a,t) = U_0, \quad u(b,t) = 0, \tag{66}$$

where $U(x,0)$ denotes the elevation of the water above the equilibrium surface at time $t = 0$. The change in water level of magnitude U_0 is centred at $x = x_c$, and d measures the steepness of the change. The smaller the value of d, the steeper the slope is.

The kp-Ritz method is used to solve the above equation with the penalty factor $\alpha = 10^5$ and scaling parameter $d_{\max} = 2.5$, and the time-step length $\Delta t = 0.01$, $\varepsilon = 1.5$, $\mu = 1/6$, $U_0 = 0.1$, and $d = 5$. The weight function is chosen as a Gauss function and the bases are chosen to be linear. Figure 3 shows the undular bore profiles at times $t = 10$, 50, 100 for the gentle slope $d = 5$.



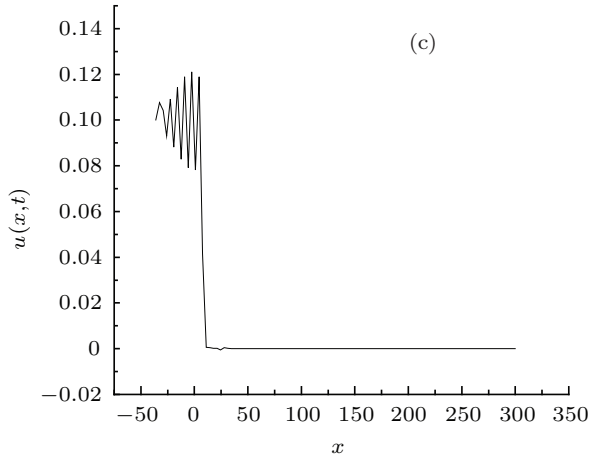


Fig. 3. Undulation profiles for slope $d = 5$ at time $t = 10$ (a), 50 (b), (100) (c) (example 3).

From these examples and obtained figures, we can conclude that numerical results derived from the kp-Ritz method to solve the EW equations are accurate and efficient.

4. Conclusion

In this paper, the nonlinear EW equation is investigated using the mesh-free kp-Ritz method. The mesh-free kernel particle estimate is employed to approximate the displacement field. A system of nonlinear discrete equations is obtained through the application of the Ritz minimization procedure to the energy expressions. The numerical results which are well consistent with the exact solutions show that the technique is accurate and efficient. A mesh-free method does not require a mesh to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. This method does not require mesh generation, which makes them conducive for solving the problems that require frequent re-meshing, such as those arising from nonlinear analysis. Due to the simplicity of its implementation, the proposed kernel particle Ritz method promises to be a potential alternative to the difference method and the finite element method for solving nonlinear EW equation. The kp-Ritz method should be extended to solve the problems of nonlinear partial differential equations which arise from the theory of solitons and other areas.

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