Analytical investigation of the boundary-triggered phase transition dynamics in a cellular automata model with a slow-to-start rule*

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Previous studies suggest that there are three different jam phases in the cellular automata automaton model with a slow-to-start rule under open boundaries. In the present paper, the dynamics of each free-flow-jam phase transition is studied. By analysing the microscopic behaviour of the traffic flow, we obtain analytical results on the phase transition dynamics. Our results can describe the detailed time evolution of the system during phase transition, while they provide good approximation for the numerical simulation data. These findings can perfectly explain the microscopic mechanism and details of the boundary-triggered phase transition dynamics.

Keywords: traffic flow, boundary-triggered phase transition, cellular automata, time evolution, analytical solution

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1. Introduction

Traffic congestion dynamics has aroused substantial interest in the field of traffic flow research. It includes the evolution mechanism of how traffic congestions emerge and dissipate, which is meaningful for both traffic flow theory and traffic management in practice. From the viewpoint of statistical physics, such phenomena are phase transitions in a one-dimensional non-equilibrium system.

Cellular automaton (CA) is one of the most important tools for traffic flow modeling due to its simplicity and effectiveness.[1−4] Thus, numerous studies on traffic flow phase transitions using the CA model have been conducted. Earlier reports indicate that phase transitions are caused mainly by the speed fluctuation of vehicles[5] on a ring-shaped road. However, later models show that open boundary condition, i.e., an entrance with variable traffic demand and a limited-capacity exit of the road, is another essential origin of phase transitions.[6,7] This conclusion has been widely verified in the real world, e.g., ramps,[8,9] traffic signals,[10] tollbooths,[11] and so on.

Theoretical discussion about boundary-triggered phase transitions can be found in Refs. [12] and [13]. An interesting finding is that there are three different jam phases in the CA model with a slow-to-start rule, namely, jam I, jam II, and jam outflow (JO) phases. The detailed characteristics of each traffic jam phase are described and explained.

The applied model in these two reports is actually an expand version of the famous Nagel–Schreckenberg model, with four rules.

(i) Acceleration: $v = \min(v + 1, v_{\max})$.

(ii) Slow down: $v = \min(v, \text{gap})$ (where gap is the number of empty sites in front of the car).

(iii) Randomization: $v = \max(v-1, 0)$ with probability

$$p = \begin{cases} p_0 \geq 0 & \text{if } v = 0, \\ p_1 = 0 & \text{if } v > 0. \end{cases}$$

Due to this rule, the stopped cars are less likely to restart, thus it is named as a 'slow-to-start' rule;

(iv) Movement: $x = x + v$.

Under the different initial conditions listed in Ref. [12], we capture the space–time plots for each of the three jam phases by numerical simulation (See Fig. 1).

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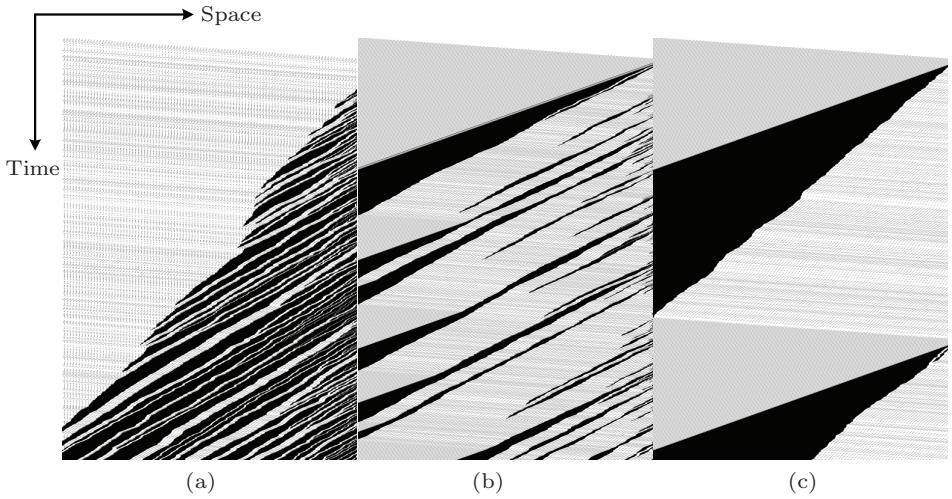


Fig. 1. Space–time plots of different jam phases of the proposed model: (a) Jam II, (b) JO, and (c) Jam I.

However, a better qualitative understanding of the numerical results is desirable and useful because there are still some open questions. During the process of phase transition, what exactly happens in the traffic flow? How does the system evolve with time? For this purpose, our work is to make a deeper investigation. The analytical results of the phase transition are successfully deduced in a good approximation.

The rest of the present paper is organized as follows. In Section 2, we state some necessary preliminaries. Afterwards, the analytical results of the boundary-triggered free-jam phase transition dynamics are deduced for each of the three jam phases. In Section 3, the presented results are verified by numerical simulations, which show that a good approximation is achieved. In Section 4 the findings of the present study are summarized.

2. Analysis of phase-transition dynamics

2.1. Preliminaries

According to the previous findings in Refs. [12] and [13], the final jam phase (Jam I, Jam II, or JO) depend on the size relationship among inflow $Q_{\rm in}$, jam outflow $Q_{\rm jo}$ (the flow runs out of a jam block), and capacity C. In a specific numerical simulation model, $Q_{\rm in}$ and C depend on the left and right boundary settings, respectively.[12,14,15] On the other hand, $Q_{\rm jo}$ varies with slow-to-start rate p_0. In the previously described model, we have the relationship:[12]

$$Q_{\rm jo} = \frac{v_{\max}\,(1-p_0)}{v_{\max}+(1-p_0)}.$$

The propagation speeds of the left and right ends of a traffic jam block are two important variables for our next discussion. For the left end, the moving speed is $v_{\rm jl} = v_{\max}Q_{\rm ji}/(v_{\max}-Q_{\rm ji})$, where $Q_{\rm ji}$ is the inflow of the jam block. The moving speed of the right boundary is $v_{\rm jr} = 1-p_0$.

Without loss of generality, the transition from free-flow to the jam phase is investigated using the following scenario. The road capacity C is initially set to be a constant, whereas the inflow is $Q_{\rm in} = 0$ (or a small value. It does not change the nature of this problem). Under such an initial configuration, the traffic flow is in the free-flow phase. After a period of time, we increase $Q_{\rm in}$ up to a value larger than C, and then the system evolves in the jam phase. Next, we analytically describe the time evolution of the system during the phase transition.

2.2. Jam II phase

In the Jam II phase, the first jam block keeps extending and new jam blocks are generated continuously. This phenomenon implies that $C < Q_{\rm jo} < Q_{\rm in}$. We divide the entire process into four stages (see Fig. 2).

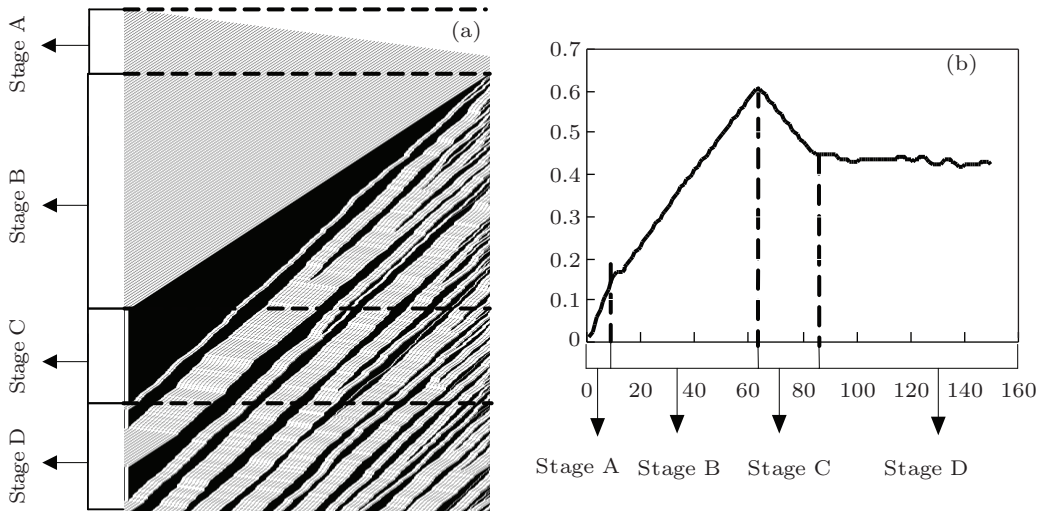


Fig. 2. Decomposition of the free-Jam II transition , showing (a) space-time plot and (b) time evolution of the global density.

Stage A This stage starts from the step when the first car runs into the road until the step when the first jam block is generated, during which the growth rate of the global density is approximately $Q_{\rm in}/L$. Since all cars run at their maximum speed, the first car arrives on the right boundary at step $L/v_{\max}$, when the global density is $Q_{\rm in}/v_{\max}$. Thus, the duration of Stage A is $T_{\rm A} = L/v_{\max}$.

Stage B The inflow of the first block is $Q_{\rm in}$ and its outflow is

$$Q_{\rm jo} = \frac{v_{\max}\,(1-p_0)}{v_{\max}+(1-p_0)}.$$

From the precondition that $Q_{\rm in} > Q_{\rm jo}$, the size of the first jam block will keep growing until it reaches the entrance of the road, when Stage B ends.

During Stage B, the growth rate is $\Delta\rho_{\rm B} = (Q_{\rm in} - C)/L$. From the above-mentioned analysis, the duration of Stage B is $T_{\rm B} = L/v_{\rm leading_l}$, where $v_{\rm leading_l}$ is the moving speed of the left end of the first jam block, which can be calculated by the formula provided in Section 2.

At the end of Stage B, the peak value of the global density is reached as $\rho_{\max} = T_{\rm B}\Delta\rho_{\rm B} + Q_{\rm in}/v_{\max}$. The size of the first jam block at this time is $L_{\rm jam}^{\max} = T_{\rm B}\,(v_{\rm leading_l} - v_{\rm leading_r})$, where $v_{\rm leading_r}$ is the moving speed of the right end of the first jam block.

Stage C When the first jam block reaches the entrance of the road, the car-injection is interrupted because the first site of the road is always occupied, which causes the global density to decrease. Given that the actual system outflow in the Jam II phase equals the capacity, the decreasing rate is $\Delta\rho_{\rm C} = C/L$. Stage C does not end until the first-generated jam block completely vanishes. Thus, Stage C lasts $T_{\rm C}$=$L_{\rm jam}^{\max}/v_{\rm leading_r}$ steps.

Figure 2 illustrates that the steady value of the global density is obtained at the last step of Stage C. Hence, the steady density is $\rho_{\rm stable} = \rho_{\max} - T_{\rm C}\Delta\rho_{\rm C}$, which can be easily simplified into

$$\rho_{\rm stable} = 1 - \frac{C}{1-p_0}.$$

Recall Ref. [14], the steady global density in the deterministic NS model ($p_0 = p_1 = 0$) has been proven to be $1-C$ by a completely different approach. Thus, we can see that the above equation is the general form for the case $p_0 > 0$.

Stage D The global density fluctuates around the steady value $\rho_{\rm stable}$.

2.3. JO phase

The free-JO phase transition requires $Q_{\rm jo} < C < Q_{\rm in}$. In the JO phase, once a jam block is formed, no new jam blocks can be generated. Thus, the road is always dominated by a single large jam block, when the system outflow is $Q_{\rm jo}$. We subsequently divide the entire procedure into four stages (see Fig. 3).

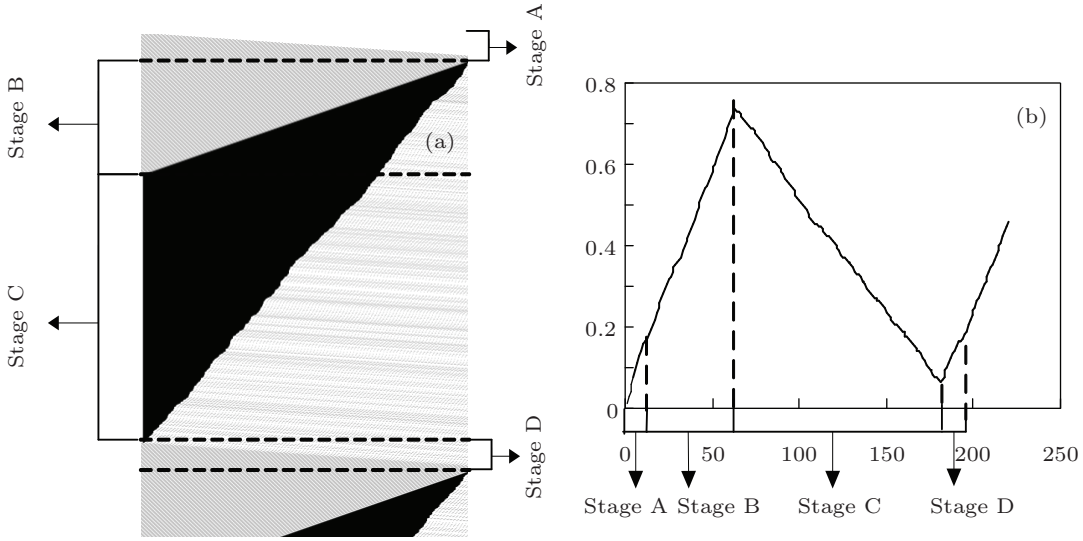


Fig. 3. Decomposition of the free-JO transition, showing (a) space–time plot and (b) time evolution of the global density.

Stage A The Stage A is exactly the same as that in the free-Jam II transition.

Stage B Considering that $Q_{\rm jo} < C$, once the first block is formed, the actual system outflow equals $Q_{\rm jo}$. Although no new jam blocks can be generated, the size of the first jam block keeps growing. We define the period from the appearance of the first jam block to its arrival at the entrance of the road as Stage B.

The growth rate of the density during Stage B is $\Delta\rho_{\rm B} = (Q_{\rm in} - Q_{\rm jo})/L$. At the end of Stage B, the peak value over a period is reached as $\rho_{\max} = T_{\rm B}\Delta\rho_{\rm B} + Q_{\rm in}/v_{\max}$. At this time, the size of the leading jam block also reaches its maximum, $L_{\rm jam}^{\max} = T_{\rm B}\,(v_{\rm leading_l} - v_{\rm leading_r})$, where $T_{\rm B} = L/v_{\rm leading_l}$ is the duration of Stage B.

Stage C The first jam block starts to dissipate at this stage, and the decrease rate of the global density can be calculated as $\Delta\rho_{\rm C} = Q_{\rm jo}/L$. At the end of Stage C, the global density reaches its minimum in one period, $\rho_{\min} = \rho_{\max} - T_{\rm C}\Delta\rho_{\rm C}$, where $T_{\rm C} = L_{\rm jam}^{\max}/v_{\rm leading_r}$ is the duration of Stage C.

The above equation can be simplified into $\rho_{\min} = Q_{\rm jo}/v_{\max}$.

Stage D At the beginning of this stage, jam blocks no longer exist. New car can run into the road freely once again. However, the actual outflow of the road is still $Q_{\rm jo}$ for a short while until the 'newly inserted' car reaches the exit of the road. Stage D is found to be quite similar to Stage A, except for the initial flow on the road. Hence, the growth rate during this stage is $(Q_{\rm in} - Q_{\rm jo})/L$ and its duration is $T_{\rm D} = L/v_{\max}$. Note that Stage B resumes after Stage D has finished, thus the JO phase actually has no steady state. This is a unique property of JO phase.

2.4. Jam I phase

The Jam I phase has many similar characteristics to the Jam II phase. The main difference is that in the Jam II phase, the first jam block keeps shrinking until it completely disappears and is replaced by the succeeding jam block. This phase transition requires $C < Q_{\rm in} < Q_{\rm jo}$. Similarly, we divide the dynamic process into three stages, and the analytical results are proposed (see Fig. 4):

Stage A This stage is exactly the same as that in previous cases. When the first jam block emerges, Stage A terminates.

Stage B According to the precondition $Q_{\rm in} < Q_{\rm jo}$, the size of the first jam block keeps decreasing. However, given that $C < Q_{\rm in}$, the jam part of the traffic flow can still propagate from the right end to the left end. When the entire road is dominated by the jam flow, Stage B terminates.

During this stage, the actual outflow equals the capacity. Thus, the growth rate is $\Delta\rho_{\rm B} = (Q_{\rm in} - C)/L$ and the duration of Stage B is $T_{\rm B} = (\rho_{\rm stable} - Q_{\rm in}/v_{\max})/\Delta\rho_{\rm B}$. In this case, $\rho_{\rm stable}$ cannot be directly calculated in a way similar to that in the previous two cases. However, many studies conclude that in the jam phase, the steady global density is irrelevant to insert rate α.[6,7,14] Thus, when we increase α to obtain $Q_{\rm in} > Q_{\rm jam}$, the steady density does not change. Hence, the steady density in this case is also $\rho_{\rm stable} = 1 - C/(1 - p_0)$.

Stage C The free-flow region completely disappears. Hence, the global density fluctuates around $\rho_{\rm stable}$.

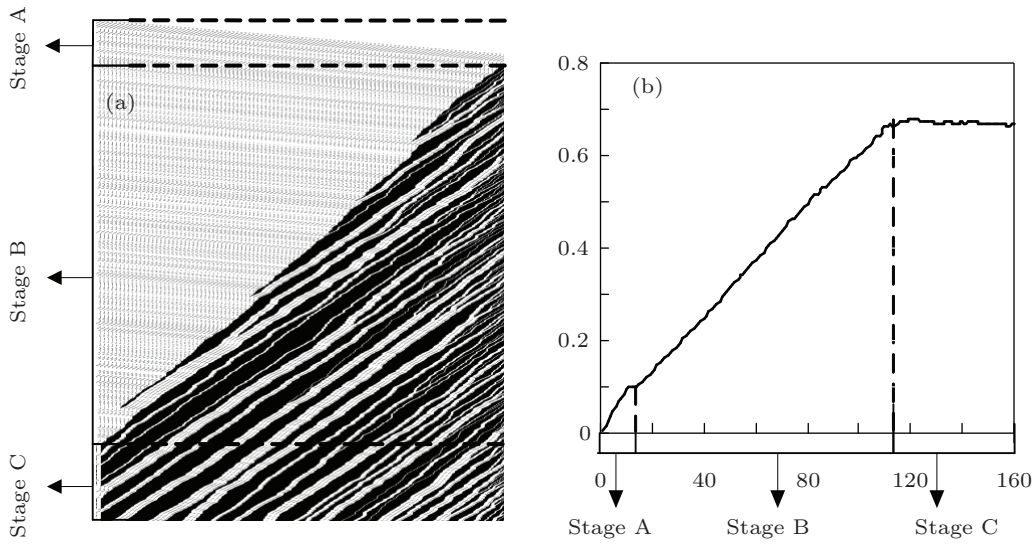


Fig. 4. Decomposition of the free-Jam I transition, showing (a) space–time plot (b) time evolution of the global density.

3. Validation of the analytical results

Note that in the previous section, we take Q_{in} and C as core variables for the analysis. However, in a 'real' numerical simulation, we can only indirectly adjust Q_{in} by insert rate α (for all widely used left boundaries, this is true), and capacity C (for the capacity, removal rate β is used). Consequently, the relationship between Q_{in} and α, and that between C and β strongly depend on the applied boundary conditions. Thus, for the validation of the analytical results, we need to specify the boundary conditions first.

The expanded left boundary[12] is used to create the inflow, in which the relationship between insert rate α and inflow is $Q_{\text{in}} = \alpha(\alpha^{v_{\max}} - 1)/(\alpha^{v_{\max}+1} - 1)$. For the right boundary, the strategy proposed in Ref. [14] is adopted. This boundary condition is a variation of the traditional stochastic right boundary, with the 'vertical queue' hypothesis in traffic flow theory adopted. With such a right boundary, we have $C = \beta$.[14]

To avoid stochastic fluctuation, we collect 100 samples to obtain the average value for each numerical simulation in the next subsections.

3.1. Jam II phase

We can verify the analytical results by the following parameters: $\alpha = 1$, $p_0 = 0.3$, $\beta = 0.4$, and the road length $L = 500$, from which we obtain $Q_{\text{in}} = 0.833$, $Q_{\text{jo}} = 0.614$, and $C = 0.4$. Figure 5 indicates that the analytical results are highly accurate (Fig. 5).

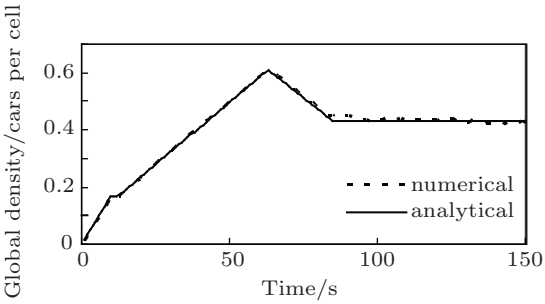


Fig. 5. Analytical and numerical results of the time evolution of global density during free-Jam II transition.

3.2. JO phase

A case study is carried out to verify our findings. In this case, the parameters are $\alpha = 1$, $p_0 = 0.7$, $\beta = 0.5$, and $L = 500$. Under these parameters, the following quantities are calculated to be $Q_{\text{in}} = 0.833$, $Q_{\text{jo}} = 0.283$, and $C = 0.5$. The comparison indicates that the analytical results have good approximation to the numerical results (Fig. 6).

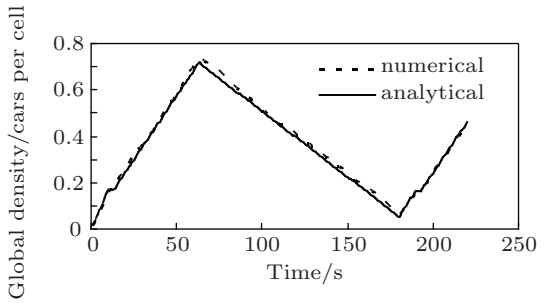


Fig. 6. Analytical and numerical results of the time evolution of global density during free-JO transition.

3.3. Jam I phase

Similarly, another simulation is conducted with the following parameters: $\alpha = 0.5$, $p_0 = 0.4$, $\beta = 0.2$, and $L = 500$. Consequently, the following quantities are calculated to be $Q_{\text{in}} = 0.492$, $Q_{\text{jo}} = 0.536$, and $C = 0.2$. The analytical and numerical results both are depicted in Fig. 7.

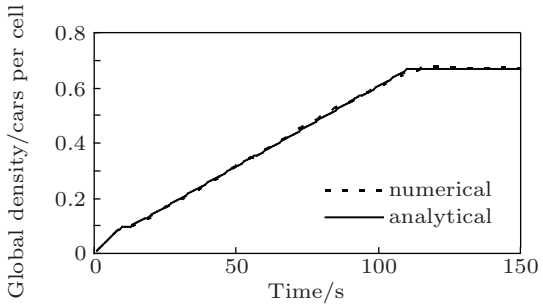


Fig. 7. Analytical and numerical results of the time evolution of global density during free-Jam I transition.

4. Conclusion

We conduct a theoretical study on the boundary-triggered freeflow-jam phase transition in an NS-based CA model with a slow-to-start rule. Previous studies revealed that such a model exhibits three totally different jam phases. We further studied this phenomenon in the present work. The highly accurate analytical results of the boundary-triggered phase transition dynamics are obtained by analysing the microscopic behaviours of the system. These results can perfectly explain the microscopic mechanism and details of the phase transition dynamics, which enable one to obtain a deeper insight into the model.

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