Nonlinear integrable couplings of a nonlinear Schrödinger–modified Korteweg de Vries hierarchy with self-consistent sources*

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By means of the Lie algebra B_2, a new extended Lie algebra F is constructed. Based on the Lie algebras B_2 and F, the nonlinear Schrödinger–modified Korteweg de Vries (NLS–mKdV) hierarchy with self-consistent sources as well as its nonlinear integrable couplings are derived. With the help of the variational identity, their Hamiltonian structures are generated.

Keywords: nonlinear integrable couplings, NLS–mKdV hierarchy, self-consistent sources

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1. Introduction

It is an important task in soliton theory to find new integrable hierarchies and their couplings systems such as those connecting with well-known physical meaning equations. Some integrable hierarchies without self-consistent sources have been obtained,[1−8] such as the AKNS hierarchy, the KN hierarchy, the Dirac hierarchy, and so on. With the development of soliton theory, people began to focus on the soliton equations with self-consistent sources.[9−14] Soliton equations with self-consistent sources have important physical meaning and are often used to express interactions between different solitary waves and are relevant to some problems of hydrodynamics, solid state physics, plasma physics, and so on.[15,16] Recently, Yu and Xia *et al.* studied the integrable couplings of hierarchies with self-consistent sources and obtained a series of integrable hierarchies with self-consistent sources and their coupling systems as well as their Hamiltonian structures.[17−20] In these papers, the coupling systems of hierarchies with self-consistent sources are linear for supplementary variables. Meanwhile, Ma proposed a general scheme to generate the nonlinear integrable couplings of hierarchies.[21] Based on the general scheme, Zhang[22] constructed some Lie algebras and derived the nonlinear integrable couplings of the Yang hierarchy and the GJ hierarchy. Yu[23] discussed the prolongation structure for nonlinear integrable couplings of a KdV soliton hierarchy. We note that there are very few researches on the nonlinear integrable couplings of hierarchies with self-consistent sources available up to now. In this paper, we will generate the nonlinear integrable couplings of hierarchy with self-consistent sources. It enriches the content of the integrable couplings of hierarchies with self-consistent sources.

The aim of the present paper is to derive the nonlinear integrable couplings of the nonlinear Schrödinger–modified Korteweg de Vries (NLS–mKdV) hierarchy with self-consistent sources. The plan of the paper is arranged as follows. In Section 2, a Lie algebra B_2 is introduced, which is different from the common Lie algebra $sl(2)$. Based on the Lie algebra B_2, the NLS–mKdV hierarchy with self-consistent sources is given. By employing the variational identity, the Hamiltonian structure of the above hierarchy is obtained. With the help of the Lie algebra B_2, an extended matrix Lie algebra F is presented

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in Section 3. By using the Lie algebra F, the nonlinear integrable couplings of the NLS–mKdV hierarchy with self-consistent sources and their Hamiltonian structures are derived. Finally, some conclusions are obtained in Section 4.

2. NLS–mKdV hierarchy with self-consistent sources

Consider the following set of Lie algebra B_2:[24]

$$e_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad e_2=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad e_3=\begin{pmatrix}1&2\\0&-1\end{pmatrix}. \tag{1}$$

Take

$$M=\begin{pmatrix}-1&0\\1&1\end{pmatrix},$$

the commutator is defined as $[e_i,e_j]=e_iMe_j-e_jMe_i$, $i,j=1,2,3$. We have

$$[e_1,e_2]=2e_2,\quad [e_1,e_3]=2e_2-2e_3,\quad [e_2,e_3]=-2e_1.$$

It is easily found that the Lie algebra B_2 is far different from the Lie algebra $sl(2)$. In terms of expression (1), a loop algebra $\tilde{B_2}$ is presented as

$$\begin{aligned}&e_i(n)=e_i\lambda^n,\quad [e_i(m),e_j(n)]=[e_i,e_j]\lambda^{m+n},\\&\deg(e_i(n))=n,\quad i,j=1,2,3.\end{aligned} \tag{2}$$

Consider an isospectral problem

$$\begin{aligned}&\varphi_x=UM\varphi,\quad \lambda_t=0,\\&U=\begin{pmatrix}\lambda+u_1+u_3&2u_3\\u_2&\lambda+u_1-u_3\end{pmatrix}.\end{aligned} \tag{3}$$

Set $V=\sum_{m\geq 0}(a_me_1(-m)+b_me_2(-m)+c_me_3(-m))$, solving the stationary zero curvature equation

$$V_x=[U,V], \tag{4}$$

one arrives at the recursion relation as follows:

$$\begin{cases}a_{mx}=2u_3b_m-2u_2c_m,\\ b_{mx}=2b_{m+1}+2u_1b_m-2u_2a_m-2u_3a_m+2c_{m+1}+2u_1c_m,\\ c_{mx}=2u_3a_m-2c_{m+1}-2u_1c_m,\\ a_0=\beta\ (\text{const})\neq 0,\quad b_0=c_0=0,\quad b_1=\beta u_2,\quad c_1=\beta u_3, a_1=0,\\ a_2=\beta u_2u_3+\dfrac{\beta}{2}u_3^2,\quad b_2=-\beta u_1u_2+\dfrac{\beta}{2}u_{3x}+\dfrac{\beta}{2}u_{2x},\quad c_2=-\beta u_1u_3-\dfrac{\beta}{2}u_{3x}.\end{cases} \tag{5}$$

For an arbitrary natural number n, denoting

$$V_+^{(n)}=\sum_{m\geq 0}^{n}[a_me_1(n-m)+b_me_2(n-m)+c_me_3(n-m)]=\lambda^nV-V_-^{(n)},$$

then Eq. (4) can be written as $-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}]$, a direct calculation reads

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=-2(b_{n+1}+c_{n+1})e_2(0)+2c_{n+1}e_3(0).$$

Taking $V^{(n)}=V_+^{(n)}+\Delta_n$, $\Delta_n=-a_ne_1(0)$, then the zero curvature equation

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{6}$$

leads to the following integrable system

$$\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}_t=\begin{pmatrix}-a_{nx}\\b_{nx}-2u_1(b_n+c_n)\\c_{nx}+2u_1c_n\end{pmatrix}=\begin{pmatrix}-\partial&0&0\\0&\partial&-\partial+2u_1\\0&-\partial-2u_1&0\end{pmatrix}\begin{pmatrix}a_n\\-c_n\\-b_n-c_n\end{pmatrix}=J_1P_{1n}, \tag{7}$$

where J_1 is the Hamiltonian operator and $P_{1n}=LP_{1n-1}$, the recurrence operator L is as follows:

$$L=\begin{pmatrix}0&-2\partial^{-1}u_1u_3-\partial^{-1}u_2\partial-2\partial^{-1}u_1u_2&2\partial^{-1}u_1u_3-\partial^{-1}u_3\partial\\-u_3&-u_1-\frac{1}{2}\partial&0\\-u_2-u_3&-\frac{1}{2}\partial&-u_1+\frac{1}{2}\partial\end{pmatrix}. \tag{8}$$

Therefore expression (7) can be written as

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t = J_1 L^n \begin{pmatrix} a_0 \\ -c_0 \\ -b_0 - c_0 \end{pmatrix}. \tag{9}$$

When taking $u_1 = 0$, $u_2 = u_3$, $n = 3$, $\beta = -4$, we obtain the mKdV equation

$$u_{2t} + 6u_2^2 u_{2x} + u_{2xxx} = 0.$$

When taking $u_1 = 0$, $u_2 = 1$, $n = 3$, we obtain the composed KdV–mKdV equation

$$u_{3t} + (\alpha + \beta u_3) u_3 u_{3x} + \gamma u_{3xxx} = 0.$$

When taking $u_1 = 0$, $u_2 = \mathrm{i}\tilde{u}_2$, $n = 3$, denoting $Q = u_2 + u_3 = u_3 + \mathrm{i}\tilde{u_2}$, we have the well-known nonlinear Schrödinger equation

$$\mathrm{i}Q_t + Q_{xx} - |Q|^2 Q = 0.$$

So we call the system (9) NLS–mKdV hierarchy.

For seeking the Hamiltonian structure of the system (9), we introduce a linear functional by

$$g_1(a, b) = 2(a_1 b_1 - a_3 b_2 - a_2 b_3 - a_3 b_3), \tag{10}$$

where $\forall\ a = a_1 e_1(0) + a_2 e_(0) + a_3 e_3(0)$, $b = b_1 e_1(0) + b_2 e_(0) + b_3 e_3(0)$. It is easy to verify that equation (10) satisfies the variational identity

$$\frac{\delta}{\delta u} g_1(V, U_\lambda) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma g_1(V, U_u), \quad \gamma = \text{const.} \tag{11}$$

Taking $V = a e_1(0) + b e_2(0) + c e_3(0)$, where $a = \sum\limits_{m \geq 0} a_m \lambda^{-m}$, $b = \sum\limits_{m \geq 0} b_m \lambda^{-m}$, $c = \sum\limits_{m \geq 0} c_m \lambda^{-m}$. By using Eq. (10), a direct calculation reads

$$\begin{aligned} &g_1(V, U_\lambda) = 2a, \quad g_1(V, U_{u_1}) = 2a, \\ &g_1(V, U_{u_2}) = -2c, \quad g_1(V, U_{u_3}) = -2(b + c). \end{aligned}$$

Substituting the above results into Eq. (11), we have

$$\frac{\delta}{\delta u}(2a) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{pmatrix} 2a \\ -2c \\ -2(b + c) \end{pmatrix}. \tag{12}$$

Comparing the coefficient of λ^{-n-1} in Eq. (12) gives

$$\frac{\delta}{\delta u}(2a_{n+1}) = (\gamma - n) \begin{pmatrix} 2a_n \\ -2c_n \\ -2(b_n + c_n) \end{pmatrix}. \tag{13}$$

Taking $n = 0$ in Eq. (13) gives $\gamma = 0$. Then we obtain

$$P_{1n} = \frac{\delta H_n}{\delta u}, \quad H_n = -\frac{a_{n+1}}{n}, \quad n \geq 1. \tag{14}$$

Therefore the integrable systems (9) can be written as

$$U_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t = J_1 \frac{\delta H_n}{\delta u} = J_1 L^{n-1} \frac{\delta H_1}{\delta u}. \tag{15}$$

We observe that $J_1 L = L^* J_1$, so the NLS–mKdV hierarchy is Liouville integrable.

Next, we will derive the NLS–mKdV hierarchy with self-consistent sources. Consider the linear problem

$$\begin{aligned} &\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = U(u, \lambda_j) M \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \\ &\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_t = V(u, \lambda_j) M \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}. \end{aligned} \tag{16}$$

The following equation was given in Ref. [11]

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0, \tag{17}$$

where α_j are constants, $\delta H_k / \delta u$ determines a finite-dimensional invariant set for the flow (16). Here we obtain[25]

$$\begin{aligned} &\frac{\delta \lambda_j}{\delta u} = \frac{1}{2} g_1 \left(\Psi_j, \frac{\partial U(u, \lambda_j)}{\partial u} \right), \\ &\Psi_j = \begin{pmatrix} \phi_{1j}\phi_{2j} - \phi_{1j}^2 & -2\phi_{1j}^2 \\ \phi_{2j}^2 & \phi_{1j}\phi_{2j} + \phi_{1j}^2 \end{pmatrix}, \\ &\quad j = 1, \ldots, N. \end{aligned} \tag{18}$$

So

$$\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} \langle \Phi_1, \Phi_2 \rangle \\ \langle \Phi_1, \Phi_1 \rangle \\ \langle \Phi_1, \Phi_1 \rangle - \langle \Phi_2, \Phi_2 \rangle \end{pmatrix}, \tag{19}$$

where $\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^{\mathrm{T}}$, $i = 1, 2$, and $\langle \cdot, \cdot \rangle$ stands for the inner product in R^N.

Then, according to Eq. (17), the NLS–mKdV hierarchy with self-consistent sources can be rewritten as follows:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_t = J_1 L^{n-1} \frac{\delta H_1}{\delta u} + J_1 \begin{pmatrix} \langle \Phi_1, \Phi_2 \rangle \\ \langle \Phi_1, \Phi_1 \rangle \\ \langle \Phi_1, \Phi_1 \rangle - \langle \Phi_2, \Phi_2 \rangle \end{pmatrix}. \tag{20}$$

When taking $u_1 = 0$, $u_2 = u_3$, $n = 3$, $\beta = -4$, we obtain the mKdV equation with self-consistent sources

$$u_{2t} + 6u_2^2 u_{2x} + u_{2xxx} = 2u_1 \sum_{j=1}^{N} \phi_{1j}^2 + (\partial - 2u_1) \sum_{j=1}^{N} \phi_{2j}^2.$$

When taking $u_1 = 0$, $u_2 = 1$, $n = 3$, we obtain the composed KdV–mKdV equation with self-consistent sources

$$u_{3t} + (\alpha + \beta u_3)u_3 u_{3x} + \gamma u_{3xxx} = (\partial + 2u_1)\sum_{j=1}^{N}\phi_{1j}^2.$$

When taking $u_1 = 0$, $u_2 = \mathrm{i}\tilde{u}_2$, $n = 3$, denoting $Q = u_2 + u_3 = u_3 + \mathrm{i}\tilde{u_2}$, we have the well-known nonlinear Schrödinger equation

$$\begin{aligned}&\mathrm{i}Q_t + Q_{xx} - |Q|^2 Q\\ &= \mathrm{i}(\partial + 4u_1)\sum_{j=1}^{N}\phi_{1j}^2 + \mathrm{i}(\partial - 2u_1)\sum_{j=1}^{N}\phi_{2j}^2.\end{aligned}$$

3. Nonlinear integrable couplings of NLS–mKdV hierarchy with self-consistent sources

In Ref. [21], Ma presented a general scheme to generate the nonlinear integrable couplings of hierarchy. Based on the Lie algebra $sl(2)$, the nonlinear integrable couplings of AKNS hierarchy are obtained. In this section, by employing the Lie algebra B_2, we will seek the nonlinear integrable couplings of the NLS–mKdV hierarchy with self-consistent sources.

First, by virtue of the Lie algebra B_2, which is mentioned in Eq. (1), we construct a new matrix Lie algebra $F = \mathrm{span}\{f_1, f_2, f_3, f_4, f_5, f_6\}$ with the basis as follows:

$$f_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix},$$

$$f_3 = \begin{pmatrix} e_3 & 0 \\ 0 & e_3 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 0 & e_1 \\ 0 & e_1 \end{pmatrix},$$

$$f_5 = \begin{pmatrix} 0 & e_2 \\ 0 & e_2 \end{pmatrix}, \quad f_6 = \begin{pmatrix} 0 & e_3 \\ 0 & e_3 \end{pmatrix}.$$

Taking

$$\bar{M} = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix},$$

the commutator is defined as $[f_i, f_j] = f_i\bar{M}f_j - f_j\bar{M}f_i$, $i, j = 1, 2, \ldots, 6$. Then we have

$$\begin{aligned}&[f_1, f_2] = 2f_2, \quad [f_1, f_3] = 2(f_2 - f_3), \quad [f_1, f_4] = 0,\\ &[f_1, f_5] = 2f_5, \quad [f_1, f_6] = 2(f_5 - f_6), \quad [f_2, f_3] = -2f_1,\\ &[f_2, f_4] = -2f_5, \quad [f_2, f_5] = 0, \quad [f_2, f_6] = -2f_4,\\ &[f_3, f_4] = 2(f_6 - f_5), \quad [f_3, f_5] = 2f_4, \quad [f_3, f_6] = 0,\\ &[f_4, f_5] = 2f_5, \quad [f_4, f_6] = 2(f_5 - f_6), \quad [f_5, f_6] = -2f_4.\end{aligned}$$

Denoting $F_1 = \mathrm{span}\{f_1, f_2, f_3\}$ and $F_2 = \mathrm{span}\{f_4, f_5, f_6\}$, then we have

$$F = F_1 \oplus F_2, \quad F_1 \cong B_2, \quad [F_1, F_2] \subset F_2. \tag{21}$$

Here we note that the sub-algebras F_1 and F_2 are both nonsemi-simple, which is very important for deriving the nonlinear integrable couplings of hierarchy. By using the Lie algebra F, a loop algebra $\tilde{F}$ is presented as

$$\begin{aligned}&\tilde{F} = \mathrm{span}\{f_i(n)\}_{i=1}^{6},\\ &f_i(n) = f_i\lambda^n, \quad i = 1, 2, 3, 4, 5, 6,\end{aligned} \tag{22}$$

along with the commutators $[f_i(m), f_j(n)] = [f_i, f_j]\lambda^{m+n}$, $\deg(f_i(n)) = n$, $m, n \in Z$, $1 \le i, j \le 6$.

Taking the following isospectral problem

$$\phi_x = U\bar{M}\phi, \quad U(u, \lambda) = (\lambda + u_1)f_1(0) + u_2 f_2(0) + u_3 f_3(0) + u_4 f_4(0) + u_5 f_5(0) + u_6 f_6(0), \tag{23}$$

set $V = \sum_{m\ge 0}\sum_{i=1}^{6} v_{im} f_i(-m)$. Solving the stationary zero curvature equation $V_x = [U, V]$ gives rise to the recursion relation as follows:

$$\begin{cases}
v_{1mx} = 2(u_3 v_{2m} - u_2 v_{3m}), \quad v_{2mx} = 2(v_{2m+1} + u_1 v_{2m} - u_2 v_{1m} - u_3 v_{1m} + v_{3m+1} + u_1 v_{3m}),\\
v_{3mx} = 2(u_3 v_{1m} - v_{3m+1} - u_1 v_{3m}), \quad v_{4mx} = 2(u_6 v_{2m} - u_2 v_{6m} + u_3 v_{5m} - u_5 v_{3m} + u_6 v_{5m} - u_5 v_{6m}),\\
v_{5mx} = 2(v_{5m+1} + u_1 v_{5m} - u_5 v_{1m} + v_{6m+1} + u_1 v_{6m} - u_6 v_{1m} + u_4 v_{2m} - u_2 v_{4m} + u_4 v_{3m} - u_3 v_{4m}\\
\qquad + u_4 v_{6m} - u_6 v_{4m} + u_4 v_{5m} - u_5 v_{4m}),\\
v_{6mx} = 2(u_6 v_{1m} - v_{6m+1} - u_1 v_{6m} + u_3 v_{4m} - u_4 v_{3m} + u_6 v_{4m} - u_4 v_{6m}),\\
v_{10} = \beta \ne 0, \quad v_{20} = v_{30} = v_{40} = v_{50} = v_{60} = 0, \quad v_{21} = \beta u_2, \quad v_{31} = \beta u_3, \quad v_{11} = v_{41} = 0, \quad v_{51} = \beta u_5,\\
v_{61} = \beta u_6, \quad v_{12} = \beta u_2 u_3 + \dfrac{\beta}{2}u_3^2, \quad v_{22} = -\beta u_1 u_2 + \dfrac{\beta}{2}u_{3x} + \dfrac{\beta}{2}u_{2x}, \quad v_{32} = -\beta u_1 u_3 - \dfrac{\beta}{2}u_{3x},\\
v_{42} = \beta(u_3 u_6 + u_2 u_6 + u_3 u_5 + u_5 u_6 + u_6^2/2), \quad v_{52} = -\beta[u_1 u_5 + u_2 u_4 + u_4 u_5 - (u_5 + u_6)_x/2],\\
v_{62} = -\beta(u_3 u_4 + u_4 u_6 + u_1 u_6 + u_{6x}/2).
\end{cases} \tag{24}$$

Denoting $V_+^{(n)} = \sum_{m=0}^{n}\sum_{i=1}^{6} v_{im} f_i(-m)$, $V_-^{(n)} = \lambda^n V - V_+^{(n)}$, it is easy to compute

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -2(v_{2n+1} + v_{3n+1})f_2(0) + 2v_{3n+1}f_3(0) - 2(v_{5n+1} + v_{6n+1})f_5(0) + 2v_{6n+1}f_6(0). \quad (25)$$

Taking $V^{(n)} = V_+^{(n)} + \Delta_n$, $\Delta_n = -v_{1n}f_1(0) - v_{4n}f_4(0)$. Thus, the zero curvature equation (6) admits

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = \begin{pmatrix} -v_{1nx} \\ v_{2nx} - 2u_1(v_{2n} + v_{3n}) \\ v_{3nx} + 2u_1 v_{3n} \\ -v_{4nx} \\ v_{5nx} - 2(u_1 + u_4)(v_{5n} + v_{6n}) - 2u_4(v_{2n} + v_{3n}) \\ v_{6nx} + 2(u_1 v_{6n} + u_4 v_{3n} + u_4 v_{6n}) \end{pmatrix} = J_2 \begin{pmatrix} 2v_{1n} + v_{4n} \\ -2v_{3n} - v_{6n} \\ -2(v_{2n} + v_{3n}) - v_{5n} - v_{6n} \\ v_{1n} + v_{4n} \\ -v_{3n} - v_{6n} \\ -v_{2n} - v_{3n} - v_{5n} - v_{6n} \end{pmatrix} = J_2 P_{2n}, \quad (26)$$

where

$$J_2 = \begin{pmatrix} -\partial & 0 & 0 & \partial & 0 & 0 \\ 0 & \partial & -\partial + 2u_1 & 0 & -\partial & \partial - 2u_1 \\ 0 & -\partial - 2u_1 & 0 & 0 & \partial + 2u_1 & 0 \\ \partial & 0 & 0 & -2\partial & 0 & 0 \\ 0 & -\partial & \partial - 2u_1 & 0 & 2\partial & -2\partial - 2u_4 + 4u_1 \\ 0 & \partial + 2u_1 & 0 & 0 & -2\partial + 2u_4 - 4u_1 & 0 \end{pmatrix}$$

is the Hamiltonian operator and $P_{2n} = \bar{L}P_{2n-1}$, the recurrence operator $\bar{L}$ is as follows:

$$\bar{L} = \begin{pmatrix} 0 & L_1 & -\partial^{-1}u_3(\partial - 2u_1) & 0 & L_2 & L_3 \\ -u_3 & -u_1 - \partial/2 & 0 & u_6 & -u_4 & 0 \\ -u_2 - u_3 & -\partial/2 & -u_1 + \partial/2 & -u_5 - u_6 & 0 & -u_4 \\ 0 & 0 & 0 & 0 & L_4 & L_5 \\ 0 & 0 & 0 & -u_3 - u_6 & -\partial/2 - u_1 - u_4 & 0 \\ 0 & 0 & 0 & L_6 & -\partial/2 & \partial/2 - u_1 - u_4 \end{pmatrix}, \quad (27)$$

where

$$L_1 = -2\partial^{-1}u_1(u_2 + u_3) - \partial^{-1}u_2\partial, \quad L_2 = -\partial^{-1}u_5\partial - 2\partial^{-1}u_1(u_5 + u_6) - 2\partial^{-1}u_4(u_2 + u_3 + u_5 + u_6),$$
$$L_3 = 2\partial^{-1}u_3u_4 - \partial^{-1}u_6(\partial - 2u_1 - 2u_4), \quad L_4 = -\partial^{-1}(u_2 + u_5)\partial - 2\partial^{-1}(u_1 + u_4)(u_2 + u_3 + u_5 + u_6),$$
$$L_5 = -\partial^{-1}(u_3 + u_6)(\partial - 2u_1 - 2u_4), \quad L_6 = -(u_2 + u_3 + u_5 + u_6).$$

Then system (28) can be rewritten as $U_t = J_2P_{2n} = J_2\bar{L}^{n-1}P_{21}$. When taking $u_4 = u_5 = u_6$, hierarchy (26) reduces to the NLS–mKdV hierarchy. Taking $n = 2$ and $\beta = 1$ in system (26), we can obtain the following equations about the supplementary variables u_4, u_5, u_6

$$\begin{cases} u_{4t} = (u_3u_6 + u_2u_6 + u_3u_5 + u_5u_6 + u_6^2/2)_x, \\ u_{5t} = -[u_1u_5 + u_2u_4 + u_4u_5 - (u_5 + u_6)_x/2]_x + 2(u_1 + u_4)[u_4(u_2 + u_3) + (u_1 + u_4)(u_5 + u_6) \\ \qquad - u_{5x}/2] + 2u_4(u_1u_2 - u_{2x}/2 + u_1u_3), \\ u_{6t} = -(u_3u_4 + u_4u_6 + u_1u_6 + u_{6x}/2)_x - 2(u_1 + u_4)(u_3u_4 + u_4u_6 + u_1u_6 + u_{6x}/2) \\ \qquad - 2u_4(u_1u_3 + u_{3x}/2). \end{cases} \quad (28)$$

Obviously, equation (28) are nonlinear equations in u_4 and u_6, so we call Eq. (26) the nonlinear integrable coupling of the NLS–mKdV hierarchy.

Next, we continue to seek the Hamiltonian structures of the nonlinear integrable couplings of the NLS–mKdV hierarchy. Similar to Eq. (10), $\forall\ a = \sum_{i=1}^{6} a_i f_i(0),\ b = \sum_{i=1}^{6} b_i f_i(0)$, the following function is introduced

$$\begin{aligned} g_2(a,b) =\ & 2[(a_1+a_4)(b_1+b_4)-(a_3+a_6)(b_2+b_5) \\ & -(a_2+a_3+a_5+a_6)(b_3+b_6) \\ & +a_1b_1-(a_2+a_3)b_2-a_3b_3]. \end{aligned} \tag{29}$$

It is easy to verify that Eq. (29) also satisfies the variational identity (11). By using Eq. (29), a direct calculation reads

$$\begin{aligned} &g_2(V,U_\lambda) = 2(2v_1+v_4), \\ &g_2(V,U_{u_1}) = 2(2v_1+v_4), \quad g_2(V,U_{u_2}) = -2(2v_3+v_6), \\ &g_2(V,U_{u_3}) = -2(2v_2+2v_3+v_5+v_6), \\ &g_2(V,U_{u_4}) = 2(v_1+v_4), \quad g_2(V,U_{u_5}) = -2(v_3+v_6), \\ &g_2(V,U_{u_6}) = -2(v_2+v_3+v_5+v_6). \end{aligned}$$

Repeat the procedures (12)–(14), we can obtain $\gamma = 0$ and

$$P_{2n} = \frac{\delta \bar{H}_n}{\delta u}, \quad \bar{H}_n = -\frac{2v_{1n}+v_{4n}}{n}, \quad n \geq 1. \tag{30}$$

Hence, the nonlinear integrable couplings of the NLS–mKdV hierarchy have the following Hamiltonian structure:

$$U_t = J_2 \frac{\delta \bar{H}_n}{\delta u}, \quad n \geq 2. \tag{31}$$

It is also verify that $J_2\bar{L} = \bar{L}^* J_2$, so the nonlinear integrable couplings of the NLS–mKdV hierarchy are also Liouville integrable.

Further, we will work out the nonlinear integrable couplings of the NLS–mKdV hierarchy with self-consistent sources. Consider the auxiliary linear problem

$$\begin{aligned} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_x &= U(u,\lambda_j)\bar{M} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}_t &= V(u,\lambda_j)\bar{M} \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \\ \psi_{4j} \end{pmatrix}. \end{aligned} \tag{32}$$

For Eq. (32), it is known that

$$\frac{\delta \lambda_j}{\delta u} = \frac{1}{2} g_2\left(\Psi_j, \frac{\partial U(u,\lambda_j)}{\partial u}\right),$$

where

$$\Psi_j = \psi_{1j}\psi_{2j}f_1(0) + \psi_{2j}^2 f_2(0) - \psi_{1j}^2 f_3(0) + \psi_{3j}\psi_{4j}f_4(0) + \psi_{4j}^2 f_5(0) - \psi_{3j}^2 f_6(0), \quad j = 1,\ldots,N. \tag{33}$$

So

$$\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \begin{pmatrix} 2\langle \Phi_1, \Phi_2\rangle + \langle \Phi_3, \Phi_4\rangle \\ 2\langle \Phi_1, \Phi_1\rangle + \langle \Phi_3, \Phi_3\rangle \\ 2\langle \Phi_1, \Phi_1\rangle - 2\langle \Phi_2, \Phi_2\rangle + \langle \Phi_3, \Phi_3\rangle - \langle \Phi_4, \Phi_4\rangle \\ \langle \Phi_1, \Phi_2\rangle + \langle \Phi_3, \Phi_4\rangle \\ \langle \Phi_1, \Phi_1\rangle + \langle \Phi_3, \Phi_3\rangle \\ \langle \Phi_1, \Phi_1\rangle - \langle \Phi_2, \Phi_2\rangle + \langle \Phi_3, \Phi_3\rangle - \langle \Phi_4, \Phi_4\rangle \end{pmatrix}, \tag{34}$$

where $\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^{\mathrm{T}}$, $i = 1,2,3,4$, and $\langle \cdot, \cdot \rangle$ stands for the inner product in R^N.

Then, according to Eq. (17), the nonlinear integrable couplings of the NLS–mKdV hierarchy with self-consistent sources can be written as follows:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}_t = J_2 \bar{L}^{n-1} \frac{\delta \bar{H}_1}{\delta u} + J_2 \begin{pmatrix} 2\langle \Phi_1, \Phi_2\rangle + \langle \Phi_3, \Phi_4\rangle \\ 2\langle \Phi_1, \Phi_1\rangle + \langle \Phi_3, \Phi_3\rangle \\ 2\langle \Phi_1, \Phi_1\rangle - 2\langle \Phi_2, \Phi_2\rangle + \langle \Phi_3, \Phi_3\rangle - \langle \Phi_4, \Phi_4\rangle \\ \langle \Phi_1, \Phi_2\rangle + \langle \Phi_3, \Phi_4\rangle \\ \langle \Phi_1, \Phi_1\rangle + \langle \Phi_3, \Phi_3\rangle \\ \langle \Phi_1, \Phi_1\rangle - \langle \Phi_2, \Phi_2\rangle + \langle \Phi_3, \Phi_3\rangle - \langle \Phi_4, \Phi_4\rangle \end{pmatrix}. \tag{35}$$

Taking $n = 2$ and $\beta = 1$ in system (35), we can obtain the following coupling equations with self-consistent sources

$$\begin{cases} u_{4t} = (u_3u_6 + u_2u_6 + u_3u_5 + u_5u_6 + u_6^2/2)_x - \partial\sum_{j=1}^{N}\psi_{3j}\psi_{4j}, \\ u_{5t} = -[u_1u_5 + u_2u_4 + u_4u_5 - (u_5+u_6)_x/2]_x + 2(u_1+u_4)[u_4(u_2+u_3) + (u_1+u_4)(u_5+u_6) \\ \quad - u_{5x}/2] + 2u_4(u_1u_2 - u_{2x}/2 + u_1u_3) + (\partial - 2u_4)\sum_{j=1}^{N}\psi_{1j}^2 - (\partial - 2u_4)\sum_{j=1}^{N}\psi_{2j}^2 \\ \quad + (\partial - 2u_4 + 2u_1)\sum_{j=1}^{N}\psi_{3j}^2 + (2u_4 - 2u_1)\sum_{j=1}^{N}\psi_{4j}^2, \\ u_{6t} = -(u_3u_4 + u_4u_6 + u_1u_6 + u_{6x}/2)_x - 2(u_1+u_4)(u_3u_4 + u_4u_6 + u_1u_6 + u_{6x}/2) \\ \quad - 2u_4(u_1u_3 + u_{3x}/2) + 2u_4\sum_{j=1}^{N}\psi_{1j}^2 - (\partial + 2u_1 - 2u_4)\psi_{3j}^2. \end{cases} \tag{36}$$

4. Conclusion

Starting from a Lie algebra, researchers can obtain some well-known integrable hierarchies. With the help of the variational identity, the Hamiltonian structure can also be generated. These studies have been done well for many researches. Recently, considerable attention has been given to the integrable couplings of hierarchies with self-consistent sources and the nonlinear integrable couplings of hierarchies without self-consistent sources. It seems that very few ones on the nonlinear integrable couplings of hierarchies with self-consistent sources are available up to now. In this paper, we generate the nonlinear integrable couplings of the NLS–mKdV hierarchy with self-consistent sources. It enriches the content of the integrable couplings of hierarchies with self-consistent sources. In fact, the method can be applied to derive the nonlinear integrable couplings of other hierarchies with self-consistent sources. Furthermore, the solutions of the nonlinear integrable couplings of hierarchies with self-consistent sources are also worth while exploring. We will study these problems in future investigations.

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