Hamilton formalism and Noether symmetry for mechanico–electrical systems with fractional derivatives*

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This paper presents extensions to the traditional calculus of variations for mechanico–electrical systems containing fractional derivatives. The Euler–Lagrange equations and the Hamilton formalism of the mechanico–electrical systems with fractional derivatives are established. The definition and the criteria for the fractional generalized Noether quasi-symmetry are presented. Furthermore, the fractional Noether theorem and conseved quantities of the systems are obtained by virtue of the invariance of the Hamiltonian action under the infinitesimal transformations. An example is presented to illustrate the application of the results.

Keywords: fractional derivative, mechanico–electrical system, Noether symmetry, Hamiltonian formulation

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1. Introduction

Fractional calculus, a generalization of ordinary differentiation and integration to an arbitrary order, was considered as a sole mathematical and theoretical subject with no applications for many years.[1] But, in recent years, the traditional calculus of variations for systems containing fractional derivatives has played a significant role in physics, engineering, biology, and pure and applied mathematics.[2−9] Besides, fractional-order controllers, such as CRONE, PID controller, and TID controller, have been introduced and designed.[10−13] Furthermore, one can find many other important applications of the fractional calculus in classical mechanics. Riewe developed a version of the Euler–Lagrange equations for both conservative and non-conservative systems with fractional derivatives.[14,15] Agrawal gave the formulation of Euler–Lagrange equations for variational problems by using the left and right fractional derivatives in the Riemann–Liouville sense.[16] Rabei *et al.* discussed the Hamilton formalism, and Zhou *et al.* studied the symmetry theories of the Hamiltonian systems with fractional derivatives.[17,18]

Symmetries and conservation laws have played a significant role in mathematics, optimal control, and mechanics.[19−26] The symmetries are useful for integrating the equations of motion and building the invariants of the systems, which can be used to reduce the orders of the equations and simplify the solutions of the problems.[27−32] Fu *et al.* have studied the symmetries and the conserved quantities of mechanico–electrical systems in the last few years, and many significant results were obtained.[33−35]

In this paper, we present some new results through a combination of the Noether symmetry and the fractional calculus of variations for mechanico–electrical systems. First, we apply the left and right Riemann–Liouville fractional derivatives to the systems to formulate the Hamiltonian equations with fractional derivatives. Then, the systems' Noether symmetry is studied, and the corresponding conserved quantities are obtained through infinitesimal transformations with respect to time, generalized coordinates, and generalized momentums.

2. Preliminaries

Several definitions of fractional derivatives have been proposed, including Riemann–Liouville, Grunwald-Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivatives.[1,36−38] Here, we formulate the problem in terms of the left and right

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Riemann–Liouville fractional derivatives, which are defined in Ref. [1]. The necessary definitions and properties are reviewed here.

Definition 1 Let f be a function defined on the interval $[a,b]$, and α be a positive real number. The left Riemann–Liouville fractional derivative (LRLFD) ${}_aD_t^\alpha$ and the right Riemann–Liouville fractional derivative (RRLFD) ${}_tD_b^\alpha$ are defined as

$$\begin{aligned}&{}_aD_t^\alpha f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t(t-\theta)^{n-\alpha-1}f(\theta)\mathrm{d}\theta,\end{aligned}\quad(1)$$

$$\begin{aligned}&{}_tD_b^\alpha f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_t^b(t-\theta)^{n-\alpha-1}f(\theta)\mathrm{d}\theta,\end{aligned}\quad(2)$$

where α is the order of the derivative, $n-1\le\alpha<n$, $n\in N$, and $\alpha\neq 0$. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$\begin{aligned}&{}_aD_t^\alpha f(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t),\\&{}_tD_b^\alpha f(t)=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t),\\&\alpha=1,2,\ldots.\end{aligned}\quad(3)$$

Theorem 1 Let f and g be two smooth functions defined on the interval $[a,b]$. Then, for $\forall t\in[a,b]$, the following properties hold:

(i) for $m>0$,

$${}_aD_t^m\left[f(t)+g(t)\right]={}_aD_t^m f(t)+{}_aD_t^m g(t);\quad(4)$$

(ii) for $m\ge n\ge 0$,

$${}_aD_t^m\left({}_aD_t^{-n}f(t)\right)={}_aD_t^{m-n}f(t);\quad(5)$$

(iii) for $0<m<1$,

$$\int_a^b({}_aD_t^m f(t))g(t)\mathrm{d}t=\int_a^b f(t)\,({}_tD_b^m g(t))\mathrm{d}t;\quad(6)$$

(iv) for $m\in(0,\infty)$ and $j\in N^*$,

$$\begin{aligned}&{}_aD_t^m\left[f(t)\cdot g(t)\right]\\&=\sum_{j=0}^{\infty}\left\{\binom{m}{j}{}_aD_t^j f(t)\cdot{}_aD_t^{m-j}g(t)\right\}.\end{aligned}\quad(7)$$

Remark 1[40] The relations between the contemporaneous and the non-contemporaneous variations for the fractional derivatives can be described as

$$\delta\,{}_aD_t^\alpha q={}_aD_t^\alpha\delta q,\quad \delta\,{}_tD_b^\beta q={}_tD_b^\beta\delta q,\quad(8)$$

$$\begin{aligned}&\Delta\,{}_aD_t^\alpha q=\delta\,{}_aD_t^\alpha q+{}_aD_t^{\alpha+1}q\cdot\Delta t,\\&\Delta\,{}_tD_b^\beta q=\delta\,{}_tD_b^\beta q-{}_tD_b^{\beta+1}q\cdot\Delta t.\end{aligned}\quad(9)$$

Following Agrawal, we can obtain the Euler–Lagrange equation of conservative systems for the fractional calculus of variation problem as

$$\frac{\partial L}{\partial q}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q}=0,\quad(10)$$

where L is a function of the form

$$L=L(t,q,{}_aD_t^\alpha q,{}_tD_b^\beta q).$$

For $\alpha=\beta=1$, we have ${}_aD_t^\alpha=\mathrm{d}/\mathrm{d}t$ and ${}_tD_b^\beta=-\mathrm{d}/\mathrm{d}t$, then equation (10) reduces to the standard Euler–Lagrange equation. This Euler–Lagrange equation (10) has been extended to the classical field systems[39] and the nonholonomic systems.[40] In 2010, Zhou *et al.* obtained the Euler–Lagrange equation for non-conservative systems with fractional derivatives

$$\frac{\partial L}{\partial q}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q}=-Q,\quad(11)$$

where Q is a non-conservative force.

3. Fractional Hamiltonian of mechanico–electrical systems

Consider a menchanico–electrical system with the Lagrangian depending on the fractional time derivatives of generalized coordinates $q=\{q_1(t),q_2(t),\ldots,q_m(t)\}$ and generalized electric quantities $e=\{e_1(t),e_2(t),\ldots,e_n(t)\}$ in the form

$$\begin{aligned}&L=L(t,q,e,{}_aD_t^\alpha q,{}_tD_b^\beta q,{}_aD_t^\alpha e,{}_tD_b^\beta e),\\&0<\alpha,\beta<1.\end{aligned}$$

Following Ref. [33] and in terms of Eq. (11), we can obtain the Euler–Lagrange equations for the mechanico–electrical system with fractional derivatives

$$\begin{aligned}&\frac{\partial L}{\partial q}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q}\\&=-\left(Q-\frac{\partial F}{\partial\,{}_aD_t^\alpha q}-\frac{\partial F}{\partial\,{}_tD_b^\beta q}\right),\end{aligned}\quad(12)$$

$$\begin{aligned}&\frac{\partial L}{\partial e}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha e}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta e}\\&=-\left(U-\frac{\partial F}{\partial\,{}_aD_t^\alpha e}-\frac{\partial F}{\partial\,{}_tD_b^\beta e}\right),\end{aligned}\quad(13)$$

where U is the generalized electromotive force. The dissipative function F of the system is composed of the electric dissipative function F_e and the dissipative function of the viscous frictional damping force F_v, i.e.

$$F=F_e({}_aD_t^\alpha e,{}_tD_b^\beta e)+F_v({}_aD_t^\alpha q,{}_tD_b^\beta q).$$

We further make the assumption that

$$\Lambda_1=Q-\frac{\partial F}{\partial\,{}_aD_t^\alpha q}-\frac{\partial F}{\partial\,{}_tD_b^\beta q},$$

$$\Lambda_2 = U - \frac{\partial F}{\partial {}_aD_t^\alpha e} - \frac{\partial F}{\partial {}_tD_b^\beta e} \tag{14}$$

are non-conservative and non-potential forces acting on the mechanico–electrical system, respectively.

Introduce the generalized momenta as

$$\begin{aligned} p_{\alpha 1} &= \frac{\partial L}{\partial {}_aD_t^\alpha q}, \quad p_{\beta 1} = \frac{\partial L}{\partial {}_tD_b^\beta q}, \\ p_{\alpha 2} &= \frac{\partial L}{\partial {}_aD_t^\alpha e}, \quad p_{\beta 2} = \frac{\partial L}{\partial {}_tD_b^\beta e}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} p_{\alpha 1} &= \{p_{\alpha 1}^1, p_{\alpha 1}^2, \ldots, p_{\alpha 1}^m\}, \\ p_{\beta 1} &= \{p_{\beta 1}^1, p_{\beta 1}^2, \ldots, p_{\beta 1}^m\}, \\ p_{\alpha 2} &= \{p_{\alpha 2}^1, p_{\alpha 2}^2, \ldots, p_{\alpha 2}^n\}, \\ p_{\beta 2} &= \{p_{\beta 2}^1, p_{\beta 2}^2, \ldots, p_{\beta 2}^n\}. \end{aligned}$$

The Hamiltonian depending on the fractional time derivatives reads as

$$\begin{aligned} H =\ & p_{\alpha 1 a}D_t^\alpha q + p_{\beta 1 t}D_b^\beta q + p_{\alpha 2 a}D_t^\alpha e \\ & + p_{\beta 2 t}D_b^\beta e - L. \end{aligned} \tag{16}$$

Calculating the total differential of this Hamiltonian, we obtain

$$\begin{aligned} \mathrm{d}H =\ & p_{\alpha 1}\mathrm{d}_a D_t^\alpha q + {}_aD_t^\alpha q \mathrm{d}p_{\alpha 1} + p_{\beta 1}\mathrm{d}_t D_b^\beta q \\ & + {}_tD_b^\beta q \mathrm{d}p_{\beta 1} + p_{\alpha 2}\mathrm{d}_a D_t^\alpha e + {}_aD_t^\alpha e \mathrm{d}p_{\alpha 2} \\ & + p_{\beta 2}\mathrm{d}_t D_b^\beta e + {}_tD_b^\beta e \mathrm{d}p_{\beta 2} - \frac{\partial L}{\partial q}\mathrm{d}q \\ & - \frac{\partial L}{\partial e}\mathrm{d}e - \frac{\partial L}{\partial {}_aD_t^\alpha q}\mathrm{d}_a D_t^\alpha q \\ & - \frac{\partial L}{\partial {}_tD_b^\beta q}\mathrm{d}_t D_b^\beta q + \frac{\partial L}{\partial {}_aD_t^\alpha e}\mathrm{d}_a D_t^\alpha e \\ & - \frac{\partial L}{\partial {}_tD_b^\beta e}\mathrm{d}_t D_b^\beta e - \frac{\partial L}{\partial t}\mathrm{d}t. \end{aligned} \tag{17}$$

Substituting the momenta from Eq. (15), we obtain

$$\begin{aligned} \mathrm{d}H =\ & {}_aD_t^\alpha q \mathrm{d}p_{\alpha 1} + {}_tD_b^\beta q \mathrm{d}p_{\beta 1} + {}_aD_t^\alpha e \mathrm{d}p_{\alpha 2} \\ & + {}_tD_b^\beta e \mathrm{d}p_{\beta 2} - \frac{\partial L}{\partial q}\mathrm{d}q \\ & - \frac{\partial L}{\partial e}\mathrm{d}e - \frac{\partial L}{\partial t}\mathrm{d}t. \end{aligned} \tag{18}$$

In virtue of the Euler–Lagrange equations (12) and (13), we have

$$\begin{aligned} \mathrm{d}H =\ & {}_aD_t^\alpha q \mathrm{d}p_{\alpha 1} + {}_tD_b^\beta q \mathrm{d}p_{\beta 1} \\ & + \left({}_tD_b^\alpha p_{\alpha 1} + {}_aD_t^\beta p_{\beta 1} + Q - \frac{\partial F}{\partial {}_aD_t^\alpha q}\right. \\ & \left. - \frac{\partial F}{\partial {}_tD_b^\beta q}\right)\mathrm{d}q + {}_aD_t^\alpha e \mathrm{d}p_{\alpha 2} + {}_tD_b^\beta e \mathrm{d}p_{\beta 2} \\ & + \left({}_tD_b^\alpha p_{\alpha 2} + {}_aD_t^\beta p_{\beta 2} + U - \frac{\partial F}{\partial {}_aD_t^\alpha e}\right. \\ & \left. - \frac{\partial F}{\partial {}_tD_b^\beta e}\right)\mathrm{d}e - \frac{\partial L}{\partial t}\mathrm{d}t. \end{aligned} \tag{19}$$

This means that the Hamiltonian is a function of the form

$$H = H(t, q, e, p_{\alpha 1}, p_{\beta 1}, p_{\alpha 2}, p_{\beta 2}). \tag{20}$$

Therefore, the total differential of this function reads as

$$\begin{aligned} \mathrm{d}H =\ & \frac{\partial H}{\partial p_{\alpha 1}}\mathrm{d}p_{\alpha 1} + \frac{\partial H}{\partial p_{\beta 1}}\mathrm{d}p_{\beta 1} + \frac{\partial H}{\partial p_{\alpha 2}}\mathrm{d}p_{\alpha 2} \\ & + \frac{\partial H}{\partial p_{\beta 2}}\mathrm{d}p_{\beta 2} + \frac{\partial H}{\partial q}\mathrm{d}q \\ & + \frac{\partial H}{\partial e}\mathrm{d}e + \frac{\partial H}{\partial t}\mathrm{d}t. \end{aligned} \tag{21}$$

Comparing Eq. (19) with Eq. (21), we obtain the following Hamilton's equations of motion

$$\begin{aligned} & \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial p_{\alpha 1}} = {}_aD_t^\alpha q, \quad \frac{\partial H}{\partial p_{\beta 1}} = {}_tD_b^\beta q, \\ & \frac{\partial H}{\partial p_{\alpha 2}} = {}_aD_t^\alpha e, \quad \frac{\partial H}{\partial p_{\beta 2}} = {}_tD_b^\beta e, \\ & \frac{\partial H}{\partial q} = {}_tD_b^\alpha p_{\alpha 1} + {}_aD_t^\beta p_{\beta 1} + Q - \frac{\partial F}{\partial {}_aD_t^\alpha q} - \frac{\partial F}{\partial {}_tD_b^\beta q}, \\ & \frac{\partial H}{\partial e} = {}_tD_b^\alpha p_{\alpha 2} + {}_aD_t^\beta p_{\beta 2} + U - \frac{\partial F}{\partial {}_aD_t^\alpha e} - \frac{\partial F}{\partial {}_tD_b^\beta e}. \end{aligned} \tag{22}$$

4. Variation of Hamiltonian action with fractional derivatives for mechanico–electrical systems

The Hamilton action of the mechanico–electrical systems with fractional derivatives can be expressed as

$$\begin{aligned} S =\ & \int_{t_1}^{t_2} L(t, q, e, {}_aD_t^\alpha q, {}_tD_b^\beta q, {}_aD_t^\alpha e, {}_tD_b^\beta e)\mathrm{d}t \\ =\ & \int_{t_1}^{t_2} [p_{\alpha 1 a}D_t^\alpha q + p_{\beta 1 t}D_b^\beta q + p_{\alpha 2 a}D_t^\alpha e + p_{\beta 2 t}D_b^\beta e \\ & - H(t, q, e, p_{\alpha 1}, p_{\beta 1}, p_{\alpha 2}, p_{\beta 2})]\mathrm{d}t. \end{aligned} \tag{23}$$

We introduce the infinitesimal transformations with respect to time, generalized coordinates, electrical quantities, and generalized momenta, i.e.,

$$\begin{aligned} & t^* = t + \Delta t, \quad q^*(t) = q(t) + \Delta q, \\ & e^*(t) = e(t) + \Delta e, \quad p_{\alpha 1}^*(t) = p_{\alpha 1}(t) + \Delta p_{\alpha 1}, \\ & p_{\beta 1}^*(t) = p_{\beta 1}(t) + \Delta p_{\beta 1}, \\ & p_{\alpha 2}^*(t) = p_{\alpha 2}(t) + \Delta p_{\alpha 2}, \\ & p_{\beta 2}^*(t) = p_{\beta 2}(t) + \Delta p_{\beta 2}, \end{aligned} \tag{24}$$

and their expansion formulae are

$$\begin{aligned}
&t^* = t + \varepsilon\xi(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&q^*(t) = q(t) + \varepsilon\eta(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&e^*(t) = e(t) + \varepsilon\zeta(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&p^*_{\alpha1}(t) = p_{\alpha1}(t) + \varepsilon\mu_{\alpha1}(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&p^*_{\beta1}(t) = p_{\beta1}(t) + \varepsilon\mu_{\beta1}(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&p^*_{\alpha2}(t) = p_{\alpha2}(t) + \varepsilon\mu_{\alpha2}(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),\\
&p^*_{\beta2}(t) = p_{\beta2}(t) + \varepsilon\mu_{\beta2}(t,q,e,p_{\alpha1},p_{\beta1},p_{\alpha2},p_{\beta2}),
\end{aligned} \tag{25}$$

where ε is a small parameter, and ξ, η, ζ, $\mu_{\alpha1}$, $\mu_{\beta1}$, $\mu_{\alpha2}$, $\mu_{\beta2}$ are infinitesimal generators of the form described above.

The variation of the Hamiltonian action with fractional derivatives for the mechanico–electrical systems can be given by the following two ways

$$\begin{aligned}
\Delta S &= \int_{t_1}^{t_2}\left(\Delta L + L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t\\
&= \int_{t_1}^{t_2}\Big[\Delta(p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q + p_{\alpha2a}D_t^\alpha e\\
&\quad + p_{\beta2t}D_b^\beta e - H) + (p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q\\
&\quad + p_{\alpha2a}D_t^\alpha e + p_{\beta2t}D_b^\beta e - H)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\Big]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\Big[p_{\alpha1}\Delta_a D_t^\alpha q + {}_aD_t^\alpha q\Delta p_{\alpha1} + p_{\beta1}\Delta_t D_b^\beta q\\
&\quad + {}_tD_b^\beta q\Delta p_{\beta1} + p_{\alpha2}\Delta_a D_t^\alpha e + {}_aD_t^\alpha e\Delta p_{\alpha2}\\
&\quad + p_{\beta2}\Delta_t D_b^\beta e + {}_tD_b^\beta e\Delta p_{\beta2} - \frac{\partial H}{\partial q}\Delta q\\
&\quad - \frac{\partial H}{\partial e}\Delta e - \frac{\partial H}{\partial p_{\alpha1}}\Delta p_{\alpha1} - \frac{\partial H}{\partial p_{\beta1}}\Delta p_{\beta1}\\
&\quad - \frac{\partial H}{\partial p_{\alpha2}}\Delta p_{\alpha2} - \frac{\partial H}{\partial p_{\beta2}}\Delta p_{\beta2} - \frac{\partial H}{\partial t}\Delta t\\
&\quad + \Big(p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q\\
&\quad + p_{\alpha2a}D_t^\alpha e + p_{\beta2t}D_b^\beta e - H\Big)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\Big]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\Big[p_{\alpha1}\Delta_a D_t^\alpha q + p_{\beta1}\Delta_t D_b^\beta q + p_{\alpha2}\Delta_a D_t^\alpha e\\
&\quad + p_{\beta2}\Delta_t D_b^\beta e + \left({}_aD_t^\alpha q - \frac{\partial H}{\partial p_{\alpha1}}\right)\Delta p_{\alpha1}\\
&\quad + \left({}_tD_b^\beta q - \frac{\partial H}{\partial p_{\beta1}}\right)\Delta p_{\beta1}\\
&\quad + \left({}_aD_t^\alpha e - \frac{\partial H}{\partial p_{\alpha2}}\right)\Delta p_{\alpha2}\\
&\quad + \left({}_tD_b^\beta e - \frac{\partial H}{\partial p_{\beta2}}\right)\Delta p_{\beta2} - \frac{\partial H}{\partial t}\Delta t\\
&\quad - \frac{\partial H}{\partial q}\Delta q - \frac{\partial H}{\partial e}\Delta e + \Big(p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q\\
&\quad + p_{\alpha2a}D_t^\alpha e + p_{\beta2t}D_b^\beta e - H\Big)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\Big]\mathrm{d}t,
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\Delta S &= \delta S + \dot{S}\Delta t\\
&= \int_{t_1}^{t_2}\left[\delta L + \frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t)\right]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\Big[\delta(p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q\\
&\quad + p_{\alpha2a}D_t^\alpha e + p_{\beta2t}D_b^\beta e - H) + \frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t)\Big]\mathrm{d}t\\
&= \int_{t_1}^{t_2}[({}_aD_t^\alpha q\delta p_{\alpha1} + p_{\alpha1}\delta_a D_t^\alpha q + {}_tD_b^\beta q\delta p_{\beta1}\\
&\quad + p_{\beta1}\delta_t D_b^\beta q + {}_aD_t^\alpha e\delta p_{\alpha2} + p_{\alpha2}\delta_a D_t^\alpha e\\
&\quad + {}_tD_b^\beta e\delta p_{\beta2} + p_{\beta2}\delta_t D_b^\beta e - \delta H)\\
&\quad + D^1(L\Delta t)]\mathrm{d}t\\
&= \int_{t_1}^{t_2}\Big\{D^1\left(p_{\alpha1}\delta_a D_t^{\alpha-1}q\right) - D^1(p_{\beta1}\delta_t D_b^{\beta-1}q)\\
&\quad + D^1\left(p_{\alpha2}\delta_a D_t^{\alpha-1}e\right) - D^1(p_{\beta2}\delta_t D_b^{\beta-1}e)\\
&\quad + D^1[(p_{\alpha1a}D_t^\alpha q + p_{\beta1t}D_b^\beta q + p_{\alpha2a}D_t^\alpha e\\
&\quad + p_{\beta2t}D_b^\beta e - H)\Delta t] + \left({}_aD_t^\alpha q - \frac{\partial H}{\partial p_{\alpha1}}\right)\delta p_{\alpha1}\\
&\quad + \left({}_tD_b^\beta q - \frac{\partial H}{\partial p_{\beta1}}\right)\delta p_{\beta1}\\
&\quad + \left({}_tD_b^\alpha p_{\alpha1} + {}_aD_t^\beta p_{\beta1} - \frac{\partial H}{\partial q}\right)\delta q\\
&\quad + \left({}_aD_t^\alpha e - \frac{\partial H}{\partial p_{\alpha2}}\right)\delta p_{\alpha2}\\
&\quad + \left({}_tD_b^\beta e - \frac{\partial H}{\partial p_{\beta2}}\right)\delta p_{\beta2}\\
&\quad + \left({}_tD_b^\alpha p_{\alpha2} + {}_aD_t^\beta p_{\beta2} - \frac{\partial H}{\partial e}\right)\delta e\Big\}\mathrm{d}t.
\end{aligned} \tag{27}$$

Notice that

$$\delta q = \Delta q - \dot{q}\Delta t = \varepsilon\left(\eta - {}_aD_t^1 q\cdot\xi\right), \tag{28a}$$

$$\delta e = \Delta e - \dot{e}\Delta t = \varepsilon\left(\zeta - {}_aD_t^1 e\cdot\xi\right), \tag{28b}$$

$$\delta p_{\alpha1} = \Delta p_{\alpha1} - \dot{p}_{\alpha1}\Delta t = \varepsilon\left(\mu_{\alpha1} - {}_aD_t^1 p_{\alpha1}\cdot\xi\right), \tag{28c}$$

$$\delta p_{\alpha2} = \Delta p_{\alpha2} - \dot{p}_{\alpha2}\Delta t = \varepsilon\left(\mu_{\alpha2} - {}_aD_t^1 p_{\alpha2}\cdot\xi\right), \tag{28d}$$

$$\delta p_{\beta1} = \Delta p_{\beta1} - \dot{p}_{\beta1}\Delta t = \varepsilon\left(\mu_{\beta1} - {}_aD_t^1 p_{\beta1}\cdot\xi\right), \tag{28e}$$

$$\delta p_{\beta2} = \Delta p_{\beta2} - \dot{p}_{\beta2}\Delta t = \varepsilon\left(\mu_{\beta2} - {}_aD_t^1 p_{\beta2}\cdot\xi\right). \tag{28f}$$

Equation (27) can be written in the form

$$\begin{aligned}\Delta S = \int_{t_1}^{t_2} \Big\{ & D^1\left(p_{\alpha 1}\delta_a D_t^{\alpha-1}q\right) - D^1(p_{\beta 1}\delta_t D_b^{\beta-1}q) \\ & + D^1\left(p_{\alpha 2}\delta_a D_t^{\alpha-1}e\right) - D^1(p_{\beta 2}\delta_t D_b^{\beta-1}e) \\ & + \varepsilon\Big[D^1((p_{\alpha 1 a}D_t^{\alpha}q + p_{\beta 1 t}D_b^{\beta}q \\ & + p_{\alpha 2 a}D_t^{\alpha}e + p_{\beta 2 t}D_b^{\beta}e - H)\xi) \\ & + \left({}_aD_t^{\alpha}q - \frac{\partial H}{\partial p_{\alpha 1}}\right)\bar{\mu}_{\alpha 1} \\ & + \left({}_tD_b^{\beta}q - \frac{\partial H}{\partial p_{\beta 1}}\right)\bar{\mu}_{\beta 1} \\ & + \left({}_tD_b^{\alpha}p_{\alpha 1} + {}_aD_t^{\beta}p_{\beta 1} - \frac{\partial H}{\partial q}\right)\bar{\eta} \\ & + \left({}_tD_b^{\alpha}p_{\alpha 2} + {}_aD_t^{\beta}p_{\beta 2} - \frac{\partial H}{\partial e}\right)\bar{\zeta} \\ & + \left({}_aD_t^{\alpha}e - \frac{\partial H}{\partial p_{\alpha 2}}\right)\bar{\mu}_{\alpha 2} \\ & + \left({}_tD_b^{\beta}e - \frac{\partial H}{\partial p_{\beta 2}}\right)\bar{\mu}_{\beta 2}\Big]\Big\}\mathrm{d}t, \end{aligned} \tag{29}$$

where

$$\begin{aligned} &\bar{\eta} = \eta - {}_aD_t^1 q\cdot\xi, \quad \bar{\zeta} = \zeta - {}_aD_t^1 e\cdot\xi, \\ &\bar{\mu}_{\alpha 1} = \mu_{\alpha 1} - {}_aD_t^1 p_{\alpha 1}\cdot\xi, \\ &\bar{\mu}_{\alpha 2} = \mu_{\alpha 2} - {}_aD_t^1 p_{\alpha 2}\cdot\xi, \\ &\bar{\mu}_{\beta 1} = \mu_{\beta 1} - {}_aD_t^1 p_{\beta 1}\cdot\xi, \\ &\bar{\mu}_{\beta 2} = \mu_{\beta 2} - {}_aD_t^1 p_{\beta 2}\cdot\xi. \end{aligned} \tag{30}$$

Equations (26) and (29) are the basic formulae for the variation of the Hamilton action with fractional derivatives for the mechanico–electrical systems.

5. Generalized Noether quasi-symmetrical transformations for mechanico–electrical systems with fractional derivatives

Definition 2 If the Hamilton action is generalized quasi-invariant under the infinitesimal transformations of group for the mechanico–electrical systems, namely, for each infinitesimal transformation, the equation

$$\Delta S = -\int_{t_1}^{t_2}\left[{}_aD_t^1(\Delta G) + \Lambda_1\delta q + \Lambda_2\delta e\right]\mathrm{d}t \tag{31}$$

holds, where $G = G(t, q, e, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q, {}_aD_t^{\alpha}e, {}_tD_b^{\beta}e)$ is the gauge function of the system. Then the infinitesimal transformations are called the generalized quasi-symmetrical transformations in the Noether sense with fractional derivatives.

Criterion 1 If the infinitesimal transformations of group (24) satisfy the following equation

$$\begin{aligned} & p_{\alpha 1}\Delta_a D_t^{\alpha}q + p_{\beta 1}\Delta_t D_b^{\beta}q + p_{\alpha 2}\Delta_a D_t^{\alpha}e \\ & + p_{\beta 2}\Delta_t D_b^{\beta}e + \left({}_aD_t^{\alpha}q - \frac{\partial H}{\partial p_{\alpha 1}}\right)\Delta p_{\alpha 1} \\ & + \left({}_tD_b^{\beta}q - \frac{\partial H}{\partial p_{\beta 1}}\right)\Delta p_{\beta 1} \\ & + \left({}_aD_t^{\alpha}e - \frac{\partial H}{\partial p_{\alpha 2}}\right)\Delta p_{\alpha 2} \\ & + \left({}_tD_b^{\beta}e - \frac{\partial H}{\partial p_{\beta 2}}\right)\Delta p_{\beta 2} \\ & - \frac{\partial H}{\partial t}\Delta t - \frac{\partial H}{\partial q}\Delta q - \frac{\partial H}{\partial e}\Delta e \\ & + (p_{\alpha 1 a}D_t^{\alpha}q + p_{\beta 1 t}D_b^{\beta}q + p_{\alpha 2 a}D_t^{\alpha} \\ & + p_{\beta 2 t}D_b^{\beta}e - H)_aD_t^1(\Delta t) \\ = & -\left[\left({}_aD_t^1(\Delta G) + \Lambda_1\delta q + \Lambda_2\delta e\right)\right], \end{aligned} \tag{32}$$

then transformations (24) are the generalized Noether quasi-symmetrical transformations for the mechanico–electrical systems with fractional derivatives.

Substituting the Hamilton's equations of motion (22) and Eqs. (28a) and (28b) into the equation given above, we obtain

$$\begin{aligned} & p_{\alpha 1}\Delta_a D_t^{\alpha}q + p_{\beta 1}\Delta_t D_b^{\beta}q + p_{\alpha 2}\Delta_a D_t^{\alpha}e + p_{\beta 2}\Delta_t D_b^{\beta}e \\ & - \left(\frac{\partial H}{\partial t} + \Lambda_{1 a}D_t^1 q + \Lambda_{2 a}D_t^1 e\right)\Delta t \\ & + (p_{\alpha 1 a}D_t^{\alpha}q + p_{\beta 1 t}D_b^{\beta}q + p_{\alpha 2 a}D_t^{\alpha}e \\ & + p_{\beta 2 t}D_b^{\beta}e - H)_aD_t^1(\Delta t) - \left(\frac{\partial H}{\partial q} - \Lambda_1\right)\Delta q \\ & - \left(\frac{\partial H}{\partial e} - \Lambda_2\right)\Delta e = -{}_aD_t^1(\Delta G). \end{aligned} \tag{33}$$

Criterion 2 There exist generalized Noether quasi-symmetry transformations for the mechanico–electrical systems if the infinitesimal transformations (25) satisfy

$$\begin{aligned} & D^1\left(p_{\alpha 1}\delta_a D_t^{\alpha-1}q\right) - D^1(p_{\beta 1}\delta_t D_b^{\beta-1}q) \\ & + D^1\left(p_{\alpha 2}\delta_a D_t^{\alpha-1}e\right) - D^1(p_{\beta 2}\delta_t D_b^{\beta-1}e) \\ & + \varepsilon\Big\{D^1[(p_{\alpha 1 a}D_t^{\alpha}q + p_{\beta 1 t}D_b^{\beta}q \\ & + p_{\alpha 2 a}D_t^{\alpha}e + p_{\beta 2 t}D_b^{\beta}e - H)\xi] \\ & + \left({}_aD_t^{\alpha}q - \frac{\partial H}{\partial p_{\alpha 1}}\right)\bar{\mu}_{\alpha 1} \\ & + \left({}_tD_b^{\beta}q - \frac{\partial H}{\partial p_{\beta 1}}\right)\bar{\mu}_{\beta 1} \end{aligned}$$

$$
\begin{aligned}
&+\left({}_tD_b^{\alpha}p_{\alpha 1}+{}_aD_t^{\beta}p_{\beta 1}-\frac{\partial H}{\partial q}\right)\bar{\eta}\\
&+\left({}_tD_b^{\alpha}p_{\alpha 2}+{}_aD_t^{\beta}p_{\beta 2}-\frac{\partial H}{\partial e}\right)\bar{\zeta}\\
&+\left({}_aD_t^{\alpha}e-\frac{\partial H}{\partial p_{\alpha 2}}\right)\bar{\mu}_{\alpha 2}\\
&+\left({}_tD_b^{\beta}e-\frac{\partial H}{\partial p_{\beta 2}}\right)\bar{\mu}_{\beta 2}\Bigg\}\\
=&-\left(\varepsilon_a D_t^1 G+\Lambda_1\delta q+\Lambda_2\delta e\right), \qquad (34)
\end{aligned}
$$

where $\Delta G=\varepsilon G$. Substituting the Hamilton's equations of motion (22) and Eqs (28a) and (28b) into the above equation, we obtain

$$
\begin{aligned}
&{}_aD_t^1(p_{\alpha 1}\delta_a D_t^{\alpha-1}q)-{}_aD_t^1(p_{\beta 1}\delta_t D_b^{\beta-1}q)\\
=&\ {}_aD_t^1(p_{\alpha 2}\delta_a D_t^{\alpha-1}e)-{}_aD_t^1(p_{\beta 2}\delta_t D_b^{\beta-1}e)\\
&+{}_aD_t^1[\varepsilon(p_{\alpha 1a}D_t^{\alpha}q+p_{\beta 1t}D_b^{\beta}q+p_{\alpha 2a}D_t^{\alpha}e\\
&+p_{\beta 2t}D_b^{\beta}e-H)\xi]\\
=&-\varepsilon_a D_t^1\left(G\right). \qquad (35)
\end{aligned}
$$

6. Noether theorem of mechanico–electrical systems with fractional derivatives

In Section 3, we worked out the fractional Hamilton's equations of the mechanico–electrical systems. Making use of these equations and some properties of the fractional derivatives, we obtain

$$
\begin{aligned}
\Delta_a D_t^{\alpha}q&=\delta_a D_t^{\alpha}q+{}_aD_t^{\alpha+1}q\cdot\Delta t\\
&=\varepsilon\left[{}_aD_t^{\alpha}\eta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1q\right)+\xi\cdot{}_aD_t^{\alpha+1}q\right],\\
\Delta_t D_b^{\beta}q&=\delta_t D_b^{\beta}q-{}_tD_b^{\beta+1}q\cdot\Delta t\\
&=\varepsilon[{}_tD_b^{\beta}\eta+{}_tD_b^{\beta}\left(\xi\cdot{}_tD_b^1q\right)-\xi\cdot{}_tD_b^{\beta+1}q],\\
\Delta_a D_t^{\alpha}e&=\delta_a D_t^{\alpha}e+{}_aD_t^{\alpha+1}e\cdot\Delta t\\
&=\varepsilon\left[{}_aD_t^{\alpha}\zeta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1e\right)+\xi\cdot{}_aD_t^{\alpha+1}e\right],\\
\Delta_t D_b^{\beta}e&=\delta_t D_b^{\beta}e-{}_tD_b^{\beta+1}e\cdot\Delta t\\
&=\varepsilon[{}_tD_b^{\beta}\zeta+{}_tD_b^{\beta}\left(\xi\cdot{}_tD_b^1e\right)-\xi\cdot{}_tD_b^{\beta+1}e].
\end{aligned}
\qquad (36)
$$

Inserting the above equations into Eq. (33), we can obtain the Noether identity of the mechanico–electrical systems, i.e.

$$
\begin{aligned}
&p_{\alpha 1}[{}_aD_t^{\alpha}\eta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1q\right)+\xi\cdot{}_aD_t^{\alpha+1}q\\
&+{}_aD_t^{\alpha}q\cdot{}_aD_t^1\xi]+p_{\beta 1}[{}_tD_b^{\beta}\eta+{}_tD_b^{\beta}\left(\xi\cdot{}_tD_b^1q\right)\\
&-\xi\cdot{}_tD_b^{\beta+1}q-{}_tD_b^{\beta}q\cdot{}_tD_b^1\xi]\\
&+p_{\alpha 2}[{}_aD_t^{\alpha}\zeta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1e\right)+\xi\cdot{}_aD_t^{\alpha+1}e\\
&+{}_aD_t^{\alpha}e\cdot{}_aD_t^1\xi]+p_{\beta 2}[{}_tD_b^{\beta}\zeta+{}_tD_b^{\beta}\left(\xi\cdot{}_tD_b^1e\right)\\
&-\xi\cdot{}_tD_b^{\beta+1}e-{}_tD_b^{\beta}e\cdot{}_tD_b^1\xi]-H_aD_t^1\xi\\
&-\left(\frac{\partial H}{\partial t}+\Lambda_1{}_aD_t^1q+\Lambda_2{}_aD_t^1e\right)\xi-\left(\frac{\partial H}{\partial q}-\Lambda_1\right)\eta\\
&-\left(\frac{\partial H}{\partial e}-\Lambda_2\right)\zeta+{}_aD_t^1(G)=0. \qquad (37)
\end{aligned}
$$

For a mechanico–electrical system, if its Noether symmetrical transformations exist, the conserved quantities in the system can be obtained by the following Noether theorem.

Theorem 2 If the infinitesimal transformation (25) is a generalized quasi-symmetrical transformation of the mechanico–electrical system, then the system possesses the following first integral

$$
\begin{aligned}
I=&\ p_{\alpha 1}\left[{}_aD_t^{\alpha-1}\eta-{}_aD_t^{\alpha-1}\left(\xi\cdot{}_aD_t^1q\right)+\xi\cdot{}_aD_t^{\alpha}q\right]\\
&-p_{\beta 1}[{}_tD_b^{\beta-1}\eta+{}_tD_b^{\beta-1}\left(\xi\cdot{}_tD_b^1q\right)-\xi\cdot{}_tD_b^{\beta}q]\\
&+p_{\alpha 2}\left[{}_aD_t^{\alpha-1}\zeta-{}_aD_t^{\alpha-1}\left(\xi\cdot{}_aD_t^1e\right)+\xi\cdot{}_aD_t^{\alpha}e\right]\\
&-p_{\beta 2}[{}_tD_b^{\beta-1}\zeta+{}_tD_b^{\beta-1}\left(\xi\cdot{}_tD_b^1e\right)-\xi\cdot{}_tD_b^{\beta}e]\\
&-H\cdot\xi+G. \qquad (38)
\end{aligned}
$$

Proof In virtue of Eqs. (8), (9), and (36), we can prove that Eq. (35) yields integral Eq. (38).

7. An example

Remark 2 In physics, if t denotes the time variable, the right Riemann–Liouville fractional derivative of $f(t)$ is interpreted as a future state of the process $f(t)$. For this reason, the right derivative is usually neglected in applications when the present state of the process does not depend on the results of the future development.

We consider a simple mechanico–electrical system with the left Riemann–Liouville fractional derivative. The Lagrangian of the system is given by

$$
L=\frac{1}{2}m\left({}_aD_t^{\alpha}q\right)^2+\frac{1}{2}L_1\left({}_aD_t^{\alpha}e\right)^2-\frac{1}{2}kq^2-\frac{1}{2}\frac{s-q}{A}e^2,
$$

where m, L_1, k, s, and A are the parameters of the system. The dissipative function of the system is

$$
F=\frac{1}{2}R\left({}_aD_t^{\alpha}e\right)^2
$$

and the generalized electromotive force of the system is

$$
U=E_0\sin\omega t,
$$

so the non-conservative forces are expressed as

$$
\Lambda_1=0,\Lambda_2=-E_0\sin\omega t+R_aD_t^{\alpha}e.
$$

From Eqs. (12) and (13), we obtain the Euler–Lagrange equations of the system as

$$-L_{1t}D_b^{\alpha}({}_aD_t^{\alpha}e)+R_aD_t^{\alpha}e+\frac{(s-q)e}{A}=E_0\sin\omega t,$$
$$m_tD_b^{\alpha}({}_aD_t^{\alpha}q)+\frac{e^2}{2A}-kq=0.$$

The generalized momenta are given by

$$p_{\alpha1}=\frac{\partial L}{\partial {}_aD_t^{\alpha}q}=m_aD_t^{\alpha}q,\quad p_{\beta1}=0,$$
$$p_{\alpha2}=\frac{\partial L}{\partial {}_aD_t^{\alpha}e}=L_{1a}D_t^{\alpha}e,\quad p_{\beta2}=0,$$

then the Hamiltonian of the system is

$$H=\frac{1}{2}m\left({}_aD_t^{\alpha}q\right)^2+\frac{1}{2}L_1\left({}_aD_t^{\alpha}e\right)^2+\frac{1}{2}kq^2+\frac{1}{2}\frac{s-q}{A}e^2.$$

From Eq. (22), we obtain the Hamilton's equations of motion

$$\frac{\partial H}{\partial t}=0,\quad \frac{\partial H}{\partial p_{\alpha1}}=m_aD_t^{\alpha}q,\quad \frac{\partial H}{\partial p_{\beta1}}=0,$$
$$\frac{\partial H}{\partial p_{\alpha2}}=L_{1a}D_t^{\alpha}e,\quad \frac{\partial H}{\partial p_{\beta2}}=0,$$
$$\frac{\partial H}{\partial q}=m_tD_b^{\alpha}({}_aD_t^{\alpha}q),$$
$$\frac{\partial H}{\partial e}=L_{1t}D_b^{\alpha}({}_aD_t^{\alpha}e)+E_0\sin\omega t-R_aD_t^{\alpha}e.$$

The Noether identity (37) gives

$$\begin{aligned}&p_{\alpha1}[{}_aD_t^{\alpha}\eta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1q\right)+\xi\cdot{}_aD_t^{\alpha+1}q\\&+{}_aD_t^{\alpha}q\cdot{}_aD_t^1\xi]+p_{\alpha2}[{}_aD_t^{\alpha}\zeta-{}_aD_t^{\alpha}\left(\xi\cdot{}_aD_t^1e\right)\\&+\xi\cdot{}_aD_t^{\alpha+1}e+{}_aD_t^{\alpha}e\cdot{}_aD_t^1\xi]\\&+\left(-E_0\sin\omega t+R_aD_t^{\alpha}e-H\right){}_aD_t^1\xi-\left(kq-\frac{e^2}{2A}\right)\eta\\&-\left(\frac{(s-q)\,e}{A}+E_0\sin\omega t-R\cdot{}_aD_t^{\alpha}e\right)\zeta\\&+{}_aD_t^1(G)=0,\end{aligned}$$

from which we can obtain the following symmetries

$$\begin{aligned}&\xi_1=0,\quad \eta_1=q,\quad \zeta_1=0,\\&G_1={}_aD_t^{-1}\left(kq^2-\frac{qe^2}{2A}\right)-m_aD_t^{-1}[({}_aD_t^{\alpha}q)^2],\\&\xi_2=0,\quad \eta_2=e,\quad \zeta_2=-\frac{m}{L_1}q,\\&G_2={}_aD_t^{-1}\left(kqe-\frac{e^3}{2A}\right)-\frac{m}{L_1}{}_aD_t^{-1}\Big[\frac{(s-q)\,eq}{A}\\&\qquad+E_0q\sin\omega t-Rq\cdot{}_aD_t^{\alpha}e\Big].\end{aligned}$$

Substituting them into Eq. (38), we obtain the following conserved quantities of the mechanico–electrical system

$$\begin{aligned}I_1=&\frac{1}{m}p_{\alpha1}\cdot{}_aD_t^{-1}p_{\alpha1}+{}_aD_t^{-1}\left(kq^2-\frac{qe^2}{2A}\right)\\&-\frac{1}{m}{}_aD_t^{-1}\left(p_{\alpha1}\right)^2,\\I_2=&\frac{1}{L_1}\left(p_{\alpha1}\cdot{}_aD_t^{-1}p_{\alpha2}-p_{\alpha2}\cdot{}_aD_t^{-1}p_{\alpha1}\right)\\&-\frac{m}{L_1}{}_aD_t^{-1}\Big[\Big(\frac{(s-q)e}{A}+E_0\sin\omega t\\&-R_aD_t^{\alpha}e\Big)q\Big]+{}_aD_t^{-1}\left(kqe-\frac{e^3}{2A}\right).\end{aligned}$$

8. Conclusion

The Euler–Lagrange equations with fractional derivatives have been extended to mechanico–electrical systems and the Hamilton formalism and the Noether symmetry of the systems have been presented in this paper. Given that many systems can be modeled more accurately using the fractional derivatives the corresponding symmetry theories will become more useful and important.

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