Mei symmetry and Mei conserved quantity of Appell equations for a variable mass holonomic system of relative motion*

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Mei symmetry and Mei conserved quantity of Appell equations for a variable mass holonomic system of relative motion are studied. The definition and criterion of the Mei symmetry of Appell equations for a variable mass holonomic system of relative motion under the infinitesimal transformations of groups are given. The structural equation of Mei symmetry of Appell equations and the expression of Mei conserved quantity deduced directly from Mei symmetry for a variable mass holonomic system of relative motion are gained. Finally, an example is given to illustrate the application of the results.

Keywords: variable mass, relative motion, Appell equations, Mei conserved quantity

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1. Introduction

The research on the symmetries and conserved quantities in dynamical systems plays an important role in modern physics, mechanics, and mathematics. In 1899, Appell constructed Appell equations in dynamical systems.[1] Appell equations have become an important kind of equation of analytical mechanics. More than ten forms of equations of motion are grouped into Lagrange systems,[2−9] Nielsen systems,[10−14] and Appell systems[15−23] in analytical mechanics. In recent years, Chinese scholars have made a series of contributions to the research on symmetries and conserved quantities and received some important results.[24−36] Some scholars have also studied the symmetries and conserved quantities of Lagrange equations in a dynamical system of relative motion.[37] However, so far the research report on the symmetries and conserved quantities of Appell equations for a variable mass holonomic system of relative motion has not been found. In this paper, we specifically study Mei symmetry and the Mei conserved quantity of Appell equations for a variable mass holonomic system of relative motion. And an example is also given to illustrate the application of the results.

2. Appell equations for a variable mass holonomic system of relative motion

Let the speed of a base-point O in the carrier be $\boldsymbol{v}$ and its angular velocity be $\boldsymbol{\omega}$. Assume that a variable mass holonomic mechanical system consists of N particles that have masses m_i at the moment t, and the infinitesimal element of mass $\mathrm{d}m_i$ that is given out by or merged into a particle at the moment $t + \mathrm{d}t$. Suppose that the configuration of the mechanical system is determined by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$, and that the particle mass is dependent on time and generalized coordinates and given as

$$m_i = m_i(t, \boldsymbol{q}), \quad (i = 1, \ldots, N). \tag{1}$$

Appell equations of the system are written as

$$\frac{\partial S_r}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + P_s, \quad (s = 1, \ldots n), \tag{2}$$

where $S_r = \frac{1}{2} m_i \boldsymbol{a}_{ir} \cdot \boldsymbol{a}_{ir}$ is the acceleration energy of the system; $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the generalized forces;

$$V^0 = M(\boldsymbol{v}_0^* + \boldsymbol{\omega} \times \boldsymbol{v}_0) \cdot \boldsymbol{r}_c' \tag{3}$$

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is the potential energy in a uniform field, with M being the total mass for the system, $\boldsymbol{r}_c'$ the vector of mass point of the relative motion system relative to the base point O, $\boldsymbol{v}_0^*$ the relative derivative of v_0;

$$V^{\omega} = -\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\theta}^0\cdot\boldsymbol{\omega} \tag{4}$$

is the potential energy of centrifugal force, with $\boldsymbol{\theta}^0$ being the inertial tensor of the system in point O;

$$Q_s^{\dot{\omega}} = -\left(\dot{\boldsymbol{\omega}}\times m_i\boldsymbol{r}_i'\right)\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{5}$$

is the generalized Coriolis inertial force;

$$\Gamma_s = \gamma_{sk}\dot{q}_k, \quad \gamma_{sk} = 2\boldsymbol{\omega}\cdot\left(m_i\frac{\partial\boldsymbol{r}_i'}{\partial q_s}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_k}\right) \tag{6}$$

is the generalized gyroscopic force;

$$P_s = \dot{m}_i\left(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i\right)\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s} \tag{7}$$

is the generalized counter-thrust.

From Eq. (2), we can solve all $\ddot{q}_s$, and have

$$\ddot{q}_s = \alpha_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right), \quad (s=1,\ldots,n). \tag{8}$$

3. Definition and criterion of Mei symmetry of Appell equations for a variable mass holonomic system of relative motion

Introduce the expansion of the infinitesimal transformation of groups of time and generalized coordinates as

$$t^* = t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad q_s^*(t^*) = q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad (s=1,\ldots,n), \tag{9}$$

where ε is an infinitesimal parameter; ξ_0 and ξ_s are infinitesimal generators. From Eq. (9), we have

$$\begin{aligned}\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} &= \frac{\mathrm{d}q_s+\varepsilon\mathrm{d}\xi_s}{\mathrm{d}t+\varepsilon\mathrm{d}\xi_0} = \dot{q}_s+\varepsilon(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+O(\varepsilon^2),\\ \frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}} &= \ddot{q}_s+\varepsilon\left[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0\right]+O(\varepsilon^2).\end{aligned} \tag{10}$$

Suppose that after transformations (9), functions S_r, Q_s, V^0, V^{ω}, $Q_s^{\dot{\omega}}$, Γ_s, and P_s are substituted by S_r^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, and P_s^* respectively, then taking the Taylor expansions of S_r^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, and P_s^* at the point of $(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ along the trajectory of motion of the system, we will have

$$\begin{aligned}S_r^* &= S_r\left(t^*,q^*,\frac{\mathrm{d}q^*}{\mathrm{d}t^*},\frac{\mathrm{d}^2q^*}{\mathrm{d}t^{*2}}\right) = S_r(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\\ &\quad+\varepsilon\left\{\frac{\partial S_r}{\partial t}\xi_0+\frac{\partial S_r}{\partial q_s}\xi_s+\frac{\partial S_r}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right.\\ &\quad\left.+\frac{\partial S_r}{\partial\ddot{q}_s}\left[\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)^{\cdot}-\ddot{q}_s\dot{\xi}_0\right]\right\}+O\left(\varepsilon^2\right),\end{aligned} \tag{11}$$

namely

$$S_r^* = S_r(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(2)}\left(S_r\right)+O\left(\varepsilon^2\right), \tag{12}$$

$$Q_s^* = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(Q_s\right)+O\left(\varepsilon^2\right), \quad (s=1,\ldots,n), \tag{13}$$

$$V^{0*} = V^0(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(V^0\right)+O\left(\varepsilon^2\right), \tag{14}$$

$$V^{\omega*} = V^{\omega}(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(V^{\omega}\right)+O\left(\varepsilon^2\right), \tag{15}$$

$$Q_s^{\dot{\omega}*} = Q_s^{\dot{\omega}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(Q_s^{\dot{\omega}}\right)+O\left(\varepsilon^2\right), \quad (s=1,\ldots,n), \tag{16}$$

$$\Gamma_s^* = \Gamma_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(\Gamma_s\right)+O\left(\varepsilon^2\right), \quad (s=1,\ldots,n), \tag{17}$$

$$P_s^* = P_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\tilde{X}^{(1)}\left(P_s\right)+O\left(\varepsilon^2\right), \quad (s=1,\ldots,n), \tag{18}$$

where

$$\tilde{X}^{(2)} = \tilde{X}^{(1)}+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)-\ddot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial\ddot{q}_s}, \tag{19}$$

$$\tilde{X}^{(1)} = X^{(0)}+\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t}-\dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_s}, \tag{20}$$

$$X^{(0)} = \xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{21}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+\alpha_s\frac{\partial}{\partial\dot{q}_s}+\dot{\alpha}_s\frac{\partial}{\partial\ddot{q}_s}. \tag{22}$$

Definition If dynamic functions S_r^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, and P_s^* which are transformed by Eq. (9) are used to replace respectively dynamic functions S_r, Q_s, V^0, V^{ω}, $Q_s^{\dot{\omega}}$, Γ_s, and P_s which are those before transformations, equation (2) remains unchanged in form, namely

$$\frac{\partial S_r^*}{\partial\ddot{q}_s} = Q_s^*-\frac{\partial}{\partial q_s}\left(V^{0*}+V^{\omega*}\right)+Q_s^{\dot{\omega}*}+\Gamma_s^*+P_s^*, \quad (s=1,\ldots,n). \tag{23}$$

Hence, this symmetry is called the Mei symmetry of Appell equations for a variable mass holonomic system of relative motion.

Substituting Eqs. (12)–(18) into Eq. (23), and neglecting terms that are of the order of $O(\varepsilon^2)$ or of higher orders from Eq. (2), we obtain

$$\begin{aligned}&\frac{\partial}{\partial\ddot{q}_s}\left[\tilde{X}^{(2)}\left(S_r\right)\right]\\&-\tilde{X}^{(1)}\left[Q_s-\frac{\partial}{\partial q_s}\left(V^0+V^{\omega}\right)+Q_s^{\dot{\omega}}+\Gamma_s+P_s\right]=0,\\&(s=1,\ldots,n).\end{aligned}\tag{24}$$

Equation (24) is called the criterion equation of the Mei symmetry for Appell equations for a variable mass holonomic system of relative motion.

Criterion For Appell equations (2) for a variable mass holonomic system of relative motion, if the infinitesimal generators ξ_0 and ξ_s make the criterion equation (24) hold true, then the invariance of Eq. (2) under the infinitesimal transformations (9) is called Mei symmetry of Appell equations (2) for a variable mass holonomic system of relative motion.

4. Structural equation of Mei symmetry of Appell equations and Mei conserved quantity deduced from Mei symmetry for a variable mass holonomic system of relative motion

Proposition If the infinitesimal generators ξ_0 and ξ_s and the gauge function $G_{\rm M}=G_{\rm M}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ of Appell equations (2) for a variable mass holonomic system of relative motion make the following structural equation hold true

$$\begin{aligned}&\tilde{X}^{(2)}(S_r)\frac{\bar{\rm d}\xi_0}{{\rm d}t}+\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_r)\right]\\&+\left(\xi_s-\dot{q}_s\xi_0\right)\tilde{E}_s\left[\tilde{X}^{(2)}(S_r)\right]\\&+\xi_0\Bigg\{\tilde{X}^{(1)}\Bigg[Q_s-\frac{\partial}{\partial q_s}\left(V^0+V^{\omega}\right)\\&+Q_s^{\dot{\omega}}+\Gamma_s+P_s\Bigg]\Bigg\}\\&\times\frac{\bar{\rm d}\alpha_s}{{\rm d}t}+\frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}=0,\end{aligned}\tag{25}$$

then the Mei conserved quantity deduced from the Mei symmetry of Appell equations for a variable mass holonomic system of relative motion is expressed as follows:

$$\begin{aligned}I_{\rm M}&=\xi_0\tilde{X}^{(2)}(S_r)+\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}\left(\xi_s-\dot{q}_s\xi_0\right)+G_{\rm M}\\&=\text{const.}\end{aligned}\tag{26}$$

Proof From Eq. (22), we have

$$\begin{aligned}\frac{\bar{\rm d}I_{\rm M}}{{\rm d}t}=&\Bigg[\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial t}+\dot{q}_s\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial q_s}\\&+\alpha_s\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}+\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\ddot{q}_s}\frac{\bar{\rm d}\alpha_s}{{\rm d}t}\Bigg]\xi_0\\&+\tilde{X}^{(2)}(S_r)\frac{\bar{\rm d}\xi_0}{{\rm d}t}+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}\right]\left(\xi_s-\dot{q}_s\xi_0\right)\\&+\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}\left(\frac{\bar{\rm d}\xi_s}{{\rm d}t}-\alpha_s\xi_0-\dot{q}_s\frac{\bar{\rm d}\xi_0}{{\rm d}t}\right)\\&+\frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}.\end{aligned}\tag{27}$$

Note that

$$\begin{aligned}\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_r)\right]=&\,\xi_0\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial t}+\xi_s\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial q_s}\\&+\left(\frac{\bar{\rm d}\xi_s}{{\rm d}t}-\dot{q}_s\frac{\bar{\rm d}\xi_0}{{\rm d}t}\right)\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s},\end{aligned}\tag{28}$$

then Eq. (27) will become

$$\begin{aligned}\frac{\bar{\rm d}I_{\rm M}}{{\rm d}t}=&\,\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_r)\right]-\xi_s\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial q_s}\\&+\xi_0\dot{q}_s\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial q_s}+\xi_0\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\ddot{q}_s}\frac{\bar{\rm d}\alpha_s}{{\rm d}t}\\=&\,\tilde{X}^{(2)}(S_r)\frac{\bar{\rm d}\xi_0}{{\rm d}t}+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}\right]\\&\times\left(\xi_s-\dot{q}_s\xi_0\right)+\frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}\\=&\,\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_r)\right]-\left(\xi_s-\dot{q}_s\xi_0\right)\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial q_s}\\&+\xi_0\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\ddot{q}_s}\frac{\bar{\rm d}\alpha_s}{{\rm d}t}\\=&\,\tilde{X}^{(2)}(S_r)\frac{\bar{\rm d}\xi_0}{{\rm d}t}+\left[\frac{\bar{\rm d}}{{\rm d}t}\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\dot{q}_s}\right]\\&\times\left(\xi_s-\dot{q}_s\xi_0\right)+\frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}.\end{aligned}$$

Using Eqs. (24) and (25), we have

$$\begin{aligned}\frac{\bar{\rm d}I_{\rm M}}{{\rm d}t}=&\,\tilde{X}^{(1)}\left[\tilde{X}^{(2)}(S_r)\right]+\xi_0\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\ddot{q}_s}\frac{\bar{\rm d}\alpha_s}{{\rm d}t}\\&+\left(\xi_s-\dot{q}_s\xi_0\right)\tilde{E}_s\left[\tilde{X}^{(2)}(S_r)\right]\\&+\xi_0\frac{\partial\tilde{X}^{(2)}(S_r)}{\partial\ddot{q}_s}\frac{\bar{\rm d}\alpha_s}{{\rm d}t}+\tilde{X}^{(2)}(S_r)\frac{\bar{\rm d}\xi_0}{{\rm d}t}+\frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}\end{aligned}$$

$$= \xi_0 \frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\left\{\frac{\partial \tilde{X}^{(2)}(S_r)}{\partial \ddot{q}_s} - \tilde{X}^{(1)}\left[Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + P_s\right]\right\} = 0.$$

5. An illustrative example

Now we use a simple example to illustrate the application of the conclusion.

As shown in Fig. 1, the particle moves without friction in plane P, which goes through a fixed straight line Oq_1 (the angle with respect to the plumb line is constant α), and rotates at angular velocity ω around the straight line Oq_1. In the plane the axis Oq_2 is perpendicular to Oq_1. The mass of the particle is $m = m_0\,\mathrm{e}^{-t}$, and the speed of the separated particle is zero relative to the particle. Now we study the Mei symmetry and the Mei conserved quantity of the system.

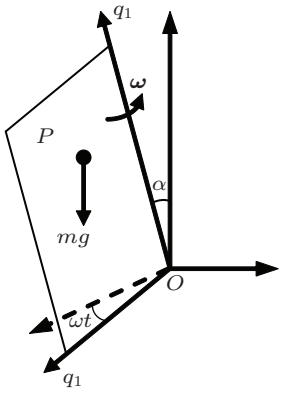


Fig. 1. Particle position.

It is easy to obtain dynamic functions of the system. The Appell function, the generalized force, the generalized Coriolis inertial force, the generalized gyroscopic force, the generalized counter-thrust, the centrifugal force potential energy, and even the force potential energy are given, respectively, as

$$S_r = \frac{1}{2}m(\ddot{q}_1^2 + \ddot{q}_2^2), \tag{29}$$

$$Q_1 = -mg\cos\alpha, \quad Q_2 = -mg\sin\alpha\sin\omega t, \tag{30}$$

$$Q_s^{\dot{\omega}} = 0, \quad (s = 1, 2), \tag{31}$$

$$\Gamma_s = 0, \quad (s = 1, 2), \tag{32}$$

$$P_1 = -m\dot{q}_1, \quad P_2 = -m\dot{q}_2, \tag{33}$$

$$V^\omega = -\frac{1}{2}m\omega^2 q_2^2, \tag{34}$$

$$V^0 = 0. \tag{35}$$

The differential equations of the system are

$$m\ddot{q}_1 + mg\cos\alpha + m\dot{q}_1 = 0, \tag{36}$$

$$m\ddot{q}_2 - m\omega^2 q_2 + mg\sin\alpha\sin\omega t + m\dot{q}_2 = 0. \tag{37}$$

By calculations, we have

$$\tilde{X}^{(2)}(S_r) = \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) - \ddot{q}_1\dot{\xi}_0\right]m\ddot{q}_1 + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) - \ddot{q}_2\dot{\xi}_0\right]m\ddot{q}_2 - \frac{m}{2}(\ddot{q}_1^2 + \ddot{q}_2^2)\xi_0. \tag{38}$$

Take the infinitesimal generator

$$\xi_0 = \xi_2 = 0, \quad \xi_1 = \ln(\dot{q}_1 + q_1 + gt\cos\alpha), \tag{39}$$

then from criterion equation (24) we will know that generator (39) is the generator of the Mei symmetry, and the system is Mei symmetric. According to structural equation (25), we have

$$G_{\mathrm{M}} = \ln(\dot{q}_1 + q_1 + gt\cos\alpha). \tag{40}$$

Using Eq. (26), we obtain Mei conserved quantity of the system as

$$I_{\mathrm{M}} = \ln(\dot{q}_1 + q_1 + gt\cos\alpha) = \mathrm{const}. \tag{41}$$

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