Epidemic spreading on a scale-free network with awareness*

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This paper presents a modified susceptible-infected-recovered (SIR) model with the effects of awareness and vaccination to study the epidemic spreading on scale-free networks based on the mean-field theory. In this model, when susceptible individuals receive awareness from their infected neighbor nodes, they will take vaccination measures. The theoretical analysis and the numerical simulations show that the existence of awareness and vaccination can significantly improve the epidemic threshold and reduce the risk of virus outbreaks. In addition, regardless of the existence of vaccination, the awareness can increase the spreading threshold and slow the spreading speed effectively. For a given awareness and a certain spreading rate, the total number of infections reduces with the increasing vaccination rate.

Keywords: scale-free network, awareness, vaccination, epidemic threshold

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1. Introduction

With the development of complex network theory,[1,2] epidemic spreading has become a hot research topic and attracted a lot of academic attention and interest. Many epidemic models have been studied, including the susceptible-infected (SI) model,[3−6] the susceptible-infected-susceptible (SIS) model,[7−9] and the susceptible-infected-removed (SIR) model.[10−12] The dynamical behavior of epidemics on scale-free (SF) networks has been studied by many researchers. Some network models with power-law degree distributions were presented to describe the disease transmission dynamics. For example, Bai and Zhou[4] investigated epidemic spreading with two major immunization strategies of the SI model on a scale-free network. The spreading of epidemics with nonlinear infectivity on scale-free networks was studied based on the SIS model[13] and the SIR model,[14] respectively. In this paper, we concentrate mainly on the spreading behavior of the SIR model on scale-free networks.

In particular, many susceptible individuals are aware of the potential risks of becoming infected and change their behavior to reduce the possibility of infection. Many researchers have attempted to study the impact of individually behavioral changes in response to awareness of disease.[15−17] For example, Funk *et al.*[16] introduced awareness on the homogeneous network to investigate how the spread of awareness affects the spread of a disease. It is known that epidemic spreading on scale-free networks is difficult to control and a virus can spread with a small infection rate. Therefore, immunization strategies have always been adopted as an effective control strategy. The most basic immunization strategies are random immunization,[18] targeted immunization,[19] and acquaintance immunization.[20] In this model, we will take vaccination measures for susceptible nodes that can receive awareness.

In this paper, susceptible nodes who are aware of the risks of becoming infected can send awareness to their susceptible neighbor nodes. In other words, susceptible nodes can receive awareness from their infected neighbor nodes. When a susceptible node receives awareness, it will take vaccination measures. In this study, we will investigate the variety of the epidemic threshold between the modified model and the standard model and analyze the effects of awareness and vaccination on epidemic spreading.

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The organization of this paper is as follows. In Section 2, the modified SIR model with awareness and vaccination on scale-free networks is presented. In Section 3, we analyze the modified SIR model and deduce the mathematical expression of the epidemic threshold. Some numerical simulations are performed in Section 4. Finally, Section 5 concludes this study.

2. The SIR model with awareness

For the SIR epidemic model on scale-free networks, each individual is represented by a node of the network and the edges are the physical interactions between individuals along which the infection spreads. Let $S_k(t)$, $I_k(t)$, and $R_k(t)$ be the densities of susceptible, infected and recovered nodes of degree k at time t. At each time step, each susceptible node is infected with probability υ if it is connected to an infected individual. On the other hand, the infected node recovers and becomes immune node with probability δ. Therefore, $\lambda = \upsilon/\delta$ (without lack of generality, we set $\delta = 1$) is considered as an effective spreading rate. If a susceptible node has $k_{\rm inf}$ infectious neighbors, then the total probability that the node becomes infected is $1-(1-\lambda)^{k_{\rm inf}}$.

These variables satisfy the following normalization condition:

$$I_k(t) + S_k(t) + R_k(t) = 1. \tag{1}$$

Global quantities are accordingly expressed by the average over the various degree classes as

$$\begin{aligned} S(t) &= \sum_k P(k) S_k(t), \\ I(t) &= \sum_k P(k) I_k(t), \\ R(t) &= \sum_k P(k) R_k(t). \end{aligned} \tag{2}$$

When susceptible individuals are aware of the potential risks of becoming infected, they will send awareness to their susceptible neighbor nodes. In other words, susceptible nodes can receive awareness from its infected neighbor nodes. Generally, the higher the degree a node has, the higher the feasibility of awareness is.[21,22] Let $k_{\rm inf}$ be the total number of infected neighbor nodes of susceptible nodes of degree k. The awareness in the susceptible part of the population $g(\rho, k)$ can be supposed to be $g(\rho, k_{\rm inf}) = 1-(1-\rho)^{k_{\rm inf}}$, the probability generating function of awareness within susceptible nodes, where the decay constant $\rho \in (0,1)$. The awareness of susceptible nodes varies with the total numbers of infected neighbor nodes. An example of awareness function $g(\rho, k_{\rm inf})$ is shown in Fig. 1.

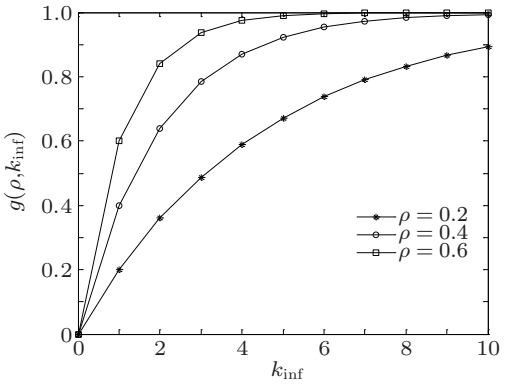


Fig. 1. Awareness function $g(\rho, k_{\rm inf})$ for different ρ.

Generally, susceptible nodes can make some responses to reduce their susceptibility when receiving awareness. In this paper, susceptible nodes can receive awareness from infected neighbor nodes and take vaccination measures with probability $\mu \in (0,1)$. In our model, the vaccinated fraction only works for the susceptible nodes.

We define λ' as a variable as follows:

$$\lambda'(\rho, k) = \lambda\left[1 - g(\rho, k)\right]. \tag{3}$$

In summary, the differential equations for modified SIR model based on the mean-field theory are given as

$$\begin{aligned} \frac{{\rm d}I_k(t)}{{\rm d}t} &= -I_k(t) + \lambda[1-g(\rho,k)](1-\mu)kS_k(t)\Theta(t), \\ \frac{{\rm d}S_k(t)}{{\rm d}t} &= -\lambda[1-g(\rho,k)](1-\mu)kS_k(t)\Theta(t), \\ \frac{{\rm d}R_k(t)}{{\rm d}t} &= I_k(t), \end{aligned} \tag{4}$$

where $0 \le \Theta(t) \le 1$ denotes the average probability of any given link point to an infected node. Considering the non-homogeneity of an uncorrelated scale-free network, $\Theta(t)$ satisfies the relation

$$\Theta(t) = \frac{1}{\langle k \rangle} \sum_k kP(k) I_k(t), \tag{5}$$

where $\langle k \rangle = \sum\limits_k kP(k)$.

When $g(\rho, k) = 0$, $\mu = 0$, the mode becomes the classical SIR model on the scale-free network.

3. Theoretical analysis of the epidemic threshold

The initial conditions of the above equations are $R_k(0) = 0$, $I_k(0) = I_k^0$, and $S_k(0) = 1 - I_k^0$. In the general case, I_k^0 is very small, then $I_k^0 \cong 0$, so we can obtain $S_k(0) \cong 1$. Under these approximately initial conditions, Eqs. (3)–(5) can be directly integrated respectively as

$$S_k(t) = \exp\left[-\lambda(1-\mu)[1-g(\rho,k)]k\int_0^t \Theta(u)\,\mathrm{d}u\right],$$
$$R_k(t) = \int_0^t I_k(u)\,\mathrm{d}u. \tag{6}$$

Defining an auxiliary function $\varphi(t)$ to represent the value of integration in Eq. (6), we have

$$\begin{aligned}\varphi(t) &= \int_0^t \Theta(u)\,\mathrm{d}u \\ &= \frac{1}{\langle k\rangle}\sum_k kP(k)\int_0^t I_k(u)\,\mathrm{d}u \\ &= \frac{1}{\langle k\rangle}\sum_k kP(k)R_k(t).\end{aligned} \tag{7}$$

Calculating the time derivative of function $\varphi(t)$, we can obtain

$$\begin{aligned}\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_k kP(k)I_k(t) \\ &= \frac{1}{\langle k\rangle}\sum_k kP(k)[1-S_k(t)-R_k(t)] \\ &= 1-\frac{1}{\langle k\rangle}\sum_k kP(k)S_k(t)-\varphi(t).\end{aligned} \tag{8}$$

In the steady state, $I_k(\infty) = 0$. We can obtain

$$\lim_{t\to\infty}\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = 0. \tag{9}$$

We can derive from Eq. (7) to Eq. (9) that

$$\begin{aligned}\varphi(\infty) &= 1-\frac{1}{\langle k\rangle}\sum_k kP(k)S_k(\infty) \\ &= 1-\frac{1}{\langle k\rangle}\sum_k kP(k) \\ &\quad\times\exp\left[-\lambda(1-\mu)[1-g(\rho,k)]k\varphi(\infty)\right] \\ &\equiv F(\varphi(\infty)).\end{aligned} \tag{10}$$

Obviously $\varphi(\infty) = 0$ is a trivial solution of Eq. (10). For an epidemic outbreak, there must be a non-zero solution of $\varphi(\infty)$, satisfying

$$\left.\frac{\mathrm{d}F(\varphi(\infty))}{\mathrm{d}\varphi(\infty)}\right|_{\varphi(\infty)=0} \geq 1. \tag{11}$$

That is,

$$\lambda(1-\mu)\frac{\langle k^2[1-g(\rho,k)]\rangle}{\langle k\rangle} \geq 1. \tag{12}$$

Further simplifying Eq. (12), we have

$$\lambda(1-\mu)\frac{\langle k^2(1-\rho)^{k_{\inf}}\rangle}{\langle k\rangle} \geq 1. \tag{13}$$

Define $R(\infty)$ as the infection degree of epidemic in the steady state and let λ_c be the epidemic threshold of scale-free network. From Eq. (13), we can obtain the mathematical expression of the epidemic threshold as follows:

$$\lambda_{\mathrm{c}} = \frac{1}{1-\mu}\frac{\langle k\rangle}{\langle k^2(1-\rho)^{k_{\inf}}\rangle}, \tag{14}$$

where

$$\langle k\rangle = \sum_k kP(k),\ \langle k^2(1-\rho)^{k_{\inf}}\rangle = \sum_k k^2(1-\rho)^{k_{\inf}}P(k).$$

If $\lambda > \lambda_{\mathrm{c}}$, the epidemic will break out and spread, and reach a special steady state at last; otherwise, the infection will die out gradually.

Next, we want to analyze the specific mathematical expression of the epidemic threshold in the BA scale-free network by using a continuous k approximation, which allows a practical substitution of a series with integrals.[12] The full connectivity distribution can be obtained as $P(k) = 2m^2k^{-3}$, where m is the minimum number of connections at each individual node. We can obtain

$$\langle k\rangle = \int_m^\infty kP(k)\,\mathrm{d}k = 2m, \tag{15}$$
$$\langle k^2\rangle = \int_m^\infty k^2P(k)\,\mathrm{d}k, \tag{16}$$

$$\begin{aligned}\langle k^2(1-\rho)^{k_{\inf}}\rangle &= \int_m^\infty k^2(1-\rho)^{k_{\inf}}P(k)\,\mathrm{d}k \\ &< \int_m^\infty k^2P(k)\,\mathrm{d}k = \langle k^2\rangle.\end{aligned} \tag{17}$$

Furthermore, we have

$$\begin{aligned}\lambda_{\mathrm{c}} &= \frac{1}{1-\mu}\frac{\langle k\rangle}{\langle k^2(1-\rho)^{k_{\inf}}\rangle} \\ &> \frac{\langle k\rangle}{\langle k^2(1-\rho)^{k_{\inf}}\rangle} \\ &> \frac{\langle k\rangle}{\langle k^2\rangle}.\end{aligned} \tag{18}$$

Statistically speaking, a virus can easily survive and cause an outbreak in an infinitely scale-free network no matter how weak its spreading capability is

when the susceptible nodes do not receive awareness. In the standard SIR model, $\lambda_c = \langle k\rangle/\langle k^2\rangle$. Formula (18) shows that the presence of awareness and the application of vaccinations can significantly increase the epidemic threshold of the network.

Finally, the number of infections is obtained as

$$\begin{aligned} R(\infty) &= \sum_k P(k)\,[1 - S_k(\infty)] \\ &= \sum_k P(k)[1 - \exp[-\lambda(1-\mu) \\ &\quad \times (1-\rho)^{k_{\inf}} k\varphi(\infty)]] \\ &< \sum_k P(k)\,[1 - \exp(-\lambda k\varphi(\infty))]\,. \end{aligned} \quad (19)$$

In the standard SIR model, $R(\infty) = 2m^2 \sum_k k^{-3}\,[1 - \exp(-\lambda k\varphi(\infty))]$. Formula (19) shows that the number of infectious is much smaller than that in the classical SIR model.

4. Simulation results and discussion

In this section, we perform the simulations on N-node BA networks[14] to verify our analysis. The parameters of generating a BA network are $N = 1000$, $m_0 = m = 3$, and $\langle k\rangle = 6$. For each simulation, the infected node is randomly chosen at the beginning.

For all susceptible nodes, we choose the awareness function as

$$g(\rho, k_{\inf}) = \begin{cases} 1 - (1-\rho)^{k_{\inf}}, & \rho \in (0,1), \\ 0, & \rho = 0, \end{cases} \quad (20)$$

where $k_{\inf}$ is the number of infected neighbor nodes.

In Fig. 2, $\rho = 0, \mu = 0$ (circles) corresponds to no awareness, and $\rho = 0.4, \mu = 0$ (forks) corresponds to no vaccination. Both awareness and vaccination exist when $\rho = 0.4, \mu = 0.4$ (stars). Through the theoretical analysis, we can obtain the spreading threshold $\lambda_c = \langle k\rangle/\langle k^2\rangle$, $\langle k\rangle/\langle k^2(1-\rho)^{k_{\inf}}\rangle$, and $\langle k\rangle/[(1-\mu)\langle k^2(1-\rho)^{k_{\inf}}\rangle]$, respectively.

Figure 2 shows that the numerical simulation is approximately consistent with the theoretical analysis. That is to say, the awareness and the vaccination have effect on the spreading threshold, which demonstrates that the proposed SIR model with awareness is reasonable. Compared with the classical SIR model with $g(\rho, k) = 0, \mu = 0$, the presence of awareness and the applying of vaccination can significant increase λ_c of the network, thereby reducing the risk of virus outbreaks. Furthermore, we can see that the presence of awareness affects the epidemic threshold regardless of the application of vaccination in Fig. 2.

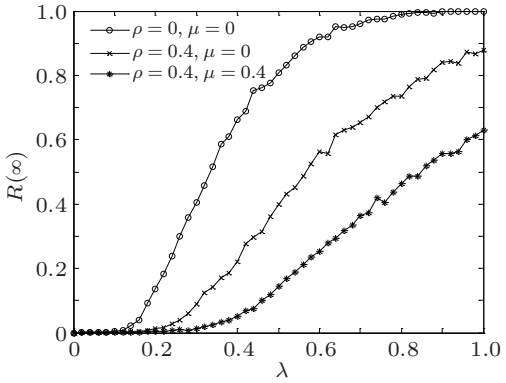


Fig. 2. The relations between $R(\infty)$ and λ for three different cases on the BA network.

In Figs. 3 and 4, the awareness parameters are $\rho = 0.2, 0.4, 0.6$ (circles, forks, stars). Figures 3 and 4 show that the awareness can increase the spreading threshold and slow the epidemic spreading speed effectively regardless of the existence of vaccination. Furthermore, we can draw the conclusion that the epidemic threshold increases with the awareness parameter ρ increasing for a fixed vaccination rate.

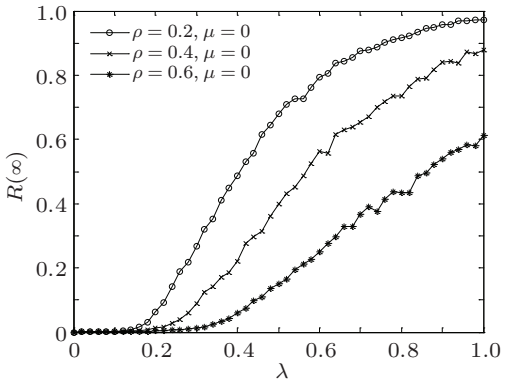


Fig. 3. The relations between $R(\infty)$ and λ for different awareness parameters with no vaccination.

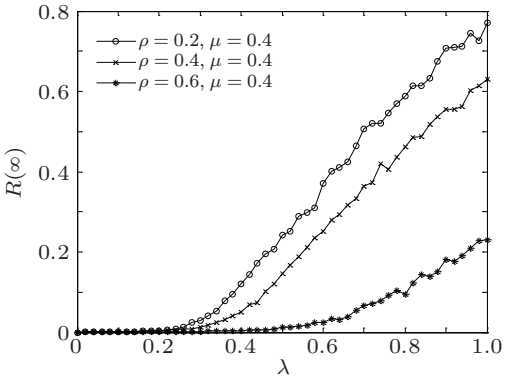


Fig. 4. The relations between $R(\infty)$ and λ for different awareness parameters with vaccination.

In Fig. 5, different vaccination rates are $\mu = 0, 0.2, 0.4, 0.6$ (squares, circles, forks, stars). Analyzing the curves in Fig. 5, we can also find that the accretion of vaccination probability can enhance the spreading threshold and weaken the outbreak of a virus when the awareness parameter takes a fixed value.

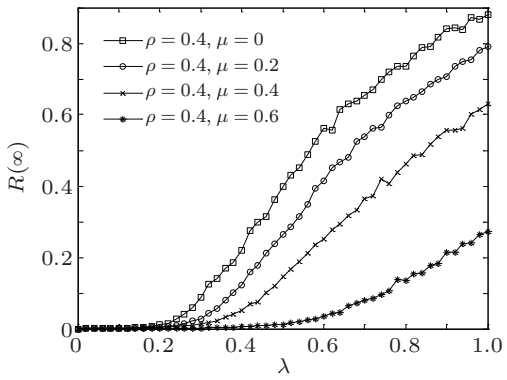


Fig. 5. The relations between $R(\infty)$ and λ for different vaccination rates.

Different spreading rates are $\lambda = 0.2, 0.4, 0.6$ (circles, forks, stars) in Fig. 6. Figure 6 shows that the total number of infections reduces with the vaccination rate increasing for given awareness and certain spreading rate, and with the epidemic spreading rate reducing.

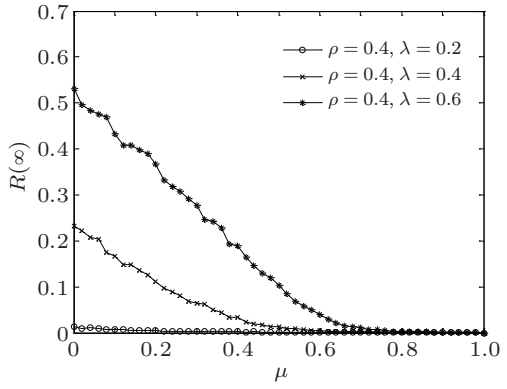


Fig. 6. The relations between $R(\infty)$ and μ with different spreading rates.

5. Conclusion

In this paper, we studied the epidemic spreading in terms of the mean-field theory on scale-free networks, and proposed a modified susceptible-infected-recovered (SIR) model with awareness. In particular, many susceptible individuals who are aware of the potential risks of becoming infected can take measures to reduce the possibility of infection. In this new model, the susceptible nodes can receive awareness from the infected neighbor nodes and take vaccination measures. The theoretical analysis and the simulation results show that the existence of awareness and the application of vaccinations can significantly reduce the risk of virus outbreak, increase the spreading threshold, and slow the spreading speed in the scale-free networks. Regardless of the application of vaccinations, the existence of awareness can increase the spreading threshold. In order to better control the epidemic spreading on networks, the effects of awareness for epidemic spreading is worthy of further study.

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