Generalized thermoelasticity of the thermal shock problem in an isotropic hollow cylinder and temperature dependent elastic moduli

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In this paper, we construct the equations of generalized thermoelasicity for a non-homogeneous isotropic hollow cylider with a variable modulus of elasticity and thermal conductivity based on the Lord and Shulman theory. The problem has been solved numerically using the finite element method. Numerical results for the displacement, the temperature, the radial stress, and the hoop stress distributions are illustrated graphically. Comparisons are made between the results predicted by the coupled theory and by the theory of generalized thermoelasticity with one relaxation time in the cases of temperature dependent and independent modulus of elasticity.

Keywords: generalized thermoelsticity, thermal shock, temperature dependent elastic moduli, finite element method

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1. Introduction

In the postwar years, we have seen a rapid development of themoelasticity stimulated by various engineering sciences.[1] Most investigations were done under the assumption of the temperature-independent material properties, which limited the applicability of the solutions obtained to certain ranges of temperature. At high temperature, the material characteristics, such as the modulus of elasticity, the Poisson's ratio, and the coefficient of thermal conductivity, are no longer constants.[2] In recent years, due to progress in various fields of science and technology, taking into consideration the real behaviour of the material characteristics becomes an actual necessity. In some investigations, they have been taken as functions of coordinates.[3,4]

The classical uncoupled theory of thermoelasticity predicts two phenomena that are not compatible with the physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type, predicting infinite speeds of propagation for heat waves.

Biot[5] introduced the theory of coupled (CD) thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, his theory still has the second shortcoming, since the heat equation for the coupled theory is also parabolic.

Lord and Shulman[6] (LS) introduced the theory of generalized thermoelasticity with one relaxation time. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is a hyperbolic one, and hence automatically eliminates the paradox of infinite speeds of propagation inherent from both the uncoupled and the coupled theories of thermoelasticity. Most often, the solutions obtained using this theory differ little qualitatively from those obtained using either the coupled or the uncoupled theories, though, the solutions differ quantitatively. However, for many problems involving steep heat gradients and when short time effects are sought, this theory gives markedly different results compared to those predicted by any of the other theories. This is the case encountered in many problems in industry, especially inside nuclear reactors where very high

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heat gradients act for very short times. This theory was extended by Dhaliwal and Sherief[7] to generalized isotropic media with the presence of heat sources lacking in the formulation of Lord and Shulman.[6] Sherief and Dhaliwal[8] solved a thermal shock problem. Recently, Othman[9,10] used this theory to study the dependence of the modulus of elasticity on the reference temperature in the two-dimensional generalized thermoelasticity in the absence and the presence of rotation respectively. Othman *et. al.*[11] have studied the transient waves caused by a line heat source with a uniform velocity inside an isotropic homogeneous thermoelastic perfectly conducting half-space permeated into a uniform magnetic field. This theory has been applied in many problems.[12,13]

The exact solution of the generalized thermoelasticity theory governing equations for a coupled and non-linear/linear problem exists only for very special and simple initial and boundary conditions. For general problems, a numerical solution technique should be used. For this reason, the finite element method (FEM) is chosen. The method of weighted residuals offers us the formulation of the finite element equations, and we obtain best approximated solutions to linear/nonlinear ordinary/partial differential equations. Applying this method basically involves three steps. The first step is to assume that the general behaviours of the unknown field variables satisfy the given differential equations. Substituting these approximating functions into the differential equations and the boundary conditions results in some errors called the residual. The residual has to vanish in an average sense over the solution domain. The second step is the time integration. The time derivatives of the unknown variables have to be determined by the former results. The third step is to solve the equations resulting from the first and the second steps by the solving algorithm of the finite element program.[14] Abbas,[15] Youssef and Abbas,[16] Abbas and Abdallah,[17] and Abbas and Othman[18] applied the finite element method in different problems.

In the present paper, we have formulated the problem of the generalized thermo-elasticity of a non-homogeneous isotropic hollow cylinder with thermal relaxation, and the modulus of elasticity is taken as a linear function of the reference temperature. The problem is solved numerically using the finite element method. Numerical results for the distributions of displacement, temperature, radial stress, and hoop stress are presented graphically. The results indicate that the effects of the reference temperature are very pronounced. Results obtained in this paper can be used to design various homogeneous thermoelastic elements under thermal load to meet special engineering requirements.

2. Formulation of problem

In the context of the generalized thermoelasticity theory, the system of equations that includes the displacement, the stress, the strain, and the temperature for a linear, homogenous, and isotropic thermoelastic continuum takes the form

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}+F_i-\gamma T_{,i}=\rho\ddot{u}_i, \tag{1}$$

the energy equation has the form

$$KT_{,ii} = \rho C_{\rm v}\left(\dot{T}+t_0\ddot{T}\right)+\left(1+t_0\frac{\partial}{\partial t}\right) \times\left(\gamma T_0\dot{u}_{i,i}-\rho Q\right), \tag{2}$$

and the constitutive equations are given by

$$\tau_{ij}=\lambda u_{i,i}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)-\gamma T\delta_{ij}, \tag{3}$$

where λ and μ are Lame's constants, ρ is the density of the medium, $C_{\rm v}$ is the specific heat at a constant strain, t is the time, t_0 is the relaxation time, T is the temperature, T_0 is the reference temperature, K is the thermal conductivity, δ_{ij} is the Kronecker symbol, τ_{ij} is the component of the stress tensor, u_i is the component of the displacement vector, F_i is the body force vector, Q is the heat source, $\gamma=(3\lambda+2\mu)\alpha_t$, and α_t is the coefficient of the linear thermal expansion.

In the cylindrical coordinate system (r,θ,z), for the axially symmetric problem, $u_r=u_r(r,z,t)$, $u_\theta=0$, $u_z=u_z(r,z,t)$. Furthermore, if only the axisymmetric plane strain problem is considered, we have $u_r=u(r,t)$ and $u_\theta=u_z=0$. The strain–displacement relations are

$$e_{rr}=\frac{\partial u}{\partial r},\quad e_{\theta\theta}=\frac{u}{r},$$
$$e_{zz}=e_{rz}=e_{r\theta}=e_{\theta z}=0. \tag{4}$$

The stress–strain relations are

$$\tau_{rr}=2\mu\frac{\partial u}{\partial r}+\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-\gamma T, \tag{5}$$

$$\tau_{\theta\theta}=2\mu\frac{u}{r}+\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-\gamma T, \tag{6}$$

$$\tau_{zz}=\lambda\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)-\gamma T,\quad \tau_{rz}=\tau_{r\theta}=\tau_{\theta z}=0. \tag{7}$$

It is assumed that there is no body force and heat source in the medium, and the equation of motion and the energy equation have the form

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{8}$$

$$\begin{aligned} & K \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \\ &= \rho C_{\rm v} \left(\frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} \right) + \gamma T_0 \left(1 + t_0 \frac{\partial}{\partial t} \right) \\ &\quad \times \left(\frac{\partial^2 u}{\partial t \partial r} + \frac{1}{r} \frac{\partial u}{\partial t} \right). \end{aligned} \tag{9}$$

In this study, we assume the non-homogeneous property of the material is characterized by

$$\begin{aligned} & \lambda = \lambda_0 f(T), \quad \mu = \mu_0 f(T), \\ & \gamma = \gamma_0 f(T), \quad K = K_0 f(T), \end{aligned} \tag{10}$$

where $f(T)$ is a given non-dimensional function of temperature. In the case of a temperature independent modulus of elasticity, $f(T) = 1$. Let $f(T) = (1 - \alpha^* T_0)$, where α^* is the parameter referred to the dependence on the temperature. In the case of the reference temperature independent modulus of elasticity and thermal conductivity, $\alpha^* = 0$, λ_0, μ_0, γ_0, and K_0 are considered to be constants. Substituting Eq. (10) into Eqs. (5)–(9), we obtain

$$\alpha \tau_{rr} = \left[2\mu_0 \frac{\partial u}{\partial r} + \lambda_0 \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma_0 T \right], \tag{11}$$

$$\alpha \tau_{\theta\theta} = \left[2\mu_0 \frac{u}{r} + \lambda_0 \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma_0 T \right], \tag{12}$$

$$\begin{aligned} & \alpha \tau_{zz} = \left[\lambda_0 \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - \gamma_0 T \right], \\ & \tau_{rz} = \tau_{r\theta} = \tau_{\theta z} = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} & (\lambda_0 + 2\mu_0) \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) - \gamma_0 \frac{\partial T}{\partial r} \\ &= \alpha \rho_0 \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} & K_0 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \\ &= \rho C_{\rm v} \alpha \left(\frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} \right) + \gamma_0 T_0 \left(1 + t_0 \frac{\partial}{\partial t} \right) \\ &\quad \times \left(\frac{\partial^2 u}{\partial t \partial r} + \frac{1}{r} \frac{\partial u}{\partial t} \right), \end{aligned} \tag{15}$$

where $\gamma_0 = (3\lambda_0 + 2\mu_0)\alpha_{\rm t}$ and $\alpha = 1/(1 - \alpha^* T_0)$. It is convenient to change the preceding equations into the dimensionless form. To do this, the dimensionless parameters are introduced as

$$(r', \quad u') = \frac{(r, u)}{c_1 \omega_1}, \quad (t', t_0') = \frac{(t, t_0)}{\omega_1},$$

$$(\tau_{rr}', \tau_{\theta\theta}', \tau_{zz}') = \frac{1}{\mu} (\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}), \quad \Theta' = \frac{\gamma T}{\rho c_1^2}, \tag{16}$$

where $c_1^2 = (\lambda_0 + 2\mu_0)/\rho$, and $\omega_1 = K_0/(\rho C_\nu c_1^2)$. From Eqs. (11)–(15), we obtain (after dropping the dash for convenience)

$$\alpha \tau_{rr} = (2 + a_1) \frac{\partial u}{\partial r} + a_1 \frac{u}{r} - (2 + a_1)\Theta, \tag{17}$$

$$\alpha \tau_{\theta\theta} = a_1 \frac{\partial u}{\partial r} + (2 + a_1) \frac{u}{r} - (2 + a_1)\Theta, \tag{18}$$

$$\alpha \tau_{zz} = a_1 \left(\frac{\partial u}{\partial r} + \frac{u}{r} \right) - (2 + a_1)\Theta, \tag{19}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} - \frac{\partial \Theta}{\partial r} = \alpha \frac{\partial^2 u}{\partial t^2}, \tag{20}$$

$$\begin{aligned} & K_0 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \\ &= \alpha \left(\frac{\partial T}{\partial t} + t_0 \frac{\partial^2 T}{\partial t^2} \right) + \varepsilon \left(1 + t_0 \frac{\partial}{\partial t} \right) \\ &\quad \times \left(\frac{\partial^2 u}{\partial t \partial r} + \frac{1}{r} \frac{\partial u}{\partial t} \right), \end{aligned} \tag{21}$$

with $a_1 = \lambda_0/\mu_0$, $\varepsilon = \gamma_0^2 T_0/(\rho_0^2 c_1^2 C_\nu)$. From the preceding description, the initial and the boundary conditions may be expressed as

$$\begin{aligned} & u(r, 0) = \frac{\partial u(r, 0)}{\partial t} = 0, \\ & \Theta(r, 0) = \frac{\partial \Theta(r, 0)}{\partial t} = 0, \\ & \tau_{rr}(a, t) = 0, \quad \tau_{rr}(b, t) = 0, \end{aligned} \tag{22}$$

$$\Theta(a, t) = H(t), \quad \frac{\partial \Theta(b, t)}{\partial r} = 0, \tag{23}$$

where a and b are the inner and the outer radii of the hollow cylinder, and $H(t)$ denotes the Heaviside unit step function.

3. Finite element method

In order to investigate the numerical solution of the thermal shock problem in a non-homogeneous isotropic hollow cylinder using the generalized thermoelasticity theory, the FEM (Reddy[19] and Cook *et al.*[20]) is adopted due to its flexibility in modeling layered structures and its capability in obtaining the full field numerical solution. The governing Eqs. (20) and (21) are coupled with the initial and the boundary conditions (22) and (23). The numerical values of the dependent variables like displacement u and temperature Θ are obtained at the interesting points,

which are called the degrees of freedom. The weak formulations of the non-dimensional governing equations are derived. A set of independent test functions consisting with the displacement δu and the temperature $\delta\Theta$ is prescribed. The governing equations are multiplied by independent weighting functions and then are integrated over the spatial domain with the boundary. Applying integration by parts and making use of the divergence theorem reduce the order of the spatial derivatives and allow the application of the boundary conditions. The same shape functions are defined piecewise on the elements. Using the Galerkin procedure, the unknown fields u and Θ and the corresponding weighting functions are approximated by the same shape functions. The last step towards the finite element discretization is to choose the element type and the associated shape functions. Three nodes of quadrilateral elements are used. The shape function is usually denoted by letter N and is usually the coefficient that appears in the interpolation polynomial. A shape function is written for each individual node of the finite element and has the property that its magnitude is 1 at that node and 0 for all other nodes in that element. We assume that the master element has its local coordinates in the range of $[-1, 1]$. In our case, the one-dimensional quadratic elements are used, which are given by linear shape functions

$$N_1 = \frac{1}{2}(1-\xi)\,, \quad N_2 = \frac{1}{2}(1+\xi)\,, \tag{24}$$

and quadratic shape functions

$$N_1 = \frac{1}{2}\left(\xi^2-\xi\right)\,,\ N_2 = 1-\xi^2,\ N_3 = \frac{1}{2}\left(\xi^2+\xi\right)\,. \tag{25}$$

On the other hand, the time derivatives of the unknown variables have to be determined by the Newmark time integration method.[20]

4. Numerical results

With an aim to illustrate the problem, we will present some numerical results. The copper material is chosen for the numerical computation. The physical data are given as $\lambda_0 = 7.76 \times 10^{10}$ kg·m^{-1}·s^{-2}, $\mu_0 = 3.86 \times 10^{10}$ kg·m^{-1}·s^{-2}, $T_0 = 293$ K, $\rho_0 = 8.954 \times 10^3$ kg·m^{-3}, $C_{\rm v} = 3.831 \times 10^2$ m^2·K^{-1}·s^{-2}, $\alpha_{\rm t} = 17.8 \times 10^{-6}$ K^{-1}, $K_0 = 3.86 \times 10^2$ kg·m·K^{-1}·s^{-3}.

The field quantities, the displacement, the temperature, and the stresses depend not only on time t and space r, but also on parameter α. It has been observed that in the Lord and Shulman theory, parameter α has a significant effect on the field quantities. Here all the variables/parameters are taken in the dimensionless forms. Figures 1–4 show the results for the displacement, the temperature, the radial stress, and the hoop stress at $t = 0.2$ for two different values of α ($\alpha = 0.0, 1.95$).

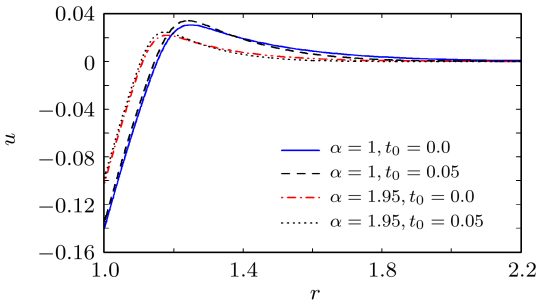


Fig. 1. (colour online) Displacement distributions.

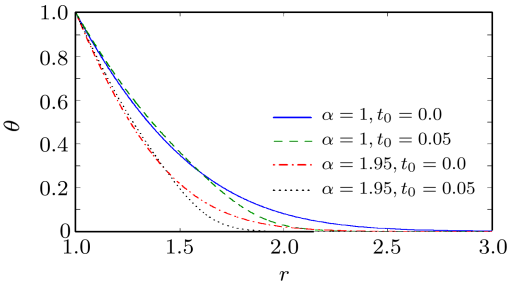


Fig. 2. (colour online) Temperature distributions.

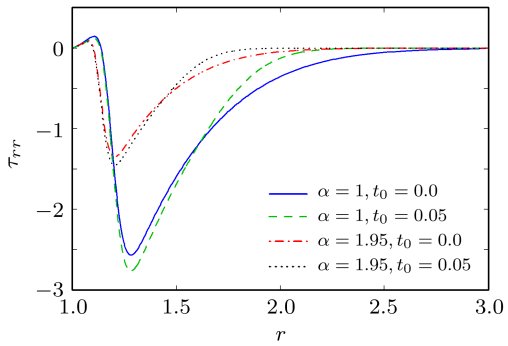


Fig. 3. (colour online) Radial stress distributions.

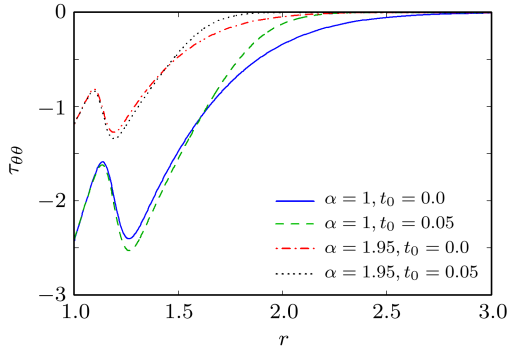


Fig. 4. (colour online) Hoop stress distributions.

Figure 1 shows that displacement u increases for $1.0 < r < 1.2$ and then starts to decrease for $r = 1.2$ (maximum). It is clear that the displacement in the case of temperature dependent elastic moduli is greater than that in the case of temperature independent elastic moduli for $1.0 < r < 1.2$, and vise versa for $r > 1.2$. Figure 2 depicts that temperature Θ decreases with the increasing r. It is clear that the temperature in the case of temperature independent elastic moduli ($\alpha = 1$) is greater than that in the case of temperature dependent elastic moduli ($\alpha = 1.95$). Figure 3 shows that radial stress τ_{rr} decreases and then starts to increase at $r = 1.0$ (minimum). It is clear that the radial stress in the case of temperature independent elastic moduli is greater than that in the case of temperature dependent elastic moduli for $1.0 < r < 1.25$, and vise versa for $r > 1.25$. Figure 4 shows that hoop stress $\tau_{\theta\theta}$ increases and then starts to decrease at $r = 1.2$ (maximum), and finally starts to increase at $r = 1.3$ (minimum). It is clear that the hoop stress in the case of temperature dependent elastic moduli is greater than that in the case of temperature independent elastic moduli.

Figures 5–8 show the results for the displacement, the temperature, the radial stress, and the hoop stress at $t = 0.2$ for four different values of t_0 ($t_0 = 0, 0.03, 0.06, 0.09$) at $\alpha = 1.95$. Figure 5 shows that displacement u in the LS theory is greater than that in the CD theory for $1.0 < r < 1.25$ and is the opposite for $r > 1.25$. Figure 6 shows that the temperature in the LS theory is greater than that in the CD theory for $1.0 < r < 1.4$ and is the opposite for $r > 1.4$. Figures 7 and 8 show that the radial and the hoop stresses in the LS theory are less than those in the CD theory for $1.0 < r < 1.4$ and are the opposite for $r > 1.4$.

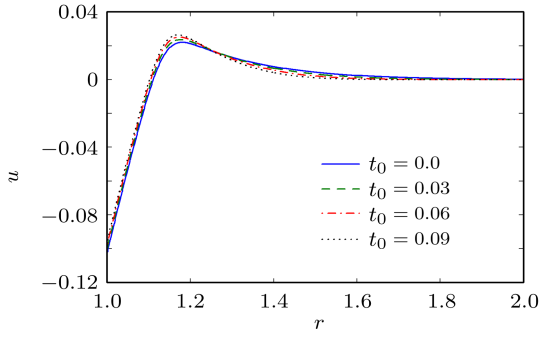


Fig. 5. (colour online) Displacement distributions.

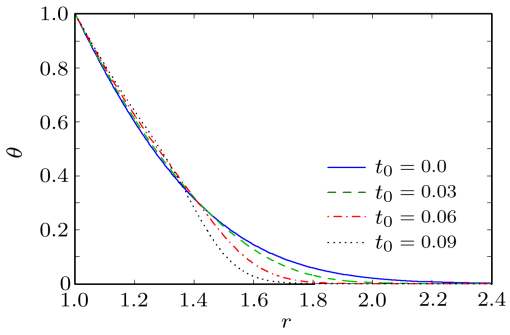


Fig. 6. (colour online) Temperature distributions.

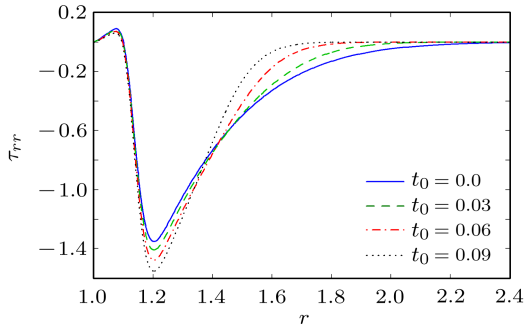


Fig. 7. (colour online) Radial stress distributions.

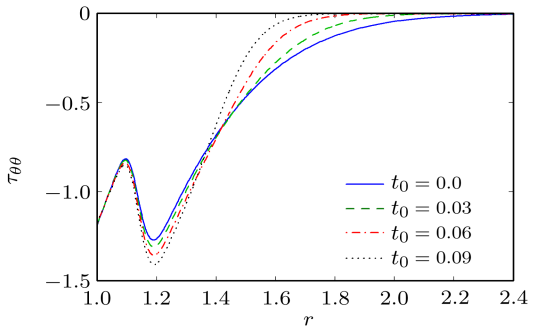


Fig. 8. (colour online) Hoop stress distributions.

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